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Appendix A: Terzaghi's (1925) diffusion equation and its discretization

A1. Derivation of the Terzaghi's governing differential equation (GDE)

$$\text{Net water flow within an element} \left(Q_{out} - Q_{in}\right) = \text{Rate of change of void ratio} \left(\frac{\partial e}{\partial t}\right) \quad \text{(A1.1)}$$

From Fig. 3.2, $Q_{out} - Q_{in} = \left(v + \frac{\partial v}{\partial z} dz\right) dxdy - vdxdy = \frac{\partial v}{\partial z} dxdydz = \frac{\partial v}{\partial z} V$ (A1.2)

($V =$ $dxdydz$=total volume of soil element)

Combining Eqs. A1.1 and A1.2 : $\frac{\partial e}{\partial t} = \frac{\partial v}{\partial z}(1+e_0) \quad \Rightarrow \frac{1}{(1+e_0)}\frac{\partial e}{\partial t} = \frac{\partial v}{\partial z}$ (A1.3)

Darcy's law $v = ki = k\frac{\partial u}{\partial z}$ where, i=hydraulic gradient (A1.4)

Spatial derivative of Eq. A1.4 provides: $\frac{\partial v}{\partial z} = \frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}$ (A1.5)

From Eqs. A1.3 and A1.5: $\frac{\partial e}{\partial t} = \frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}(1+e_0)$ (A1.6)

Using chain-rule of differentiation and the definition of coefficient of volume compressibility,

$$\frac{1}{(1+e_0)}\frac{\partial e}{\partial t} = \frac{1}{(1+e_0)}\frac{\partial e}{\partial \sigma'}\frac{\partial \sigma'}{\partial t} = -m_v\frac{\partial \sigma'}{\partial t} \quad \text{(A1.7)}$$

Combining Eqs. A1.6 and A1.7 and the assumption of constant load provides:

$$\frac{k}{m_v\gamma_w}\frac{\partial^2 u}{\partial z^2} = \frac{\partial \sigma'}{\partial t} = \frac{\partial u}{\partial t} \quad ; \sigma' = q - u \quad \text{(A1.8)}$$

GDE for homogenous soil: $c_v\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \quad \left(c_v = \frac{k}{m_v\gamma_w}\right)$ (A1.9)

GDE for layered soil: $\frac{k_j}{m_{vj}\gamma_w}\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}$; j represents the layer number (A1.10)

A2. Compact Matrix form of the algebraic equations:

The discretised equations for homogenous (Eq. 3.14b) and layered soil (Eqs. 3.20, 3.21, and 3.22) can be expressed as:

$$\underset{\text{Known Matrix}}{G} \times \underset{\substack{\text{Unknown Vector}\\ \text{@ arbitrary time step}}}{U^{t+\Delta t}} = \underset{\substack{\text{Known Vector}\\ \text{@ arbitrary time step}}}{H^{t}} \qquad \text{(A2.1)}$$

For homogenous soil:

$$\underbrace{\begin{bmatrix} (1+2\lambda\phi) & -\lambda\phi & 0 & 0 & 0 & 0 & 0 \\ -\lambda\phi & (1+2\lambda\phi) & -\lambda\phi & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -\lambda\phi & (1+2\lambda\phi) & -\lambda\phi & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\lambda\phi & (1+2\lambda\phi) & -\lambda\phi \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\lambda\phi & (1+2\lambda\phi) \end{bmatrix}}_{G} \times \underbrace{\begin{Bmatrix} u_2^{t+\Delta t} \\ u_3^{t+\Delta t} \\ \cdots \\ u_i^{t+\Delta t} \\ \cdots \\ u_{n-2}^{t+\Delta t} \\ u_{n-1}^{t+\Delta t} \end{Bmatrix}}_{U^{t+\Delta t}} = \underbrace{\begin{Bmatrix} \lambda(1-\phi)u_1^t + (1-2\lambda(1-\phi))u_2^t + \lambda(1-\phi)u_3^t \\ \lambda(1-\phi)u_2^t + (1-2\lambda(1-\phi))u_3^t + \lambda(1-\phi)u_4^t \\ \cdots \\ \lambda(1-\phi)u_{i-1}^t + (1-2\lambda)(1-\phi)u_1^t + \lambda(1-\phi)u_{i+1}^t \\ \cdots \\ \lambda_2(1-\phi)u_{n-3}^t + (1-2\lambda_2(1-\phi))u_{n-2}^t + \lambda(1-\phi)u_{n-1}^t \\ \lambda(1-\phi)u_{n-2}^t + (1-2\lambda(1-\phi))u_{n-1}^t + \lambda(1-\phi)u_n^t \end{Bmatrix}}_{H^t} \qquad \text{(A2.2)}$$

For layered soil:

$$\underbrace{\begin{bmatrix} (1+2\lambda_1\phi) & -\lambda_1\phi & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1\phi & (1+2\lambda_1\phi) & -\lambda_1\phi & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -\mu\delta_1\phi & (1+2\mu\phi) & -\mu\delta_2\phi & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\lambda_2\phi & (1+2\lambda_2\phi) & -\lambda_2\phi \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\lambda_2\phi & (1+2\lambda_2\phi) \end{bmatrix}}_{G} \times \underbrace{\begin{Bmatrix} u_2^{t+\Delta t} \\ u_3^{t+\Delta t} \\ \cdots \\ u_m^{t+\Delta t} \\ \cdots \\ u_{n-2}^{t+\Delta t} \\ u_{n-1}^{t+\Delta t} \end{Bmatrix}}_{U^{t+\Delta t}} = \underbrace{\begin{Bmatrix} \lambda_1(1-\phi)u_1^t + (1-2\lambda_2(1-\phi))u_2^t + \lambda_1(1-\phi)u_3^t \\ \lambda_1(1-\phi)u_2^t + (1-2\lambda_2(1-\phi))u_3^t + \lambda_1(1-\phi)u_4^t \\ \cdots \\ \mu\delta_1(1-\phi)u_2^t + (1-2\mu(1-\phi))u_m^t + \mu\delta_2(1-\phi)u_{m+1}^t \\ \cdots \\ \lambda_2(1-\phi)u_{n-3}^t + (1-2\lambda_2(1-\phi))u_{n-2}^t + \lambda_2(1-\phi)u_{n-1}^t \\ \lambda_2(1-\phi)u_{n-2}^t + (1-2\lambda_2(1-\phi))u_{n-1}^t + \lambda_2(1-\phi)u_n^t \end{Bmatrix}}_{H^t} \begin{matrix} \text{1st layer} \\ \\ \\ \text{Interface} \\ \\ \text{2nd layer} \\ \\ \end{matrix} \qquad \text{(A2.3)}$$

Appendix B: Construction and discretization of the GDEs considering stress dependencies, nonlinear flow, and ramp loading

B1. Construction of the GDEs with stress-dependent parameters, Darcian flow, and constant loading

Darcian flow states (Eq. A1.3): $v = k \cdot i \Rightarrow k\frac{\partial u}{\partial z}$ (B1.1)

Spatial derivative of Eq. (B1.1) provides: $\frac{\partial v}{\partial z} = \frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2} + \frac{\partial k}{\partial z}i$ (B1.2)

According to Eq. 4.1a: $e = A - C_c \log \sigma'$ (B1.3)

Temporal derivative of Eq. (B1.3): $\frac{\partial e}{\partial t} = \frac{-C_c}{2.3\sigma'}\frac{\partial \sigma'}{\partial t}$ (B1.4)

Substituting the expression of $\frac{\partial e}{\partial t}$ (from Eq. A3)provides: $\frac{\partial v}{\partial z} = \frac{-C_c}{2.3(1+e_0)\sigma'}\frac{\partial \sigma'}{\partial t}$ (B1.5)

Equating Eqs. (B1.2) and (B1.5): $\frac{-C_c}{2.3(1+e_0)\sigma'}\frac{\partial \sigma'}{\partial t} = \left(\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2} + \frac{\partial k}{\partial z}i\right)$ (B1.6)

According to Eq. 4.2: $k = 10^{\frac{A-B}{M}}(\sigma')^{-\frac{C_c}{M}}$ (B1.7)

Spatial derivative of Eq. (B1.7) provides: $\frac{\partial k}{\partial z} = 10^{\frac{A-B}{M}}\left(\frac{-C_c}{M}\right)(\sigma')^{-\frac{C_c}{M}-1}\frac{\partial \sigma'}{\partial z}$ (B1.8)

Combining Eqs. (B1.6), (B1.7), and (B1.8) provides the following generalized dissipation equation where the flow is non-linear and the material parameters are stress-dependent.

$$\frac{2.3\sigma'}{C_c}\frac{(1+e_0)}{\gamma_w}10^{\frac{A-B}{M}}(\sigma')^{1-\frac{C_c}{M}}\frac{\partial^2(u)}{\partial z^2} + \frac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c}\left(\frac{-C_c}{M}\right)(\sigma')^{-\frac{C_c}{M}}\left(\frac{\partial \sigma'}{\partial z}\right)i = \frac{\partial u}{\partial t} \quad \text{(B1.9)}$$

The modified consolidation equation can be illustrated as:

$$C_n(\sigma')^{\beta}\frac{\partial^2 u}{\partial z^2} + C_n\gamma_w(1-\beta)(\sigma')^{\beta-1}\left(\frac{\partial u}{\partial z}\right)i = \frac{\partial u}{\partial t} \quad \text{(B1.10)}$$

Case A (Quasi-Permeability variation): $X_1 \dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial u}{\partial t}$ (B1.11)

Case B (Total-Permeability variation): $X_1 \dfrac{\partial^2 u}{\partial z^2} + Y_1 \dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial t}$ (B1.12)

where $C_n = \dfrac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c \gamma_w}$ $X_1 = C_n(\sigma')^{\beta}; Y_1 = C_n \gamma_w (1-\beta)(\sigma')^{\beta-1} i$

B2. Discretization of the GDEs formed in B1

Case A: $X_1 \dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial u}{\partial t}$ (B2.1)

Discretization by CNI scheme : $X_{1@i}\left[\dfrac{1}{2}\left\{\left.\dfrac{\partial^2 u}{\partial z^2}\right|_i^{t+\Delta t} + \left.\dfrac{\partial^2 u}{\partial z^2}\right|_i^{t}\right\}\right] = \left.\dfrac{\partial u}{\partial t}\right|_i$ (B2.2)

$\Rightarrow X_{1@i}\left[\dfrac{1}{2}\left\{\left(\dfrac{u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t}}{\Delta z^2}\right) + \left(\dfrac{u_{i-1}^{t} - 2u_i^{t} + u_{i+1}^{t}}{\Delta z^2}\right)\right\}\right] = \left(\dfrac{u_i^{t+\Delta t} - u_i^t}{\Delta t}\right)$ (B2.3)

$\Rightarrow L_i u_{i-1}^{t+\Delta t} + 2(-L_i + 1)u_i^{t+\Delta t} + L_i u_{i+1}^{t+\Delta t} = -L_i u_{i-1}^t + 2(L_i - 1)u_i^t - L_i u_{i+1}^t$ (B2.4)

where $X_{1@i}^t(z,t) = C_n(\sigma_i'^{@t})E_i'^{@t}$; $C_n = \dfrac{2.3(1+e_0)}{C_c \gamma_w}$; $L_i^t = X_{1@i}^t \dfrac{\Delta t}{\Delta z^2}$

Case B: $X_1 \dfrac{\partial^2 u}{\partial z^2} + Y_1 \dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial t}$ (B2.5)

Discretization by CNI scheme- $X_{1@i}\left[\dfrac{1}{2}\left\{\left.\dfrac{\partial^2 u}{\partial z^2}\right|_i^{t+\Delta t} + \left.\dfrac{\partial^2 u}{\partial z^2}\right|_i^{t}\right\}\right] + Y_{1@i}\left\{\left.\dfrac{\partial u}{\partial z}\right|_i^{t+\Delta t}\right\} = \left.\dfrac{\partial u}{\partial t}\right|_i$ (B2.6)

$\Rightarrow X_{1@i}\left[\dfrac{1}{2}\left\{\left(\dfrac{u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t}}{\Delta z^2}\right) + \left(\dfrac{u_{i-1}^{t} - 2u_i^{t} + u_{i+1}^{t}}{\Delta z^2}\right)\right\}\right] + Y_{1@i}\left[\dfrac{u_{i+1}^{t+\Delta t} - u_i^{t+\Delta t}}{\Delta z}\right] = \left(\dfrac{u_i^{t+\Delta t} - u_i^t}{\Delta t}\right)$ (B2.7)

$$\Longrightarrow \left[\frac{L_i}{2} \left\{ \left(u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t} \right) + \left(u_{i-1}^{t} - 2u_i^{t} + u_{i+1}^{t} \right) \right\} \right] + P_i \left[u_{i+1}^{t+\Delta t} - u_i^{t+\Delta t} \right] = \left(u_i^{t+\Delta t} - u_i^{t} \right) \quad \text{(B2.8)}$$

$$\Longrightarrow L_i u_{i-1}^{t+\Delta t} + 2\left(L_i + P_i^t + 1 \right) u_i^{t+\Delta t} - \left(L_i + 2P_i^t \right) u_{i+1}^{t+\Delta t} = -L_i 2u_{i-1}^{t} + 2\left(L_i + 1 \right) u_i^{t} - L_i 2u_{i+1}^{t} \quad \text{(B2.9)}$$

where, $X_{1@i}^{t}(z,t) = C_n \left(\sigma_i'^{@t} \right)^{\beta} E_i'^{@t}$; $Y_{1@i}^{t}(z,t) = C_n \gamma_w (1-\beta) \left(\sigma_i'^{@t} \right)^{\beta-1} E_i^{@t}$

$C_n = \dfrac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c \gamma_w}$; $L_i^t(z,t) = \dfrac{X_{1@i}^{t} \Delta t}{\Delta z^2}$; $P_i^t(z,t) = \dfrac{Y_{1@i}^{t} \Delta t}{\Delta z}$

In Matrix form, Eqs. (B2.4) and (B2.9) can be recast as: (B2.10) $G \times U^{t+\Delta t} = H^t$

where the field variable, U^t, is expressed as: $U^{t+\Delta t} = \{u_2^{t+\Delta t} \quad u_3^{t+\Delta t} \quad \cdots \quad u_i^{t+\Delta t} \quad \cdots \quad u_{n-1}^{t+\Delta t} \quad u_n^{t+\Delta t}\}^T$

$$\text{For Case A}: G = \begin{bmatrix} 2(-L_2+1) & L_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_3 & 2(-L_4+1) & L_4 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & L_{i-1} & 2(-L_i+1) & L_{i+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & L_{n-2} & 2(-L_{n-1}+1) & L_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & L_n & 2(-L_n+1) \end{bmatrix}$$

$$\text{For Case B}: G = \begin{bmatrix} 2(-L_2+P_2+1) & (L_2+2P_2) & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_3 & 2(-L_4+P_4+1) & (L_5+2P_5) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & L_{i-1} & 2(-L_i+P_i+1) & (L_{i+1}+2P_{i+1}) & \cdots & \cdots & L_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & L_{n-2} & 2(-L_{n-1}+P_{n-1}+1) & (L_n+2P_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & L_n & 2(-L_{n+1}+P_{n+1}+1) \end{bmatrix}$$

$$\text{For Case A}: H^t = \begin{Bmatrix} 2(L_2^t+1)u_2^t - L_2^t 2u_3^t \\ -L_3^t 2u_2^t + 2(L_3^t+1)u_3^t - L_3^t 2u_4^t \\ \vdots \\ -L_i^t 2u_{i-1}^t + 2(L_i^t+1)u_i^t - L_i^t 2u_{i+1}^t \\ \vdots \\ -L_{n-1}^t 2u_{n-2}^t + 2(L_{n-1}^t+1)u_{n-1}^t - L_{n-1}^t 2u_n^t \\ -L_n^t 2u_{n-1}^t + 2(L_n^t+1)u_n^t \end{Bmatrix} \quad \text{For Case B}: H^t = \begin{Bmatrix} 2(L_2^t-1)u_2^t - L_2^t u_3^t \\ -L_3^t u_2^t + 2(L_3^t-1)u_3^t - L_3^t u_4^t \\ \vdots \\ -L_i^t u_{i-1}^t + 2(L_i^t-1)u_i^t - L_i^t u_{i+1}^t \\ \vdots \\ -L_{n-1}^t u_{n-2}^t + 2(L_{n-1}^t-1)u_{n-1}^t - L_i^t u_n^t \\ -L_n^t u_{n-1}^t + 2(L_n^t-1)u_n^t \end{Bmatrix}$$

B3. Construction of the GDEs with stress-dependent parameters, non-Darcian flow, and ramp loading

According to Eq. 4.9: $v = k_u\left(i-(1-a)i_1\left[1-\exp\left(\frac{-i\theta}{i_1}\right)\right]\right)$; (a,θ,i_1) model parameters (B3.1)

Spatial derivative of Eq. (B3.1) provides: $$\frac{\partial v}{\partial z} = \frac{k}{\gamma_w}E\frac{\partial^2 u}{\partial z^2} + \frac{\partial k}{\partial z}E' \quad \text{(B3.2)}$$

$$\text{where, } E = \left\{1-(1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}, E' = \left\{i-(1-a)i_1\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}$$

Equating Eqs. (B1.4) and (B3.2): $$\frac{k}{\gamma_w}E'\frac{\partial^2 u}{\partial z^2} + \frac{\partial k}{\partial z}E = \frac{-C_c}{2.3(1+e_0)\sigma'}\frac{\partial \sigma'}{\partial t} \quad \text{(B3.3)}$$

Combining Eqs. (B1.7), (B1.8), and (B3.3) provides the following generalized dissipation equation where the flow is non-linear and the material parameters are stress-dependent.

$$\frac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c\gamma_w}(\sigma')^{1-\frac{C_c}{M}}E\frac{\partial^2 u}{\partial z^2} + \frac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c}\left(\frac{-C_c}{M}\right)(\sigma')^{\frac{-C_c}{M}}E'\frac{\partial \sigma'}{\partial z} = -\frac{\partial \sigma'}{\partial t} \quad \text{(B3.4)}$$

Case A (Finite value of M):

In a more compact form, Eq. (B3.4) can be stated as below:

$$C_n(\sigma')^{\beta}E\frac{\partial^2 u}{\partial z^2} + C_n\gamma_w(1-\beta)(\sigma')^{\beta-1}E'\frac{\partial u}{\partial z} = \frac{\partial \sigma'}{\partial t} \quad \text{(B3.5)}$$

$$\Rightarrow X\frac{\partial^2 u}{\partial z^2} + Y\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial q}{\partial t} \quad \text{(B3.6)}$$

where, $C_n = \frac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c\gamma_w}$; $\beta = 1-\frac{C_c}{M}$; $X = C_n(\sigma')^{\beta}E$; $Y = C_n\gamma_w(1-\beta)(\sigma')^{\beta-1}E'$

Case B (infinite *M* i.e., constant permeability):

For $M \rightarrow \infty$, $\beta = 1$, $\therefore$Eq A13 becomes: $C_n(\sigma')E\dfrac{\partial^2 u}{\partial z^2} = -\dfrac{\partial \sigma'}{\partial t}$ (B3.7)

$$\Rightarrow X\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} - \frac{\partial q}{\partial t} \quad \text{where, } C_n = \frac{2.3(1+e_0)}{C_c\gamma_w};\ X = C_n(\sigma')E \qquad \text{(B3.8)}$$

B4. Discretization of the GDEs formed in B3

Case A: $$X\frac{\partial^2 u}{\partial z^2} + Y\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial q}{\partial t} \qquad \text{(B3.7)}$$

Discretization by CNI scheme- $$X_i^t\left[\frac{1}{2}\left\{\left.\frac{\partial^2 u}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u}{\partial z^2}\right|_i^t\right\}\right] + Y_i^t\left\{\left.\frac{\partial u}{\partial z}\right|_i^{t+\Delta t}\right\} = \left.\frac{\partial u}{\partial t}\right|_i - \left.\frac{\partial q}{\partial t}\right|_i \qquad \text{(B4.1)}$$

$$\Rightarrow X_i^t\left[\frac{1}{2}\left\{\left(\frac{u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t}}{\Delta z^2}\right) + \left(\frac{u_{i-1}^t - 2u_i^t + u_{i+1}^t}{\Delta z^2}\right)\right\}\right] + Y_i^t\left[\frac{u_{i+1}^{t+\Delta t} - u_i^{t+\Delta t}}{\Delta z}\right] = \left(\frac{u_i^{t+\Delta t} - u_i^t}{\Delta t}\right) - \left(\frac{q_i^{t+\Delta t} - q_i^t}{\Delta t}\right) \qquad \text{(B4.2)}$$

$$\Rightarrow L_i^t u_{i-1}^{t+\Delta t} + 2(L_i^t + P_i^t + 1)u_i^{t+\Delta t} - (L_i^t + 2P_i^t)u_{i+1}^{t+\Delta t} = -L_i^t 2u_{i-1}^t + 2(L_i^t + 1)u_i^t - L_i^t 2u_{i+1}^t - 2(q_i^{t+\Delta t} - q_i^t) \qquad \text{(B4.3)}$$

where, $$X_i^t(z,t) = C_n\left(\sigma_i'^{@t}\right)^\beta E_i'^{@t};\ Y_i^t(z,t) = C_n\gamma_w(1-\beta)\left(\sigma_i'^{@t}\right)^{\beta-1}E_i^{@t}$$

$$E_i'^{@t} = \left\{1 - (1-\alpha)\theta\left[\exp\left(\frac{-\theta}{i_T\gamma_w}\frac{u_{i+1}^t - u_i^t}{\Delta z}\right)\right]\right\};\ E_i^{@t} = \left\{i - (1-\alpha)i_T\left[\exp\left(\frac{-\theta}{i_T\gamma_w}\frac{u_{i+1}^t - u_i^t}{\Delta z}\right)\right]\right\}$$

$$C_n = \frac{2.3(1+e_0)10^{\frac{A-B}{M}}}{C_c\gamma_w}; \qquad L_i^t(z,t) = \frac{X_i^t\Delta t}{\Delta z^2};\quad P_i^t(z,t) = \frac{Y_i^t\Delta t}{\Delta z}$$

Case B: $$X\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} - \frac{\partial q}{\partial t} \qquad \text{(B4.4)}$$

Discretization by CNI scheme : $$X_i^t\left[\frac{1}{2}\left\{\left.\frac{\partial^2 u}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u}{\partial z^2}\right|_i^t\right\}\right] = \left.\frac{\partial u}{\partial t}\right|_i - \left.\frac{\partial q}{\partial t}\right|_i \qquad \text{(B4.5)}$$

$$X_i^t\left[\frac{1}{2}\left\{\left(\frac{u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t}}{\Delta z^2}\right) + \left(\frac{u_{i-1}^t - 2u_i^t + u_{i+1}^t}{\Delta z^2}\right)\right\}\right] = \left(\frac{u_i^{t+\Delta t} - u_i^t}{\Delta t}\right) - \left(\frac{q_i^{t+\Delta t} - q_i^t}{\Delta t}\right) \qquad \text{(B4.6)}$$

$$L_i^t u_{i-1}^{t+\Delta t} + 2(-L_i^t + 1)u_i^{t+\Delta t} + L_i^t u_{i+1}^{t+\Delta t} = -L_i^t u_{i-1}^t + 2(L_i^t - 1)u_i^t - L_i^t u_{i+1}^t - 2(q_i^{t+\Delta t} - q_i^t) \qquad \text{(B4.7)}$$

where $X_i^t(z,t) = C_n\left(\sigma_i'^{@t}\right)E_i'^{@t}$; $C_n = \dfrac{2.3(1+e_0)}{C_c\gamma_w}$; $L_i^t = X_i^t\dfrac{\Delta t}{\Delta z^2}$

In Matrix form, Eqs. (B4.3) and (B4.7) can be recast as: (B4.8) $G \times U^{t+\Delta t} = H^t$

where the field variable, U^t, is expressed as: $U^{t+\Delta t} = \{u_2^{t+\Delta t} \quad u_3^{t+\Delta t} \quad \cdots \quad u_i^{t+\Delta t} \quad \cdots \quad u_{n-1}^{t+\Delta t} \quad u_n^{t+\Delta t}\}^T$

$$\text{For Case A: } G^t = \begin{bmatrix} 2(-L_2^t + P_2^t + 1) & (L_2^t + 2P_i^t) & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_3^t & 2(-L_3^t + P_3^t + 1) & (L_3^t + 2P_3^t) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & L_i^t & 2(-L_i^t + P_i^t + 1) & (L_i^t + 2P_i^t) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & L_{n-1}^t & 2(-L_{n-1}^t + P_{n-1}^t + 1) & (L_{n-1}^t + 2P_{n-1}^t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & L_n^t & 2(-L_n^t + P_n^t + 1) \end{bmatrix}$$

$$\text{For Case B: } G^t = \begin{bmatrix} 2(-L_2^t + 1) & L_2^t & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_3^t & 2(-L_3^t + 1) & L_3^t & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & L_i^t & 2(-L_i^t + 1) & L_i^t & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & L_{n-1}^t & 2(-L_{n-1}^t + 1) & L_{n-1}^t \\ \cdots & \cdots & \cdots & \cdots & \cdots & L_n^t & 2(-L_n^t + 1) \end{bmatrix}$$

$$\text{For Case A: } H^t = \begin{Bmatrix} 2(L_2^t + 1)u_2^t - L_2^t 2u_3^t - 2(q_2^{t+\Delta t} - q_2^t) \\ -L_3^t 2u_2^t + 2(L_3^t + 1)u_3^t - L_3^t 2u_4^t - 2(q_3^{t+\Delta t} - q_3^t) \\ \vdots \\ -L_i^t 2u_{i-1}^t + 2(L_i^t + 1)u_i^t - L_i^t 2u_{i+1}^t - 2(q_i^{t+\Delta t} - q_i^t) \\ \vdots \\ -L_{n-1}^t 2u_{n-2}^t + 2(L_{n-1}^t + 1)u_{n-1}^t - L_{n-1}^t 2u_n^t - 2(q_{n-1}^{t+\Delta t} - q_{n-1}^t) \\ -L_n^t 2u_{n-1}^t + 2(L_n^t + 1)u_n^t - 2(q_n^{t+\Delta t} - q_n^t) \end{Bmatrix} \quad \text{For Case B: } H^t = \begin{Bmatrix} 2(L_2^t - 1)u_2^t - L_2^t u_3^t - 2(q_2^{t+\Delta t} - q_2^t) \\ -L_3^t u_2^t + 2(L_3^t - 1)u_3^t - L_3^t u_4^t - 2(q_3^{t+\Delta t} - q_3^t) \\ \vdots \\ -L_i^t u_{i-1}^t + 2(L_i^t - 1)u_i^t - L_i^t u_{i+1}^t - 2(q_i^{t+\Delta t} - q_i^t) \\ \vdots \\ -L_{n-1}^t u_{n-2}^t + 2(L_{n-1}^t - 1)u_{n-1}^t - L_i^t u_n^t - 2(q_{n-1}^{t+\Delta t} - q_{n-1}^t) \\ -L_n^t u_{n-1}^t + 2(L_n^t - 1)u_n^t - 2(q_i^{t+\Delta t} - q_i^t) \end{Bmatrix}$$

Appendix C: Modification and discretization of the coupled consolidation equation considering non-Darcian flow, periodic loading, and semi-permeable drainage boundaries

C1. Derivation of the governing differential equations

Total stress=Effective stress+ Excess Pore water pressure $\sigma = \sigma' + u$ (C1.1)

Differentiating of Eq. C1.1: $\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma'}{\partial t} + \frac{\partial u}{\partial t}$ (C1.2)

Assuming small strain $\sigma' = -\frac{1}{m_v}\frac{\partial s}{\partial z}$ (Biot's 1941) (C1.3)

From equations 5C1.2 and C1.3 $\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial t} = \frac{1}{m_v}\frac{\partial^2 s}{\partial z \partial t}$ (C1.4)

The rate of volume change in terms of flux (Appendix 1a) $\frac{\partial V}{\partial t} = \frac{\partial v}{\partial z}V$ (C1.5)

For non-Darcian flow $v = k\left(i - (1-a)i_1\left[1 - \exp\left(\frac{-i\theta}{i_1}\right)\right]\right)$ (Chapter 4) (C1.6)

Differentiating Eq. C1.6 in spatial direction while assuming permeability is constant in both spatial and temporal direction $\frac{\partial v}{\partial z} = \frac{k}{\gamma_w}\left\{1 - (1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}\frac{\partial^2 u}{\partial z^2}$ (C1.7)

From equation C1.5 and C1.7 $\frac{\partial V}{\partial t} = \frac{k}{\gamma_w}\left\{1 - (1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}\frac{\partial^2 u}{\partial z^2}V$ (C1.8)

The rate of volume change (in terms of settlement) $\frac{\partial V}{\partial t} = -\frac{\partial}{\partial t}\frac{\partial s}{\partial z}V$ (C1.9)

From equations C1.8 and C1.9 $\frac{\partial}{\partial t}\left(\frac{\partial s}{\partial z}\right) = -\frac{k}{\gamma_w}\left\{1 - (1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}\frac{\partial^2 u}{\partial z^2}$

(C1.10)

From C1.3 and C1.10 $-\frac{1}{m_v}\frac{\partial \sigma'}{\partial t} = \frac{k}{\gamma_w}\left\{1 - (1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}\frac{\partial^2 u}{\partial z^2}$ (C1.11)

$$\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial t} = \frac{k}{m_v\gamma_w}\underbrace{\left\{1 - (1-a)\theta\left[\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u}{\partial z}\right)\right]\right\}}_{E}\frac{\partial^2 u}{\partial z^2} \quad \text{(C1.12)}$$

Eq. C1.4 and C1.12 are used to calculate the PWP and settlement of the soil in spatial and temporal direction in coupled mannar.

C2. Discretization of the governing differential equation into algebraic form

Eq. C1.12 can be written as: $\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial t} = X \frac{\partial^2 u}{\partial z^2}$ where $X = \frac{k}{m_v \gamma_w} \cdot E$ (C2.1)

$$X \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} - \frac{\partial q}{\partial t} \quad \text{(C2.2)}$$

Discretization by CNI scheme : $X_i^t \left[\frac{1}{2} \left\{ \frac{\partial^2 u}{\partial z^2}\bigg|_i^{t+\Delta t} + \frac{\partial^2 u}{\partial z^2}\bigg|_i^{t} \right\} \right] = \frac{\partial u}{\partial t}\bigg|_i - \frac{\partial q}{\partial t}\bigg|_i$ (C2.3)

$$X_i^t \left[\frac{1}{2} \left\{ \left(\frac{u_{i-1}^{t+\Delta t} - 2u_i^{t+\Delta t} + u_{i+1}^{t+\Delta t}}{\Delta z^2} \right) + \left(\frac{u_{i-1}^t - 2u_i^t + u_{i+1}^t}{\Delta z^2} \right) \right\} \right] = \left(\frac{u_i^{t+\Delta t} - u_i^t}{\Delta t} \right) - \left(\frac{q_i^{t+\Delta t} - q_i^t}{\Delta t} \right) \quad \text{(C2.4)}$$

$$L_i^t u_{i-1}^{t+\Delta t} + 2\left(-L_i^t + 1\right) u_i^{t+\Delta t} + L_i^t u_{i+1}^{t+\Delta t} = -L_i^t u_{i-1}^t + 2\left(L_i^t - 1\right) u_i^t - L_i^t u_{i+1}^t - 2\left(q_i^{t+\Delta t} - q_i^t\right) \quad \text{(C2.5)}$$

where $X_i^t(z,t) = \frac{k}{m_v \gamma_w} E_i'^{@t}$; $L_i^t = X_i^t \frac{\Delta t}{\Delta z^2}$

From Eq. C1.4 $\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial t} = \frac{1}{m_v} \frac{\partial^2 s}{\partial z \partial t}$ (C2.6)

Discretization by CNI scheme : $\frac{1}{m_v} \left[\frac{\partial^2 s}{\partial z \partial t} \right] = \frac{\partial u}{\partial t}\bigg|_i - \frac{\partial q}{\partial t}\bigg|_i$ (C2.7)

$$\frac{1}{m_v} \left[\frac{s_{i+1}^{t+\Delta t} - s_i^{t+\Delta t} - s_{i+1}^t + s_i^t}{\Delta z \Delta t} \right] = \left(\frac{u_i^{t+\Delta t} - u_i^t}{\Delta t} \right) - \left(\frac{q_i^{t+\Delta t} - q_i^t}{\Delta t} \right) \quad \text{(C2.8)}$$

$-\psi s_{i+1}^{t+\Delta t} + u_i^{t+\Delta t} + \psi s_i^{t+\Delta t} = u_i^{t+\Delta t} - \psi s_{i+1}^{t+\Delta t} + \psi s_i^{t+\Delta t} + q_i^{t+\Delta t} - q_i^{t+\Delta t}$,where $\psi = \frac{1}{m_v \Delta z}$ (C2.9)

In Matrix form, Eqs. (C2.5) and (C2.9) can be recast as

G^t	$\times$	$U^{t+\Delta t}$	$=$	H^t	(C2.10)
Known Matrix		Unknown Vector		Known Vector	
@ arbitrary time step		@ arbitrary time step		@ arbitrary time step	

$$G^t = \begin{bmatrix} 2(1-aL_1^t) & 0 & L_1^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \psi & 0 & -\psi & 0 & 0 & 0 & 0 & 0 & 0 \\ L_2^t & 0 & 2(1-L_2^t) & 0 & L_2^t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \psi & 0 & -\psi & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & L_i^t & 0 & 2(1-L_i^t) & 0 & L_i^t & 0 & \cdots & \cdots \\ \cdots & \cdots & 0 & 0 & 1 & \psi & 0 & -\psi & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & L_n^t & 0 & 2(1-L_n^t) & 0 & L_n^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \psi & 0 & -\psi \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{n+1}^t & 0 & 2(1-L_{n+1}^t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \psi \end{bmatrix}$$

$$U^{t+\Delta t} = \begin{bmatrix} u_1^{t+\Delta t} \\ s_1^{t+\Delta t} \\ u_2^{t+\Delta t} \\ s_2^{t+\Delta t} \\ \cdots \\ u_i^{t+\Delta t} \\ s_i^{t+\Delta t} \\ \cdots \\ u_n^{t+\Delta t} \\ s_n^{t+\Delta t} \\ u_{n+1}^{t+\Delta t} \\ s_{n+1}^{t+\Delta t} \end{bmatrix} \quad H^t = \begin{Bmatrix} 2(L_1^t - 1)u_1^t - L_1^t u_2^t - 2(q_1^{t+\Delta t} - q_1^t) \\ u_1^t + \psi s_1^t - \psi s_2^t - (q_1^{t+\Delta t} - q_1^t) \\ -L_1^t u_1^t + 2(L_1^t - 1)u_2^t - L_1^t u_3^t - 2(q_2^{t+\Delta t} - q_2^t) \\ u_2^t + \psi s_2^t - \psi s_3^t - (q_2^{t+\Delta t} - q_2^t) \\ \cdots \\ -L_1^t u_{i-1}^t + 2(L_1^t - 1)u_i^t - L_1^t u_{i+1}^t - 2(q_i^{t+\Delta t} - q_i^t) \\ u_i^t + \psi s_i^t - \psi s_{i+1}^t - (q_i^{t+\Delta t} - q_i^t) \\ \cdots \\ -L_1^t u_{n-1}^t + 2(L_1^t - 1)u_n^t - L_1^t u_{n+1}^t - 2(q_n^{t+\Delta t} - q_n^t) \\ u_n^t + \psi s_n^t - \psi s_{n+1}^t - (q_n^{t+\Delta t} - q_n^t) \\ -L_1^t u_n^t + 2(L_1^t - 1)u_{n+1}^t - 2(q_{n+1}^{t+\Delta t} - q_{n+1}^t) \\ u_{n+1}^t + \psi s_{n+1}^t - (q_{n+1}^{t+\Delta t} - q_{n+1}^t) \end{Bmatrix}$$

$$a = 0.5\left(1 + \frac{\Delta z R_T}{H}\right); b = 0.5\left(1 + \frac{\Delta z R_B}{H}\right)$$

Appendix-D: Forms of the coefficient and Jacobian matrices and the vectors for different methods

D1. Linearized form of non-Darcian consolidation through CNI scheme (Method A1)

$$\underset{\text{@ arbitrary time step}}{\underset{\text{Known Matrix}}{G^t}} \times \underset{\text{@ arbitrary time step}}{\underset{\text{Unknown Vector}}{U^{t+\Delta t}}} = \underset{\text{@ arbitrary time step}}{\underset{\text{Known Vector}}{H^t}} \qquad \text{D1.1}$$

$$G^t = \begin{bmatrix} \left(1+\lambda E_i^t\right) & -0.5\lambda E_i^t & 0 & 0 & 0 & 0 & 0 \\ -0.5\lambda E_i^t & \left(1+\lambda E_i^t\right) & -0.5\lambda E_i^t & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -0.5\lambda E_i^t & \left(1+\lambda E_i^t\right) & -0.5\lambda E_i^t & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -0.5\lambda E_i^t & \left(1+\lambda E_i^t\right) & -0.5\lambda E_i^t \\ \cdots & \cdots & \cdots & \cdots & \cdots & -0.5\lambda E_i^t & \left(1+\lambda E_i^t\right) \end{bmatrix}$$

$$U^{t+\Delta t} = \begin{Bmatrix} u_2^{t+\Delta t} \\ u_3^{t+\Delta t} \\ \cdots \\ u_i^{t+\Delta t} \\ \cdots \\ u_{n-2}^{t+\Delta t} \\ u_{n-1}^{t+\Delta t} \end{Bmatrix} H^t = \begin{Bmatrix} 0.5\lambda u_1^t + (1-\lambda)u_2^t + 0.5\lambda u_3^t - \left(q^{t+\Delta t} - q^t\right) \\ 0.5\lambda u_2^t + (1-\lambda)u_3^t + 0.5\lambda u_4^t - \left(q^{t+\Delta t} - q^t\right) \\ \cdots \\ 0.5\lambda u_{i-1}^t + (1-\lambda)u_1^t + 0.5\lambda u_{i+1}^t - \left(q^{t+\Delta t} - q^t\right) \\ \cdots \\ 0.5\lambda_2 u_{n-3}^t + (1-\lambda_2)u_{n-2}^t + 0.5\lambda u_{n-1}^t - \left(q^{t+\Delta t} - q^t\right) \\ 0.5\lambda u_{n-2}^t + (1-\lambda)u_{n-1}^t + 0.5\lambda u_n^t - \left(q^{t+\Delta t} - q^t\right) \end{Bmatrix}$$

D2. The form of Jacobian Matrix (*J*) and *G* vector for nonlinear CNI scheme (Method A2)

$$\boldsymbol{J}_{ij}\left(\text{Jacobian Matrix}\right) = \frac{\partial \boldsymbol{g}_i}{\partial u_j} \text{ and } \boldsymbol{G}_i = \boldsymbol{g}_i \qquad \text{(D2.1)}$$

Note that, i and j runs from (i) 2 to n for PTPB and (ii) 2 to n+1 for PTIB. Therefore, for $t+\Delta t$:

$$
\boldsymbol{J} = \begin{bmatrix}
\frac{\partial \boldsymbol{g}_2}{\partial u_2^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_2}{\partial u_3^{t+\Delta t}} & 0 & 0 & 0 & 0 \\
\frac{\partial \boldsymbol{g}_3}{\partial u_2^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_3}{\partial u_3^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_3}{\partial u_4^{t+\Delta t}} & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \frac{\partial \boldsymbol{g}_{n-1}}{\partial u_{n-2}^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_{n-1}}{\partial u_{n-1}^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_{n-1}}{\partial u_n^{t+\Delta t}} \\
0 & 0 & 0 & 0 & \frac{\partial \boldsymbol{g}_n}{\partial u_{n-1}^{t+\Delta t}} & \frac{\partial \boldsymbol{g}_n}{\partial u_n^{t+\Delta t}}
\end{bmatrix} ; \boldsymbol{G} = \begin{Bmatrix}
\boldsymbol{g}_2\left(u_1^t, u_2^t, u_3^t, u_1^{t+\Delta t}, u_2^{t+\Delta t}, u_3^{t+\Delta t}\right) \\
\boldsymbol{g}_3\left(u_2^t, u_3^t, u_4^t, u_2^{t+\Delta t}, u_3^{t+\Delta t}, u_4^{t+\Delta t}\right) \\
\vdots \\
\vdots \\
\boldsymbol{g}_{n-1}\left(u_{n-2}^t, u_{n-1}^t, u_n^t, u_{n-2}^{t+\Delta t}, u_{n-1}^{t+\Delta t}, u_n^{t+\Delta t}\right) \\
\boldsymbol{g}_n\left(u_{n-1}^t, u_n^t, u_{n+1}^t, u_{n-1}^{t+\Delta t}, u_n^{t+\Delta t}, u_{n+1}^{t+\Delta t}\right)
\end{Bmatrix}
$$

$$
\boldsymbol{g}_2 : \lambda\left(u_1^t - 2u_2^t + u_3^t + u_1^{t+\Delta t} - 2u_2^{t+\Delta t} + u_3^{t+\Delta t}\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{u_3^{t+\Delta t} - u_1^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - u_2^{t+\Delta t} + u_2^t\right) = 0 \quad \text{(D2.2)}
$$

$$
\boldsymbol{g}_3 : \lambda\left(u_2^t - 2u_3^t + u_4^t + u_2^{t+\Delta t} - 2u_3^{t+\Delta t} + u_4^{t+\Delta t}\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{u_4^{t+\Delta t} - u_2^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - u_3^{t+\Delta t} + u_3^t\right) = 0 \quad \text{(D2.2)}
$$

$\vdots$

$\vdots$

$$
\boldsymbol{g}_n : \lambda\left(u_{n-1}^t - 2u_n^t + u_{n+1}^t + u_{n-1}^{t+\Delta t} - 2u_n^{t+\Delta t} + u_{n+1}^{t+\Delta t}\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{u_{n+1}^{t+\Delta t} - u_{n-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - u_n^{t+\Delta t} + u_n^t\right) = 0 \quad \text{(D2.3)}
$$

$$
\frac{\partial \mathbf{g}_i}{\partial u_{i-1}^{t+\Delta t}} = \lambda\left[2\Delta z E_i^{t+\Delta t} + \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^t - 2u_{i-1}^t + u_{i-1}^t\right)\left\{k_2\left(1 - 2E_i^{t+\Delta t}\right)\right\}\right] \quad \text{(D2.4)}
$$

$$
\frac{\partial \mathbf{g}_i}{\partial u_{i+1}^{t+\Delta t}} = \lambda\left[2\Delta z E_i^{t+\Delta t} - \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^t - 2u_{i-1}^t + u_{i-1}^t\right)\left\{k_2\left(1 - 2E_i^{t+\Delta t}\right)\right\}\right] \quad \text{(D2.5)}
$$

$$
\frac{\partial \mathbf{g}_i}{\partial u_i^{t+\Delta t}} = -\Delta z\left(1 + 4E_i^{t+\Delta t}\lambda\right) \text{where, } E_i^{t+\Delta t} = 1 - b_1 \exp\left\{b_2\left(\frac{u_{i+1}^{t+\Delta t} - u_{i-1}^{t+\Delta t}}{2\Delta z}\right)\right\} \quad \text{(D2.6)}
$$

D3. The form of Jacobian Matrices (J_1 and J_2) and G vector for nonlinear BKC method (Method B2)

$$J_{1ij} = \frac{\partial h_{1i}}{\partial p_j} \text{ and } J_{2ij} = \frac{\partial h_{2i}}{\partial q_j} \tag{D3.1}$$

$$J_1 = \begin{bmatrix} \frac{\partial h_{12}}{\partial p_2^{t+\Delta t}} & \frac{\partial h_{12}}{\partial p_3^{t+\Delta t}} & 0 & 0 & 0 & 0 \\ \frac{\partial h_{13}}{\partial p_2^{t+\Delta t}} & \frac{\partial h_{13}}{\partial p_3^{t+\Delta t}} & \frac{\partial h_{13}}{\partial p_4^{t+\Delta t}} & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{\partial h_{1(n-1)}}{\partial p_{n-2}^{t+\Delta t}} & \frac{\partial h_{1(n-1)}}{\partial p_{n-1}^{t+\Delta t}} & \frac{\partial h_{1(n-1)}}{\partial p_n^{t+\Delta t}} \\ 0 & 0 & 0 & 0 & \frac{\partial h_{1n}}{\partial p_{n-1}^{t+\Delta t}} & \frac{\partial h_{1n}}{\partial p_n^{t+\Delta t}} \end{bmatrix}; G_1 = \begin{Bmatrix} h_{12}\left(p_2^t, p_3^t, p_1^{t+\Delta t}, p_2^{t+\Delta t}, p_3^{t+\Delta t}\right) \\ h_{13}\left(p_3^t, p_4^t, p_2^{t+\Delta t}, p_3^{t+\Delta t}, p_4^{t+\Delta t}\right) \\ \vdots \\ \vdots \\ h_{1(n-1)}\left(p_{n-1}^t, p_n^t, p_{n-2}^{t+\Delta t}, p_{n-1}^{t+\Delta t}, p_n^{t+\Delta t}\right) \\ h_{1n}\left(p_n^t, p_{n+1}^t, p_{n-1}^{t+\Delta t}, p_n^{t+\Delta t}, p_{n+1}^{t+\Delta t}\right) \end{Bmatrix}$$

$$h_{12} : \lambda\left(p_1^{t+\Delta t} - p_2^{t+\Delta t} - p_2^t + p_3^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{p_3^{t+\Delta t} - p_1^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - p_2^{t+\Delta t} + p_2^t\right) = 0 \tag{D3.2}$$

$$h_{13} : \lambda\left(p_2^{t+\Delta t} - p_3^{t+\Delta t} - p_3^t + p_4^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{p_4^{t+\Delta t} - p_2^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - p_3^{t+\Delta t} + p_3^t\right) = 0 \tag{D3.3}$$

$\vdots$

$\vdots$

$$h_{1n} : \lambda\left(p_{n-1}^{t+\Delta t} - p_n^{t+\Delta t} - p_n^t + p_{n+1}^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{p_{n+1}^{t+\Delta t} - p_{n-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - p_n^{t+\Delta t} + p_n^t\right) = 0 \tag{D3.4}$$

$$\frac{\partial h_{1i}}{\partial p_{i-1}^{t+\Delta t}} = \lambda\left[2\Delta z E_{1i}^{t+\Delta t} + \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^t - 2u_{i-1}^t + u_{i-1}^t\right)\left\{b_2\left(1 - 2E_{1i}^{t+\Delta t}\right)\right\}\right] \tag{D3.5}$$

$$\frac{\partial h_{1i}}{\partial p_{i+1}^{t+\Delta t}} = \lambda\left[2\Delta z E_{1i}^{t+\Delta t} - \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^t - 2u_{i-1}^t + u_{i-1}^t\right)\left\{b_2\left(1 - 2E_{1i}^{t+\Delta t}\right)\right\}\right] \tag{D3.6}$$

where, $\frac{\partial \boldsymbol{h}_{1i}}{\partial p_i^{t+\Delta t}} = -\Delta z\left(1+2E_i^{t+\Delta t}\lambda\right) M_{1i}^{t+\Delta t} = \frac{1}{2}\left[1 - b_1 \exp\left\{b_2\left(\frac{p_{i+1}^{t+\Delta t} - p_{i-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right]$ (D3.7)

$$\boldsymbol{J}_2 = \begin{bmatrix} \frac{\partial \boldsymbol{h}_{22}}{\partial r_{n-1}^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{22}}{\partial r_n^{t+\Delta t}} & 0 & 0 & 0 & 0 \\ \frac{\partial \boldsymbol{h}_{23}}{\partial r_{n-2}^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{23}}{\partial r_{n-1}^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{23}}{\partial r_n^{t+\Delta t}} & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{\partial \boldsymbol{h}_{2(n-1)}}{\partial r_2^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{2(n-1)}}{\partial r_3^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{2(n-1)}}{\partial r_4^{t+\Delta t}} \\ 0 & 0 & 0 & 0 & \frac{\partial \boldsymbol{h}_{2n}}{\partial r_2^{t+\Delta t}} & \frac{\partial \boldsymbol{h}_{2n}}{\partial r_3^{t+\Delta t}} \end{bmatrix}; \boldsymbol{G}_2 = \begin{Bmatrix} \mathbf{h}_{22}\left(r_{n-1}^t, r_n^t, r_{n-1}^{t+\Delta t}, r_n^{t+\Delta t}, r_{n+1}^{t+\Delta t}\right) \\ \mathbf{h}_{23}\left(r_{n-2}^t, r_{n-1}^t, r_{n-2}^{t+\Delta t}, r_{n-1}^{t+\Delta t}, r_n^{t+\Delta t}\right) \\ \vdots \\ \vdots \\ \mathbf{h}_{2(n-1)}\left(r_2^t, r_3^t, r_2^{t+\Delta t}, r_3^{t+\Delta t}, r_4^{t+\Delta t}\right) \\ \mathbf{h}_{2n}\left(r_1^t, r_2^t, r_1^{t+\Delta t}, r_2^{t+\Delta t}, r_3^{t+\Delta t}\right) \end{Bmatrix}$$

$\boldsymbol{h}_{22}: \lambda\left(r_{n+1}^{t+\Delta t} - r_n^{t+\Delta t} - r_n^t + r_{n-1}^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{r_{n+1}^{t+\Delta t} - r_{n-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - r_n^{t+\Delta t} + r_n^t\right) = 0$ (D3.8)

$\boldsymbol{h}_{23}: \lambda\left(r_n^{t+\Delta t} - r_{n-1}^{t+\Delta t} - r_{n-1}^t + r_{n-2}^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{r_n^{t+\Delta t} - r_{n-2}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - r_{n-1}^{t+\Delta t} + r_{n-1}^t\right) = 0$ (D3.9)

$\vdots$

$\vdots$

$\boldsymbol{h}_{2(\text{n-1})}: \lambda\left(r_4^{t+\Delta t} - r_3^{t+\Delta t} - r_3^t + r_2^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{r_4^{t+\Delta t} - r_2^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - r_3^{t+\Delta t} + r_3^t\right) = 0$ (D3.10)

$\boldsymbol{h}_{2n}: \lambda\left(r_3^{t+\Delta t} - r_2^{t+\Delta t} - r_2^t + r_1^t\right)\left[1 - b_1 \exp\left\{b_2\left(\frac{r_3^{t+\Delta t} - r_1^{t+\Delta t}}{2\Delta z}\right)\right\}\right] + 2\left(\sigma^{t+\Delta t} - \sigma^t - r_2^{t+\Delta t} + r_2^t\right) = 0$ (D3.11)

$\frac{\partial \boldsymbol{h}_{2i}}{\partial r_{i-1}^{t+\Delta t}} = \lambda\left[2\Delta z M_{2i}^{t+\Delta t} + (r_{i-1}^{t+\Delta t} - r_i^{t+\Delta t} - r_i^t + r_{i+1}^t)\left[b_2\left(1 - 2M_{2i}^{t+\Delta t}\right)\right]\right]$ (D3.12)

$\frac{\partial \boldsymbol{h}_{2i}}{\partial r_{i+1}^{t+\Delta t}} = \lambda\left[2\Delta z M_{2i}^{t+\Delta t} - (r_{i-1}^{t+\Delta t} - r_i^{t+\Delta t} - r_i^t + r_{i+1}^t)\left[b_2\left(1 - 2M_{2i}^{t+\Delta t}\right)\right]\right]$ (D3.13)

$$\frac{\partial \boldsymbol{h}_{2i}}{\partial r_i^{t+\Delta t}} = -\Delta z\left(1+2M_{2i}^{t+\Delta t}\lambda\right) \text{ where, } M_{2i}^{t+\Delta t} = \frac{1}{2}\left[1-b_1\exp\left\{b_2\left(\frac{r_{i+1}^{t+\Delta t}-r_{i-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] \quad \text{(D3.14)}$$

$$\frac{\partial \boldsymbol{h}_{2i}}{\partial r_{i-1}^{t+\Delta t}} = \lambda\left[2\Delta z E_{1i}^{t+\Delta t} + \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^{t} - 2u_{i-1}^{t} + u_{i-1}^{t}\right)\left\{b_2\left(1-2E_{1i}^{t+\Delta t}\right)\right\}\right] \quad \text{(D3.15)}$$

$$\frac{\partial \boldsymbol{h}_{2i}}{\partial r_{i+1}^{t+\Delta t}} = \lambda\left[2\Delta z E_{1i}^{t+\Delta t} - \left(u_{i-1}^{t+\Delta t} - 2u_{i-1}^{t+\Delta t} + u_{i-1}^{t+\Delta t} + u_{i-1}^{t} - 2u_{i-1}^{t} + u_{i-1}^{t}\right)\left\{b_2\left(1-2E_{1i}^{t+\Delta t}\right)\right\}\right] \quad \text{(D3.16)}$$

$$\text{where, } \frac{\partial \boldsymbol{h}_{2i}}{\partial r_i^{t+\Delta t}} = -\Delta z\left(1+2E_i^{t+\Delta t}\lambda\right) \; M_{1i}^{t+\Delta t} = \frac{1}{2}\left[1-b_1\exp\left\{b_2\left(\frac{p_{i+1}^{t+\Delta t}-p_{i-1}^{t+\Delta t}}{2\Delta z}\right)\right\}\right] \quad \text{(D3.17)}$$

Appendix E: Derivation and discretization of the GDEs representing unsaturated consolidation

E1. Coefficient of volume change or Compressibility Parameters (CP)

The deformation in the vadose zone is envisaged by considering the following relationship:

Soil-Structure constitutive phase: $\frac{\partial V_v}{V_0} = m_{1k}^s d(\sigma - u_a) + m_2^s d(u_a - u_w)$; CP→ (m_{1k}^s, m_2^s) (E1.1)

Water constitutive phase: $\frac{\partial (V_w - V_d)}{V_0} = m_{1k}^w d(\sigma - u_a) + m_2^w d(u_a - u_w)$;CP→$(m_{1k}^w, m_2^w)$ (E1.2)

Air constitutive phase: $\frac{\partial (V_a + V_d)}{V_0} = m_{1k}^a d(\sigma - u_a) + m_2^a d(u_a - u_w)$; CP→$(m_{1k}^a, m_2^a)$ (E1.3)

Here, (i) $m_{1k}^s = \frac{1}{V_0} \cdot \frac{\partial V_v}{\partial(\sigma - u_a)}$; $m_{1k}^w = \frac{1}{V_0} \cdot \frac{\partial (V_w - V_d)}{\partial(\sigma - u_a)}$; $m_{1k}^a = \frac{1}{V_0} \cdot \frac{\partial (V_a + V_d)}{\partial(\sigma - u_a)}$

(ii) $m_2^s = \frac{\partial (V_v / V_0)}{\partial (u_a - u_w)}$; $m_2^w = \frac{\partial (V_w / V_0)}{\partial (u_a - u_w)}$; $m_2^a = \frac{\partial (V_a / V_0)}{\partial (u_a - u_w)}$

m_{1k}^i and m_2^i are the coefficients of overall volume change with respect to $(\sigma - u_a)$ and $(u_a - u_w)$, respectively; the superscript i denotes either soil-structure (s), water (w), or air (a) phases. Here, V_a and V_d indicate the volume of free air and dissolved air, respectively. Fig. B1.1a shows the phase diagram including the dissolved air (the dissolution of air in water is controlled by the solubility coefficient, h) before the occurrence of dissipation process.

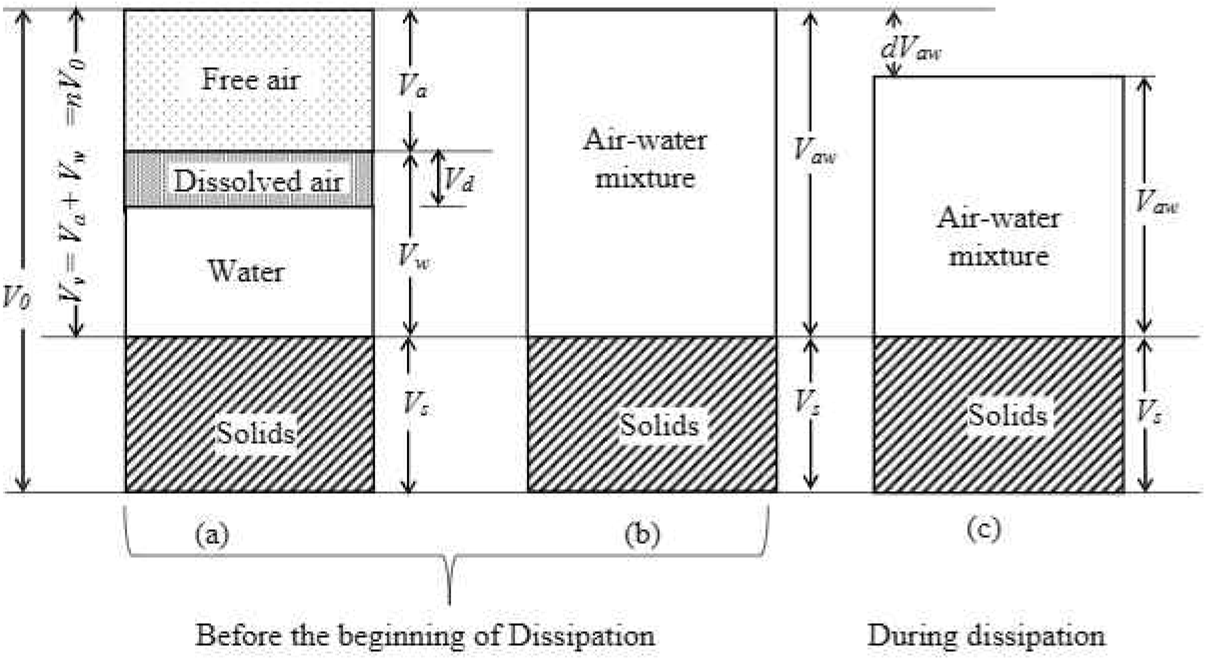

Fig B1.1 Phase diagram of the unsaturated soil system (a-b) at the onset of dissipation and (c) during the dissipation

The volume proportion in terms of the total void volume can be written as:

Free air volume	V_a=(1-S)(V_a+V_w)	(E1.4a)
Dissolved air volume	V_d=hS(V_a+V_w)	(E1.4b)
Total air volume	$V_a + V_d = (1 - S + hS) \cdot n \cdot V_0$	(E1.4c)
Water volume	V_w=S(V_a+V_w)	(E1.4d)

Compressibility of air-water mixture

Upon the application of external load, undrained boundary condition develops instantly. Unlike saturated soil, unsaturated soil assembly deforms even during undrained loading due to the compression of the air-water mixture. Figs. B1.1b and B1.1c illustrate the speculative two-phase diagram of the unsaturated soil before and during the dissipation process, respectively. The compressibility of the air-water mixture (β_{aw}) can be derived as:

$$\beta_{aw} = -\frac{1}{V_{aw}}\frac{dV_{aw}}{d\sigma} = -\frac{1}{V_{aw}}\frac{d(V_a+V_w)}{d\sigma} = -\frac{1}{V_{aw}}\left(\frac{d(V_w-V_d)}{d\sigma}+\frac{d(V_a+V_d)}{d\sigma}\right) \quad \text{(E1.5)}$$

$$= -\frac{1}{V_{aw}}\left(\frac{d(V_w-V_d)}{du_w}\frac{du_w}{d\sigma}+\frac{d(V_a+V_d)}{du_a}\frac{du_a}{d\sigma}\right)$$

$$= -\frac{1}{V_{aw}}\left(\frac{dV_w}{du_w}\frac{du_w}{d\sigma}+\frac{d(V_a+V_d)}{du_a}\frac{du_a}{d\sigma}\right) \text{ (Considering constant volume of the dissolved air)}$$

$$= -\frac{1}{V_w}\frac{dV_w}{du_w}\cdot\frac{V_w}{V_{aw}}\cdot\frac{du_w}{d\sigma} - \frac{1}{V_a+V_d}\frac{d(V_a+V_d)}{du_a}\cdot\frac{V_a+V_d}{V_{aw}}\cdot\frac{du_a}{d\sigma}$$

$$= \underbrace{S\beta_w\frac{du_w}{d\sigma}}_{\text{Compressibility of the Water Portion}} + \underbrace{(1-S+hS)\beta_a\frac{du_a}{d\sigma}}_{\text{Compressibility of the Air Portion}} ;\ \beta_w = -\frac{1}{V_w}\frac{dV_w}{du_w} \text{ and}$$

$$\beta_a = -\frac{1}{V_a+V_d}\frac{d(V_a+V_d)}{du_a} \text{ Applying Boyle's Law: } \beta_a = \frac{1}{\bar{u}_a};$$

$$\underbrace{\bar{u}_a}_{\text{Absolute Pr.}} = \underbrace{u_a}_{\text{Gauge Pr.}} + \underbrace{u_{atm}}_{\text{Atmospheric Pr.}} \text{; Pr. =Pressure} \quad \text{(E1.6)}$$

Without any external influences (load application or climate change), the pore air pressure is at the atmospheric pressure. Generally, pressure gauge ignores the atmospheric pressure. The measured gauge pressure is therefore, relative to the atmospheric pressure.

$\frac{du_w}{d\sigma}(=B_w)$, and $\frac{du_a}{d\sigma}(=B_a)$ are the pore pressure parameters corresponding to the water and the air phase, respectively; these parameters indicate the rate of excess pore pressure increment with respect to the applied load. The value of B_a and B_w approach towards unity as the matric suction diminishes. Furthermore, $B_w > B_a$; this implies that upon application of certain load, u_{w0} develops higher than u_{a0}.

Each term of the respective phase, consists of three components: (a) volume proportion, (b) volumetric strain with respect to change in respective pore pressure, and (c) change in pore pressure in response to total stress change. It is to be noted that β_{aw} is function of: (a) compressibility of water (β_w) and air (β_a) phases, (b) degree of saturation (S), (c) coefficient of solubility (h), and (d) pore pressure parameters. In the present analysis, dissolution of air in water is ignored; hence, h is taken to be zero.

E2. Deriving the initial excess pore water and pore air pressures

From Eq. (B1.1) $m_{1k}^s d(\sigma-u_a)+m_2^s d(u_a-u_w)=\frac{\partial V_v}{V_0}=\frac{1}{V_{aw}}\frac{dV_{aw}}{d\sigma}\cdot\frac{V_{aw}}{V_0}\cdot d\sigma=\beta_{aw}\cdot n\cdot d\sigma$ (E2.1)

(Considering small strain)

$$=\left(S\beta_w\frac{du_w}{d\sigma}+(1-S+hS)\beta_a\frac{du_a}{d\sigma}\right)nd\sigma$$

$$=Sn\beta_w du_w+(1-S+hS)n\beta_a du_a$$

$$\Rightarrow m_{1k}^s d\sigma-m_{1k}^s du_a+m_2^s du_a-m_2^s du_w=Sn\beta_w du_w+(1-S+hS)n\beta_a du_a \quad \text{(E2.2)}$$

$$\Rightarrow(Sn\beta_w+m_2^s)du_w+\left(m_{1k}^s-m_2^s+(1-S+hS)n\beta_a\right)du_a=m_{1k}^s d\sigma \quad \text{(E2.3)}$$

From Eq. (B1.3) $m_{1k}^a d(\sigma-u_a)+m_2^a d(u_a-u_w)=\frac{\partial(V_a+V_d)}{V_0}$ (E2.4)

$$=\frac{1}{(V_a+V_d)}\frac{d(V_a+V_d)}{du_a}\cdot\frac{(V_a+V_d)}{V_v}\cdot\frac{V_v}{V_0}du_a$$

$$=\beta_a\cdot(1-S+hS)\cdot n\cdot du_a$$

$$\Rightarrow m_{1k}^{a}d\sigma - m_{1k}^{a}du_a + m_2^a du_a - m_2^a du_w = \beta_a \cdot (1 - S + hS) \cdot n \cdot du_a \tag{E2.5}$$

$$\Rightarrow m_2^a du_w + \left((1 - S + hS) n\beta_a + m_{1k}^a - m_2^a\right) du_a = m_{1k}^a d\sigma \tag{E2.6}$$

$$\begin{bmatrix} (Sn\beta_w + m_2^w) & m_{1k}^w - m_2^w + (1 - S + hS) n\beta_a \\ m_2^a & (1 - S + hS) n\beta_a + m_{1k}^a - m_2^a \end{bmatrix} \begin{Bmatrix} du_w \\ du_a \end{Bmatrix} = \begin{Bmatrix} m_{1k}^w d\sigma \\ m_{1k}^a d\sigma \end{Bmatrix} \tag{E2.7}$$

The computed values of du_w and du_a are treated as u_{a0} and u_{w0}, respectively.

E3. Derivation of the dissipation equations related to the water and air phase

Dissipation equation concerning the water phase

Two different approaches for determining

the net flux of water per unit volume of soil:

1) Considering the incompressibility of water phase

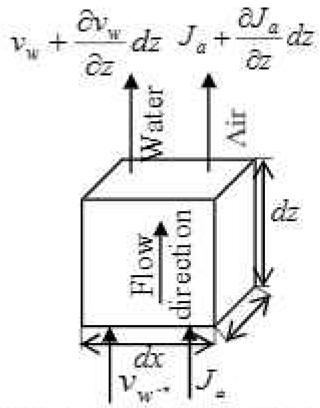

Fig E3.1 Flow of water and air phase

Rate of change of stored water= Net water flow in or out of an element, as shown in Fig. E3.1

$$\frac{\partial (V_w - V_d)}{\partial t} = \left(v_w + \frac{\partial v_w}{\partial z} dz\right) dxdy - v_w dxdy = \frac{\partial v_w}{\partial z} dzdydx = \frac{\partial v_w}{\partial z} V_0 \tag{E3.1}$$

$$\Rightarrow \frac{1}{V_0} \frac{\partial (V_w - V_d)}{\partial t} = \frac{\partial v_w}{\partial z} = -\frac{\partial}{\partial z}\left(k_w\left[\frac{1}{\gamma_w}\frac{\partial u_w}{\partial z}\right]\right) = -\left[\frac{k_w}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2}\right] \tag{E3.2}$$

2) Differentiating the water phase constitutive relation

$$\frac{1}{V_0} \frac{\partial (V_w - V_d)}{\partial t} = m_{1k}^w \frac{d(\sigma - u_a)}{dt} + m_2^w \frac{d(u_a - u_w)}{dt} \tag{E3.3}$$

II. Equating Eqs. B3.2 and B3.3, the following relation is obtained:

$$m_{1k}^{w}\frac{\partial(\sigma-u_a)}{\partial t}+m_2^{w}\frac{\partial(u_a-u_w)}{\partial t}=-\frac{k_w}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2} \qquad \text{(E3.3)}$$

$$m_2^{w}\frac{\partial u_w}{\partial t}+\left(m_{1k}^{w}-m_2^{w}\right)\frac{\partial u_a}{\partial t}=\frac{k_w}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2} \qquad \text{(E3.4)}$$

$$\frac{\partial u_w}{\partial t}+C_w\frac{\partial u_a}{\partial t}=c_v^{w}\frac{\partial^2 u_w}{\partial z^2}\ ;\ C_w=\frac{(m_{1k}^{w}-m_2^{w})}{m_2^{w}}\ ;\ c_v^{w}=\frac{k_w}{\gamma_w m_2^{w}}\ \text{(B3.5)} \qquad \text{(E3.5)}$$

Dissipation equation concerning the air phase

I. Two different approaches for determining the net mass rate of air flowing across the element:

1) Mass rates of air entering and leaving the element within a small time period

$$\frac{\partial\left[(V_a+V_d)\cdot\rho_a\right]}{\partial t}=\frac{\partial M_a}{\partial t}=\left(J_a+\frac{\partial J_a}{\partial z}dz\right)dxdy-J_a dxdy=\frac{\partial J_a}{\partial z}V_0;\quad \begin{matrix} M_a\rightarrow\text{Mass of air}\\ \rho_a\rightarrow\text{Density of air}\end{matrix} \qquad \text{(E3.6)}$$

$$\frac{\rho_a}{V_0}\frac{\partial V_a}{\partial t}+\frac{V_a+V_d}{V_0}\frac{\partial\rho_a}{\partial t}=\frac{\partial J_a}{\partial z}=\frac{\partial(-D_a\cdot\partial u_a/\partial z)}{\partial z}=-D_a\cdot\frac{\partial^2 u_a}{\partial z^2}\ (\because D_a\ \text{spatially-independent}) \qquad \text{(E3.7)}$$

$$\frac{\rho_a}{V_0}\frac{\partial V_a}{\partial t}+(1-S+hS)\cdot n\cdot\frac{\partial\rho_a}{\partial t}=-D_a\cdot\frac{\partial^2 u_a}{\partial z^2} \qquad \text{(E3.8)}$$

The three state variables (pressure, volume, temperature) of the air phase are related followingly, in accordance to the ideal gas law:

$$\overline{u_a}\cdot V_a=mRT_k\quad\Rightarrow\overline{u_a}\cdot M_a/\rho_a=mRT_k\quad\Rightarrow\rho_a=\frac{M_a}{mRT_k}\overline{u_a}=\frac{\omega_a}{RT_k}\overline{u_a} \qquad \text{(E3.9)}$$

Substituting the expression of ρ_a

$$\frac{\omega_a}{RV_0T_k}\overline{u_a}\frac{\partial V_a}{\partial t}+(1-S+hS)\cdot n\cdot\frac{\omega_a}{RT_k}\cdot\frac{\partial u_a}{\partial t}=-D_a\cdot\frac{\partial^2 u_a}{\partial z^2} \qquad \text{(E3.10)}$$

This leads to: $\frac{1}{V_0}\frac{\partial V_a}{\partial t} = -\frac{RT_k D_a}{\omega_a \cdot \overline{u_a}} \cdot \frac{\partial^2 u_a}{\partial z^2} - \frac{(1-S+hS)\cdot n}{\overline{u_a}} \cdot \frac{\partial u_a}{\partial t}$ (E3.11)

2) Differentiating the air phase constitutive relation

$$\frac{1}{V_0}\frac{\partial V_a}{\partial t} = \frac{1}{V_0}\frac{\partial (V_a - V_d)}{\partial t} = m_{1k}^a \frac{d(\sigma - u_a)}{dt} + m_2^a \frac{d(u_a - u_w)}{dt} \quad \text{(E3.12)}$$

II. Equating Eqs. B3.11 and B3.12 the following relation is obtained:

$$m_{1k}^a \frac{d(\sigma - u_a)}{dt} + m_2^a \frac{d(u_a - u_w)}{dt} = -\frac{RT_k D_a}{\omega_a \cdot \overline{u_a}} \cdot \frac{\partial^2 u_a}{\partial z^2} - \frac{(1-S+hS)\cdot n}{\overline{u_a}} \cdot \frac{\partial u_a}{\partial t} \quad \text{(E3.13)}$$

$$\left(m_2^a - m_{1k}^a + \frac{(1-S+hS)\cdot n}{\overline{u_a}} \right) \frac{\partial u_a}{\partial t} - m_2^a \frac{\partial u_w}{\partial t} = -\frac{RT_k D_a}{\omega_a \cdot \overline{u_a}} \cdot \frac{\partial^2 u_a}{\partial z^2} \quad \text{(E3.14)}$$

$$\frac{\partial u_a}{\partial t} - \frac{m_2^a}{m_2^a - m_{1k}^a + (1-S+hS)\cdot n \cdot \overline{u_a}^{-1}} \frac{\partial u_w}{\partial t} = -\frac{RT_k D_a}{\omega_a \cdot \overline{u_a}\left(m_2^a - m_{1k}^a\right) - \omega_a \cdot (1-S+hS)\cdot n} \cdot \frac{\partial^2 u_a}{\partial z^2} \quad \text{(E3.15)}$$

$$\frac{\partial u_a}{\partial t} + C_a \frac{\partial u_w}{\partial t} = c_v^a \frac{\partial^2 u_a}{\partial z^2} \quad \text{(E3.16)}$$

$$C_a = \frac{m_2^a}{m_2^a - m_{1k}^a + \frac{(1-S+hS)n}{\overline{u_a}}} \quad \text{and} \quad c_v^a = \frac{RT_k D_a \omega_a^{-1}}{\overline{u_a}\left(m_2^a - m_{1k}^a\right) - (1-S+hS)\cdot n}$$

Converting space derivative PDEs of a phase into time derivative PDEs

From Eq. (B3.5 and B3.16): (Water phase) $\frac{\partial u_w}{\partial t} = \frac{1}{1 - C_a C_w}\left(c_v^w \frac{\partial^2 u_w}{\partial z^2} - C_w c_v^a \frac{\partial^2 u_a}{\partial z^2} \right)$ (E3.17a)

(Air phase) $\frac{\partial u_a}{\partial t} = \frac{1}{1 - C_a C_w}\left(c_v^a \frac{\partial^2 u_a}{\partial z^2} - C_a c_v^w \frac{\partial^2 u_w}{\partial z^2} \right)$ (E3.17b)

E4. Numerical Crank-Nicolson Finite Difference Scheme

Water Phase: Discritised the Eq. E3.17a by CN scheme

$$\left.\frac{\partial u_w}{\partial t}\right|_{@i^{th}node} = \frac{1}{1-C_aC_w}\left(0.5c_v^w\left(\left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^t\right) - 0.5C_wc_v^w\left(\left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^t\right)\right) \quad \text{E4.1}$$

<u>User Defined</u>
Δt = time increment
Δz = step increment

$$\frac{u_{w(i+1)}^{t+\Delta t} - u_{w(i)}^t}{\Delta t} = \frac{0.5}{1-C_aC_w}\left[c_v^w\left(\frac{u_{w(i+1)}^{t+\Delta t} - 2u_{w(i)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^t}{\Delta z^2}\right) + -C_wc_v^a\left(\frac{u_{a(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{a(i+1)}^t - 2u_{a(i)}^t + u_{a(i-1)}^t}{\Delta z^2}\right)\right] \quad \text{E4.2}$$

$$((1+\lambda_1)u_{w(i)}^{t+\Delta t} - 0.5\lambda_1(u_{w(i+1)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t})) - 0.5\lambda_2(u_{a(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}) = ((1-\lambda_1)u_{w(i)}^t + 0.5\lambda_1(u_{w(i+1)}^t + u_{w(i-1)}^t)) - 0.5\lambda_2(u_{a(i+1)}^t - 2u_{a(i)}^t + u_{a(i-1)}^t) \quad \text{E4.3}$$

Air Phase: Discritised the Eq. E3.17b by CN scheme

$$\left.\frac{\partial u_a}{\partial t}\right|_{@i^{th}node} = \frac{1}{1-C_aC_w}\left(0.5c_v^a\left(\left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^t\right) - 0.5C_ac_v^w\left(\left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^t\right)\right) \quad \text{E4.4}$$

$$\frac{u_{a(i+1)}^{t+\Delta t} - u_{a(i)}^t}{\Delta t} = \frac{0.5}{1-C_aC_w}\left[c_v^a\left(\frac{u_{a(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{a(i+1)}^t - 2u_{a(i)}^t + u_{a(i-1)}^t}{\Delta z^2}\right) + -C_ac_v^w\left(\frac{u_{w(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^t}{\Delta z^2}\right)\right] \quad \text{E4.5}$$

$$((1+\lambda_3)u_{a(i)}^{t+\Delta t} - 0.5\lambda_3(u_{a(i+1)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t})) - 0.5\lambda_4(u_{w(i+1)}^{t+\Delta t} - 2u_{w(i)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t}) = ((1-\lambda_3)u_{a(i)}^t + 0.5\lambda_3(u_{a(i+1)}^t + u_{a(i-1)}^t)) - 0.5\lambda_4(u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^t) \quad \text{E4.6}$$

Where: $\lambda_1 = \frac{c_v^w \Delta t}{\Delta z^2(1-C_aC_w)}$, $\lambda_2 = \frac{C_wc_v^a \Delta t}{\Delta z^2(1-C_aC_w)}$, $\lambda_3 = \frac{c_v^a \Delta t}{\Delta z^2(1-C_aC_w)}$ and $\lambda_4 = \frac{C_ac_v^w \Delta t}{\Delta z^2(1-C_aC_w)}$

Matrix representation: of the simultaneous set of linear equations: G (known matrix)$\times U^{t+\Delta t}$ (unknown vector)$=H^{t}$ (known vector)

$$G=\begin{bmatrix} 1+\lambda_1 & -b\lambda_2 & -0.5\lambda_1 & 0.5\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5\lambda_1 & 0.5\lambda_2 & 1+\lambda_1 & -\lambda_2 & -0.5\lambda_1 & 0.5\lambda_2 & 0 & 0 & 0 & 0 \\ -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -0.5\lambda_1 & 0.5\lambda_2 & 1+\lambda_1 & -\lambda_2 & -0.5\lambda_1 & 0.5\lambda_2 & \cdots & \cdots \\ \cdots & \cdots & -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & -0.5\lambda_1 & 0.5\lambda_2 & 1+\lambda_1 & -\lambda_2 & -0.5\lambda_1 & 0.5\lambda_2 \\ 0 & 0 & 0 & 0 & -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda_1 & 0.5\lambda_2 & 1+\lambda_1 & -\lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 \end{bmatrix} \quad U^{t+\Delta t}=\begin{bmatrix} u_{w(2)}^{t+\Delta t} \\ u_{a(2)}^{t+\Delta t} \\ u_{w(3)}^{t+\Delta t} \\ u_{a(3)}^{t+\Delta t} \\ \cdots \\ u_{w(i)}^{t+\Delta t} \\ u_{a(i)}^{t+\Delta t} \\ \cdots \\ u_{w(n-1)}^{t+\Delta t} \\ u_{a(n-1)}^{t+\Delta t} \\ u_{w(n)}^{t+\Delta t} \\ u_{a(n)}^{t+\Delta t} \end{bmatrix}$$

$$H=\begin{bmatrix} 1-\lambda_1 & \lambda_2 & 0.5\lambda_1 & -0.5\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_4 & 1-\lambda_3 & -0.5\lambda_4 & +0.5\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5\lambda_1 & -0.5\lambda_2 & 1-\lambda_1 & \lambda_2 & 0.5\lambda_1 & -0.5\lambda_2 & 0 & 0 & 0 & 0 \\ -0.5\lambda_4 & 0.5\lambda_3 & \lambda_4 & 1-\lambda_3 & -0.5\lambda_4 & 0.5\lambda_3 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0.5\lambda_1 & -0.5\lambda_2 & 1-\lambda_1 & \lambda_2 & 0.5\lambda_1 & -0.5\lambda_2 & \cdots & \cdots \\ \cdots & \cdots & -0.5\lambda_4 & 0.5\lambda_3 & \lambda_4 & 1-\lambda_3 & -0.5\lambda_4 & 0.5\lambda_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0.5\lambda_1 & -0.5\lambda_2 & 1-\lambda_1 & \lambda_2 & 0.5\lambda_1 & -0.5\lambda_2 \\ 0 & 0 & 0 & 0 & -0.5\lambda_4 & 0.5\lambda_3 & \lambda_4 & 1-\lambda_3 & -0.5\lambda_4 & 0.5\lambda_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5\lambda_1 & -0.5\lambda_2 & 1-\lambda_1 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda_4 & 0.5\lambda_3 & \lambda_4 & 1-\lambda_3 \end{bmatrix} \times \begin{bmatrix} u_{w(2)}^{t} \\ u_{a(2)}^{t} \\ u_{w(3)}^{t} \\ u_{a(3)}^{t} \\ \cdots \\ u_{w(i)}^{t} \\ u_{a(i)}^{t} \\ \cdots \\ u_{w(n-1)}^{t} \\ u_{a(n-1)}^{t} \\ u_{w(n)}^{t} \\ u_{a(n)}^{t} \end{bmatrix}$$

Appendix F: Modifications of the unsaturated GDEs to instill flexible drainage, and nonlinear flow

F1. Derivation of the dissipation equations related to the water and air phase

Dissipation equation concerning the water phase

1) Considering the incompressibility of water phase

Considering the Eq. E3.1

$$\Rightarrow \frac{1}{V_0}\frac{\partial(V_w - V_d)}{\partial t} = \frac{\partial v_w}{\partial z} = -\frac{\partial}{\partial z}\left(k_{ws}\left[\frac{1}{\gamma_w}\frac{\partial u_w}{\partial z}\right] - (1-a)i_1\left(1 - e^{\frac{-\theta}{\gamma_w i_1}\frac{\partial u_w}{\partial z}}\right)\right) \quad \text{(F1.1)}$$

$$= -\left[\frac{k_{ws}}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2} - \frac{(1-a)k_{ws}\theta}{\gamma_w}\exp\left(\frac{-\theta}{\gamma_w i_1}\frac{\partial u_w}{\partial z}\right)\frac{\partial^2 u_w}{\partial z^2}\right]$$

2) Differentiating the water phase constitutive relation

$$\frac{1}{V_0}\frac{\partial(V_w - V_d)}{\partial t} = m_{1k}^w \frac{d(\sigma - u_a)}{dt} + m_2^w \frac{d(u_a - u_w)}{dt} \quad \text{(F1.2)}$$

II. Equating Eqs. F1.1 and F1.2 the following relation is obtained:

$$m_{1k}^w \frac{\partial(\sigma - u_a)}{\partial t} + m_2^w \frac{\partial(u_a - u_w)}{\partial t} = -\frac{k_{ws}}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2}\left(1 - (1-a)\theta\exp\left(\frac{-\theta}{\gamma_w i_1}\frac{\partial u_w}{\partial z}\right)\right) \quad \text{(F1.3)}$$

$$\frac{\partial u_w}{\partial t} + C_w\frac{\partial u_a}{\partial t} = c_v^w \frac{\partial^2 u_w}{\partial z^2}\left(1 - (1-a)\theta\exp\left(\frac{-\theta}{\gamma_w i_1}\frac{\partial u_w}{\partial z}\right)\right); C_w = \frac{(m_{1k}^w - m_2^w)}{m_2^w}, c_v^w = \frac{k_{ws}}{\gamma_w m_2^w} \quad \text{(F1.4)}$$

From Eq (F1.4 and E3.16)

$$\frac{\partial u_w}{\partial t} = \frac{1}{1 - C_a C_w}\left(c_v^w \frac{\partial^2 u_w}{\partial z^2}\left(1 - (1-a_1)\theta\left[1 - \exp\left(\frac{-\theta}{i_1\gamma_w}\frac{\partial u_w}{\partial z}\right)\right]\right) - C_w c_v^a \frac{\partial^2 u_a}{\partial z^2}\right) \quad \text{(F1.5)}$$

F2. Numerical Crank-Nicolson Finite Difference Scheme

Forming Algebraic equations for any arbitrary i^{th} node

Water Phase

$$\left.\frac{\partial u_w}{\partial t}\right|_{@i^{th}node} = \frac{1}{1-C_aC_w}\left(0.5E_i^t c_v^w\left(\left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^t\right) - 0.5C_w c_v^w\left(\left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^t\right)\right); \quad \begin{array}{r}\underline{\text{User Defined}}\\ \Delta t = \text{time increment}\\ \Delta z = \text{step increment}\end{array} \quad \text{(F2.1)}$$

$$\frac{u_{w(i+1)}^{t+\Delta t} - u_{w(i)}^t}{\Delta t} = \frac{0.5}{1-C_aC_w}\left[E_i^t c_v^w\left(\frac{u_{w(i+1)}^{t+\Delta t} - 2u_{w(i)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^t}{\Delta z^2}\right) + -C_w c_v^a\left(\frac{u_{a(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{a(i+1)}^t - 2u_{a(i)}^t + u_{a(i-1)}^t}{\Delta z^2}\right)\right] \quad \text{(F2.2)}$$

$$\left(1+\lambda_{1(i)}^t\right)u_{w(i)}^{t+\Delta t} - 0.5\lambda_{1(i)}^t\left(u_{w(i+1)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t}\right) + 0.5\lambda_2\left(u_{a(i+1)}^{t+\Delta t} - u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}\right) = \left(1-\lambda_{1(i)}^t\right)u_{w(i)}^t + 0.5\lambda_{1(i)}^t\left(u_{w(i+1)}^t + u_{w(i-1)}^t\right) - \left(u_{a(i+1)}^t - u_{a(i)}^t + u_{a(i-1)}^t\right) \quad \text{(F2.3)}$$

Air Phase

$$\left.\frac{\partial u_a}{\partial t}\right|_{@i^{th}node} = \frac{0.5}{1-C_aC_w}\left(c_v^a\left(\left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_a}{\partial z^2}\right|_i^t\right) - C_a c_v^w\left(\left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^{t+\Delta t} + \left.\frac{\partial^2 u_w}{\partial z^2}\right|_i^t\right)\right) \quad \text{(F2.4)}$$

$$\frac{u_{a(i+1)}^{t+\Delta t} - u_{a(i)}^t}{\Delta t} = \frac{0.5}{1-C_aC_w}\left[c_v^a\left(\frac{u_{a(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{a(i+1)}^t - 2u_{a(i)}^t + u_{a(i-1)}^t}{\Delta z^2}\right) - C_a c_v^w\left(\frac{u_{w(i+1)}^{t+\Delta t} - 2u_{a(i)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}}{\Delta z^2} + \frac{u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^t}{\Delta z^2}\right)\right] \quad \text{(F2.5)}$$

$$\Rightarrow (1+\lambda_3)u_{a(i)}^{t+\Delta t} - 0.5\lambda_3\left(u_{a(i+1)}^{t+\Delta t} + u_{a(i-1)}^{t+\Delta t}\right) + 0.5\lambda_4\left(u_{w(i+1)}^{t+\Delta t} - 2u_{w(i)}^{t+\Delta t} + u_{w(i-1)}^{t+\Delta t}\right) = (1-\lambda_3)u_{a(i)}^t + 0.5\lambda_3\left(u_{a(i+1)}^t + u_{a(i-1)}^t\right) - 0.5\lambda_4\left(u_{w(i+1)}^t - 2u_{w(i)}^t + u_{w(i-1)}^{t+\Delta t}\right) \quad \text{(F2.6)}$$

$$\lambda_{1(i)}^t = \frac{E_i^t c_v^w \Delta t}{\Delta z^2\left(1-C_aC_w\right)}, \lambda_2 = \frac{C_w c_v^a \Delta t}{\Delta z^2\left(1-C_aC_w\right)}, \lambda_3 = \frac{c_v^a \Delta t}{\Delta z^2\left(1-C_aC_w\right)}, \; E_i^t = \left(1-(1-a)\theta\left[1-\exp\left(\frac{-\theta}{i_1\gamma_w}\frac{u_{w(i+1)}^t - u_{w(i-1)}^t}{2\Delta z}\right)\right]\right)$$

Matrix representation: of the simultaneous set of linear equations: G (known matrix)$\times U^{t+\Delta t}$ (unknown vector)$=H^t$ (known vector)

$$G^t_i=\begin{bmatrix}
1+a\lambda^t_{1(1)} & -b\lambda_2 & -0.5\lambda^t_{1(1)} & 0.5\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0\\
a\lambda_4 & 1+b\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0\\
-0.5\lambda^t_{1(2)} & 0.5\lambda_2 & 1+\lambda^t_{1(2)} & -\lambda_2 & -0.5\lambda^t_{1(2)} & 0.5\lambda_2 & 0 & 0 & 0 & 0\\
-0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & -0.5\lambda^t_{1(i)} & 0.5\lambda_2 & 1+\lambda^t_{1(i)} & -\lambda_2 & -0.5\lambda^t_{1(i)} & 0.5\lambda_2 & \cdots & \cdots\\
\cdots & \cdots & -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3 & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & -0.5\lambda^t_{1(n)} & 0.5\lambda_2 & 1+\lambda^t_{1(n)} & -\lambda_2 & -0.5\lambda^t_{1(n)} & 0.5\lambda_2\\
0 & 0 & 0 & 0 & -0.5\lambda_4 & -0.5\lambda_3 & \lambda_4 & 1+\lambda_3 & -0.5\lambda_4 & -0.5\lambda_3\\
0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda^t_{1(n+1)} & 0.5\lambda_2 & 1+c\lambda^t_{1(n+1)} & -d\lambda_2\\
0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda_4 & -0.5\lambda_3 & c\lambda_4 & 1+d\lambda_3
\end{bmatrix}
\qquad
U^{t+\Delta t}=\begin{bmatrix}
u^{t+\Delta t}_{w(1)}\\ u^{t+\Delta t}_{a(1)}\\ u^{t+\Delta t}_{w(2)}\\ u^{t+\Delta t}_{a(2)}\\ \cdots\\ u^{t+\Delta t}_{w(i)}\\ u^{t+\Delta t}_{a(i)}\\ \cdots\\ u^{t+\Delta t}_{w(n)}\\ u^{t+\Delta t}_{a(n)}\\ u^{t+\Delta t}_{w(n+1)}\\ u^{t+\Delta t}_{a(n+1)}
\end{bmatrix}$$

$$
H^{t}=\begin{bmatrix}
1-a\lambda^{t}{}_{1(1)} & +b\lambda_{2} & 0.5\lambda^{t}{}_{1(1)} & -0.5\lambda_{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
a\lambda_{4} & 1-b\lambda_{3} & -0.5\lambda_{4} & +0.5\lambda_{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5\lambda^{t}{}_{1(2)} & -0.5\lambda_{2} & 1-\lambda^{t}{}_{1(2)} & \lambda_{2} & 0.5\lambda^{t}{}_{1(2)} & -0.5\lambda_{2} & 0 & 0 & 0 & 0 \\
-0.5\lambda_{4} & 0.5\lambda_{3} & \lambda_{4} & 1-\lambda_{3} & -0.5\lambda_{4} & 0.5\lambda_{3} & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & 0.5\lambda^{t}{}_{1(i)} & -0.5\lambda_{2} & 1-\lambda^{t}{}_{1(i)} & \lambda_{2} & 0.5\lambda^{t}{}_{1(i)} & -0.5\lambda_{2} & \cdots & \cdots \\
\cdots & \cdots & -0.5\lambda_{4} & 0.5\lambda_{3} & \lambda_{4} & 1-\lambda_{3} & -0.5\lambda_{4} & 0.5\lambda_{3} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0.5\lambda^{t}{}_{1(n)} & -0.5\lambda_{2} & 1-\lambda^{t}{}_{1(n)} & \lambda_{2} & 0.5\lambda^{t}{}_{1(n)} & -0.5\lambda_{2} \\
0 & 0 & 0 & 0 & -0.5\lambda_{4} & 0.5\lambda_{3} & \lambda_{4} & 1-\lambda_{3} & -0.5\lambda_{4} & 0.5\lambda_{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5\lambda^{t}{}_{1(n+1)} & -0.5\lambda_{2} & 1-c\lambda^{t}{}_{1(n+1)} & d\lambda_{2} \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5\lambda_{4} & 0.5\lambda_{3} & c\lambda_{4} & 1-d\lambda_{3}
\end{bmatrix}\times\begin{bmatrix}
u^{t}_{w(1)} \\ u^{t}_{a(1)} \\ u^{t}_{w(2)} \\ u^{t}_{a(2)} \\ \cdots \\ u^{t}_{w(i)} \\ u^{t}_{a(i)} \\ \cdots \\ u^{t}_{w(n)} \\ u^{t}_{a(n)} \\ u^{t}_{w(n+1)} \\ u^{t}_{a(n+1)}
\end{bmatrix}
$$

Where , $a=0.5\left(1+\frac{\Delta z R_{Tw}}{H}\right)$ $b=0.5\left(1+\frac{\Delta z R_{Ta}}{H}\right)$, $c=0.5\left(1+\frac{\Delta z R_{\mathrm{B}w}}{H}\right)$ and $d=0.5\left(1+\frac{\Delta z R_{Ba}}{H}\right)$

LIST OF PUBLICATIONS

PUBLISHED/ ACCEPTED JOURNALS

- Singh, A. and Chakraborty, M. (2021). Numerical analysis of 1-D consolidation of two-layered soils with different initial excess pore water pressure. *Geotechnical and Geological Engineering*.
- Singh, A . Singh, D. and Chakraborty, M. (2021). Effect of various initial excess pore water pressure distributions on 1-D consolidation of clays. *International Journal of Geotechnical Engineering*, 16 (1), 123-132.
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