Nonsmooth Analysis of Interval-Valued Functions and Optimization Methods for Set-Valued Maps

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by

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Chapter 7

Conclusions and Future Directions

This chapter presents a compilation of the main conclusions drawn from this thesis and provides a summary of potential directions for future research.

7.1 General Conclusion

The key conclusions of this thesis are as follows.

- In Chapter 1, we have provided several results on inequalities of intervals, set of intervals, closeness, boundedness, and convex hull of a set of intervals. These are helpful in developing the theories of IVFs and IOPs.

- An analysis of gH-subdifferentiability of IVFs has been given in Chapter 2. Further, a Fritz-John-Type and Karush-Kuhn-Tucker condition for IOPs are studied.

- A study on interval-valued value function has been made in Chapter 3. The Dini-Hadamard ϵ subdifferentiability of interval-valued functions has been given in Chapter 4.

- Rigorous study on the optimality conditions of unconstraint and constraint IOPs with the help of proposed theories in Chapter 2, 3, and 4 has been provided.

- In Chapter 5 and 6, the Newton method and Quasi-Newton method have been provided to solve unconstrained set-valued optimization problems. These methods find weakly minimal solutions to set-valued optimization problems.
- The algorithms and their convergence analysis have been discussed in detail.
- We have provided the numerical results for solving set-valued optimization problems having finite cardinality in Chapter 5 and 6.

7.2 Contribution of the Thesis

This thesis focused on analyzing the theories of IVFs and IOPs through the concept of generalized subdifferentials of IVFs. The chapter-wise contribution of this dissertation is outlined below.

Chapter 1 presents the basic definitions, fundamental results, and literature related to the work in this thesis.

In Chapter 2, we have studied the calculus for gH-subdifferential of convex interval-valued functions (IVFs). A relation is given on the gH-directional derivative of the maximum of finitely many comparable IVFs. With the help of existing calculus on the gH-subdifferential of an IVF, a Fritz-John-type and a KKT-type efficiency condition for weak efficient solutions of IOPs has been derived. In the sequel, a characterization of the weak, efficient solutions of nonconvex composite IOPs by applying the proposed concepts is reported. Various examples have been given in support of the proposed study. This chapter corresponds to the work listed in the "List of Publications" (see 2).

Chapter 3 discussed the concept of interval-valued value function. The notion of a

Lagrangian interval-valued function (IVF) has been studied. Subsequently, a relation between the saddle point and efficient solutions of the corresponding IOP is given. Further, a characterization of the gH-subdifferential set of an interval-valued value function is provided. A saddle point efficiency interpretation of an interval-valued value function is studied based on the saddle point criterion of the Lagrangian IVF has been reported. In the sequel, a relation of gH-subdifferential set of an interval-valued value function with the stability of a solution to an IOP is shown. Furthermore, the gH-subdifferential set of an interval-valued value function is used to prove a relation concerning the efficiency of a solution to an IOP under certain restrictions. Lastly, an example illustrating the application of proposed results on the interval-valued value function is given. This chapter associated with the work published in the "List of Publications" (see 1).

In Chapter 4, the concepts of gH-Dini Hadamard ϵ-subdifferential and $\mathbf{H}_\epsilon$-subgradient for interval-valued function (IVF) has been proposed. After that, interrelations between gH-subgradient and gH-Dini Hadamard ϵ-subgradient, and between gH-Fréchet derivative and gH-Dini Hadamard ϵ-subdifferential is reported. Next, the concepts of $\mathbf{H}_\epsilon$-subgradient, sponge of a set around a point, and gH-calm IVF at a point have been studied. Various necessary and sufficient conditions for obtaining an ϵ-efficient solution to an interval optimization problem (IOP) are derived with the help of gH-Dini Hadamard ϵ-subgradient of an IVF. Finally, an application of proposed results is discussed in the sparsity regularizer for IOPs. This chapter is based on the work published in the "List of Publications" (see 3).

In Chapter 5, the Newton method for unconstrained set optimization problems has been proposed. The Gerstewitz scalarizing function was used to identify a necessary optimality condition for weakly minimal solutions of the considered problem with respect to lower set-less ordering. Next, a sequence of iterative points has been obtained that exhibits local convergence to a point that satisfies the derived necessary optimality condition for weakly minimal points. The partition set concept has been used to transform a given problem into a family of vector optimization problems. A step-wise algorithm of the entire process is provided. The well-definedness and convergence of the proposed method have been analyzed, followed by the convergence of the proposed algorithm under some regularity condition of the stationary points. Next, the proposed method's local superlinear convergence and local quadratic convergence under uniform continuity of the Hessian and Lipschitz continuity of the Hessian has been derived. Lastly, the performance of the proposed method in comparison to the steepest descent method (see [1]) has been performed. This chapter is associated with the work published in the "List of Publications" (see 7).

In Chapter 6, a quasi-Newton method for unconstrained set optimization problems has been proposed. The approach for finding the weakly minimal solutions to a given problem is motivated in accordance with the theories of Chapter 5. However, the process in this chapter involves the computation of the approximate Hessian of every objective function, which is done by a quasi-Newton scheme for vector optimization problems. The proposed quasi-Newton method for set optimization problems is not a direct extension of that for vector optimization problems, as the selected vector optimization problem varies across the iterates. The well-definedness and convergence of the proposed method have been analyzed. We have obtained a local superlinear convergence of the proposed method under uniform continuity of the Hessian approximation function. Lastly, some numerical examples are given to exhibit the performance of the proposed method with the existing steepest-descent method. This chapter corresponds to the work published in the "List of Publications" (see 8).

7.3 Future Scopes of Studies

The research presented in this thesis has various possible opportunities for extension. The following are some potential future problems one can address.

- The proposed results in Chapter 4 may be effective in analysing the duality of non-convex IVFs via augmented Lagrangian IVFs. The Lagrangian IVF can be constructed with the help of supporting cones to the epigraphic of a usual perturbed IVF. With the help of results on IVFs in Chapter 2, 3, one may try to propose a general proximal gradient method and/or adaptive iterative reweighted method for IOPs to find efficient solutions to the nonconvex composite IVFs.

- In Chapter 5 and 6, we have used the lower set less relation to compare images of the set-valued objective function of (SOP). Future research can be carried out with other set relations [86]. Moreover, as a further direction, research can be performed on various other scalarizing functionals, such as separating functionals [117] with uniform level sets, Hiriart-Urruty functional [120], and Drummond-Svaiter functional [97].

- Further, research work can be extended to different step size conditions, such as strong Wolfe or Armijo-Wolfe conditions, and a comparison of the performance of the method can be observed. Additional versions of Newton's and quasi-Newton's methods can be performed for set optimization problems. Moreover, a future comparison study can show the performance profile of these methods with different scalarizing functionals.

- The concepts of interval-valued and set-valued optimization problems are extensively useful in a noise-uncertain environment, control theory, game theory, etc. The differential equations usually appear in incomplete information on demand for a product and changes in the environment in which the proposed study in this thesis will play a key role.