A STUDY OF FUZZY SOFT TOPOLOGIES, FUZZY TOPOLOGIES GENERATED BY FUZZY RELATIONS AND REPRESENTABILITY OF SOME FUZZY RELATIONS

Thesis submitted in partial fulfillment
for the Award of Degree

Doctor of Philosophy

by

Seema Mishra

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI- 221 005
INDIA

Enrollment No. 12612EN004

2016

Chapter 7

Representability of fuzzy biorders and fuzzy weak orders

7.1 Introduction

The idea of providing a representation for a binary relation $\mathcal{R}$ between two non empty sets A and X was formulated by Guttman[46] in 1944, by proposing two functions $f : A \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that

$$a\mathcal{R}x \Leftrightarrow f(a) > g(x),$$

for each $a \in A$ and $x \in X$. In 1969, Ducamp and Falmagne[36] showed that if A and X both are finite, then the existence of such type of functions f and g for a binary relation $\mathcal{R}$ between A and X is equivalent to the following condition:

$$a\mathcal{R}x \text{ and } b\mathcal{R}y \Rightarrow a\mathcal{R}y \text{ or } b\mathcal{R}x, \tag{7.1}$$

for each $a, b \in A$ and $x, y \in X$.

A relation satisfying the condition (7.1) (called as Ferrers condition[79, 91]) is said to be a biorder. It has been proved by Doignon et al.[34] that the condition (7.1) on $\mathcal{R}$ is equivalent to the condition

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}, \tag{7.2}$$

The contents of this chapter, in the form of a research paper, has been published in 'Internat. J. Uncertain., Fuzziness Knowledge-Based Systems,(6), **24**(2016)917-935'.

where $\mathcal{R}^d$ denotes the dual of $\mathcal{R}$. They also studied representability of biorders and interval orders. Representations of different types of ordering have been studied by several authors(see e.g., [19, 82, 86]).

The fuzzy analogues of the Ferrers conditions (7.1) and (7.2), which are given as follows:

$$\min\{\mathcal{R}(a,x),\mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y),\mathcal{R}(b,x)\}, \quad \text{for each } a,b \in A \text{ and } x,y \in X \tag{7.3}$$

and

$$\mathcal{R} o_T \mathcal{R}^d o_T \mathcal{R} \subseteq \mathcal{R} \tag{7.4}$$

are no more equivalent in general. A condition under which (7.3) and (7.4) are equivalent has been provided by Fodor[41]. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, corresponding to these two non-equivalent Ferrers conditions(also called $T-$Ferrers conditions)[41, 97]. Comparative studies of $T-$Ferrers relations, fuzzy biorders and fuzzy interval orders have been carried out by several authors(see e.g., [28, 30]). Baets and Walle[10] had introduced the notion of $T-$fuzzy interval orders and studied two particular types of $T-$fuzzy interval orders: Weak and Strong fuzzy interval orders.

The study of fuzzy weak orders with respect to a left continuous t-norm T and their representability by the residual implication operator associated with T (called $T-$representable fuzzy weak orders) has been done by several authors(see e.g., [11, 98]). Characterizations for a T_M-representable(also called Gödel representable) fuzzy weak orders and for the fuzzy relation which can be written as the union or intersection of a finite family of T_M-representable(or Gödel representable) fuzzy weak orders have been obtained by Baets et al.[11]. Characterizations for a T_P-representable fuzzy weak order and for finite intersections of fuzzy weak orders with respect to any left continuous t-norm T have been obtained by Sali et al.[98].

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the $\alpha-$cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52].

In this chapter, we have studied representability of both fuzzy biorders and fuzzy weak orders. It is observed that union of a finite family of fuzzy weak orders with respect to T is a fuzzy quasi-transitive relation with respect to T. In the last theorem, we have obtained a characterization for a T_L-representable fuzzy weak order.

Now we recall some definitions and results which will be used throughout the chapter.

Definition 7.1. [123] Let $\mathcal{R}$ be a fuzzy relation between A and X. Then for $\alpha \in [0,1]$, the α-*cut* $\mathcal{R}_\alpha$ is given by

$$\mathcal{R}_\alpha = \{(a,x) \in A \times X : \mathcal{R}(a,x) \geq \alpha\}.$$

Note that each $\mathcal{R}_\alpha$ is a binary relation between A and X.

Definition 7.2. [84] Let $\mathcal{R}$ be a fuzzy relation on A. Then its *strict part* $P_\mathcal{R}$ is the fuzzy relation on A given by:

$$P_\mathcal{R}(a,b) = \begin{cases} \mathcal{R}(a,b), & \text{if } \mathcal{R}(a,b) > \mathcal{R}(b,a) \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7.3. [11] Let T be a t-norm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy weak order* with respect to T if it is:

1. strongly S_M-complete;
2. $T-$ transitive.

Definition 7.4. [11] Let T be a t-norm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy quasi order* with respect to T if it is:

1. reflexive;
2. $T-$transitive.

Definition 7.5. [11] Let T be a t-norm and S be its dual t-conorm. Then a fuzzy relation $\mathcal{R}$ on A is called a *fuzzy quasi-transitive* relation with respect to T if it is:

1. strongly S_M- complete;
2. negatively $S-$transitive.

Definition 7.6. [34] Let $\mathcal{R}$ and $\mathcal{Q}$ be two binary relations between A and X, X and Y respectively. Then the *composition* $\mathcal{R}\mathcal{Q}$ is the binary relation between A and Y given by

$$\mathcal{R}\mathcal{Q} = \{(a, y) : \text{there exists}\, x \in X \,\text{such that}\, (a, x) \in \mathcal{R} \,\text{and}\, (x, y) \in \mathcal{Q}\}.$$

Definition 7.7. [61] Let $\mathcal{R}$ and $\mathcal{Q}$ be two fuzzy relations between A and X, X and Y respectively and T be a t-norm. Then the *fuzzy composition* $\mathcal{R}o_T\mathcal{Q}$ with respect to T is the fuzzy relation between A and Y given by

$$(\mathcal{R}o_T\mathcal{Q})(a, y) = \sup_{x\in X} T\{\mathcal{R}(a, x), \mathcal{Q}(x, y)\}, \text{for each}\, (a, y) \in A \times Y.$$

Definition 7.8. [34, 79, 91] A binary relation $\mathcal{R}$ between A and X is said to satisfy the *Ferrers property* if

$$(a, x) \in \mathcal{R} \,\text{and}\, (b, y) \in \mathcal{R} \Rightarrow (a, y) \in \mathcal{R} \,\text{or}\, (b, x) \in \mathcal{R}, \tag{7.5}$$

for each $a, b \in A$ and $x, y \in X$. Equivalently,

$$\mathcal{R}\mathcal{R}^d\mathcal{R} \subseteq \mathcal{R}. \tag{7.6}$$

Definition 7.9. [34] A binary relation $\mathcal{R}$ between A and X is said to be a *biorder* if $\mathcal{R}$ satisfies the Ferrers property.

7.2 Fuzzy biorder and its representability

Corresponding to (7.5) and (7.6), in case of fuzzy relations on A we have the following:

Definition 7.10. [42] Let T be a t-norm and S be a t-conorm. Then a fuzzy relation $\mathcal{R}$ on A is said to satisfy *type1 $T-S$ Ferrers property* if $T\{\mathcal{R}(a, b), \mathcal{R}(c, d)\} \leq S\{\mathcal{R}(a, d), \mathcal{R}(c, b)\}$, for each $a, b, c, d \in A$. In case T is a t-norm and S is the corresponding t-conorm(or dual t-conorm), then the type1 $T-S$ Ferrers property is simply called type1 T Ferrers property.

Definition 7.11. [31] A fuzzy relation $\mathcal{R}$ on A is said to satisfy *type2 T Ferrers property* if $\mathcal{R}o_T\mathcal{R}^d o_T\mathcal{R} \subseteq \mathcal{R}$.

Now we state the following important result in this context, given by Fodor[42], which shows under which condition, the two types of Ferrers properties turn out to be equivalent.

Proposition 7.12. *[41] The following statements are equivalent for a fuzzy relation on A:*

1. *A type1 T Ferrers relation is also type2 T Ferrers.*
2. *A type2 T Ferrers relation is also type1 T Ferrers.*
3. *T is rotational invariant(i.e., $T(x,y) \leq z \Leftrightarrow T(x, 1-z) \leq (1-y)$, for each $x, y, z \in [0,1]$).*

From the above proposition, it is clear that type1 T Ferrers property and type2 T Ferrers property are not equivalent in general. Keeping this in view, in literature, there are two different definitions of fuzzy biorders, one corresponding to type1 T_M Ferrers property of a fuzzy relation between A and X and another corresponding to type2 T Ferrers property of a fuzzy relation on A(cf.[41, 97]), which we have called here as type1 T_M biorder and type2 T biorder respectively.

Definition 7.13. [97] A fuzzy relation $\mathcal{R}$ between A and X is said to be *type1 T_M biorder* if $\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}$, for every $a, b \in A$ and $x, y \in X$.

Definition 7.14. [41] A fuzzy relation $\mathcal{R}$ on A is said to satisfy *type2 T biorders* if $\mathcal{R} o_T \mathcal{R}^d o_T \mathcal{R} \subseteq \mathcal{R}$.

Proposition 7.15. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then $\mathcal{R}$ is a type1 T_M biorder iff each $\mathcal{R}_\alpha$ is a biorder between A and X.*

Proof. Let $\mathcal{R}$ be a type1 T_M biorder between A and X. Then

$$\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\},$$

for each $a, b \in A$ and $x, y \in X$.

We have to show that $\mathcal{R}_\alpha$ is a biorder between A and X, for each $\alpha \in [0,1]$. Assume the contrary. Let for some $\alpha \in [0,1]$, there exist $a, b \in A$ and $x, y \in X$

such that

$$\begin{aligned} & a\mathcal{R}_\alpha x, b\mathcal{R}_\alpha y, a(\mathcal{R}_\alpha)^c y, b(\mathcal{R}_\alpha)^c x \\ \Rightarrow\ & \mathcal{R}(a,x) \geq \alpha, \mathcal{R}(b,y) \geq \alpha, \mathcal{R}(a,y) < \alpha, \mathcal{R}(b,x) < \alpha \\ \Rightarrow\ & \min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \geq \alpha > \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}, \end{aligned}$$

which is a contradiction.

Conversely, assume that each $\mathcal{R}_\alpha$ is a biorder between A and X, for $\alpha \in [0,1]$. We have to show that $\mathcal{R}$ is a type1 T_M biorder between A and X i.e,

$$\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}$$

for each $a, b \in A$ and $x, y \in X$. For this, we need to consider the following cases:
Case 1: If $\mathcal{R}(a,x) = 0$ or $\mathcal{R}(b,y) = 0$. Then the above inequality is obviously satisfied.
Case 2: If $\mathcal{R}(a,x) = \beta \neq 0$ and $\mathcal{R}(b,y) = \gamma \neq 0$. Set $\delta = \min\{\beta, \gamma\}$. Then

$$\begin{aligned} & \mathcal{R}(a,x) \geq \delta \,\text{and}\, \mathcal{R}(b,y) \geq \delta \\ \Rightarrow\ & a\mathcal{R}_\delta x \text{ and } b\mathcal{R}_\delta y \\ \Rightarrow\ & a\mathcal{R}_\delta y \text{ or } b\mathcal{R}_\delta x \quad (\text{Since } \mathcal{R}_\delta \text{ is a biorder}) \\ \Rightarrow\ & \mathcal{R}(a,y) \geq \delta \text{ or } \mathcal{R}(b,x) \geq \delta \\ \Rightarrow\ & \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\} \geq \delta = \min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\}. \end{aligned}$$

Thus $\mathcal{R}$ is a type1 T_M biorder. □

The following Example 7.1 shows that Proposition 7.15 is not true if we replace type1 T_M biorder with type2 T_M biorder or type2 T_{nM} biorder(which is equivalent to type1 T_{nM} biorder using Proposition 7.12, as T_{nM} is the largest rotational invariant t-norm[28]).

Example 7.1. *Consider the following fuzzy relation $\mathcal{R}$ between A and X, where $A = X = \{a, b\}$ as follows:*

$\mathcal{R}$	a	b
a	*1*	*0.5*
b	*0.5*	*0.6*

1. *It can be checked easily that $\mathcal{R}$ is a type2 T_M biorder.*

2. *It can also be verified that $\mathcal{R}$ is a type1 T_{nM} biorder. Further, since T_{nM} is the largest rotational invariant t-norm(cf.[28]) and hence in view of Proposition 7.12, it is also a type2 T_{nM} biorder.*

Now if we take $\alpha = 0.6$, then $(a,a) \in \mathcal{R}_\alpha$ and $(b,b) \in \mathcal{R}_\alpha$, but (a,b) and (b,a) both do not belong to $\mathcal{R}_\alpha$, which implies that $\mathcal{R}_{0.6}$ is not a biorder.

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the $\alpha-$cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52]. Motivated by these facts and keeping in view the Proposition 7.15, **from now onwards we mean fuzzy biorders in the sense of Definition 7.13** i.e., a fuzzy relation $\mathcal{R}$ between A and X will be called a fuzzy biorder relation if $\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}$, for each $a, b \in A$ and $x, y \in X$.

In [41], It has been shown that if $\mathcal{R}$ is a fuzzy relation on A, then $\mathcal{R}$ is a fuzzy biorder if and only if $\mathcal{R}^d$ is a fuzzy biorder. In the following proposition, we show that this result also holds good if we take a fuzzy relation between A and X.

Proposition 7.16. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then $\mathcal{R}$ is a fuzzy biorder between A and X iff $\mathcal{R}^d$ is a fuzzy biorder between X and A.*

Proof. Let $\mathcal{R}$ be a fuzzy biorder between A and X. So, we have

$$\min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \leq \max\{\mathcal{R}(a,y), \mathcal{R}(b,x)\}, \tag{7.7}$$

for each $a, b \in A$ and $x, y \in X$. Now,

$$\begin{aligned}
\min\{\mathcal{R}^d(x,a), \mathcal{R}^d(y,b)\} &= \min\{1-\mathcal{R}(a,x), 1-\mathcal{R}(b,y)\} \\
&= 1-\max\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} \\
&\leq 1-\min\{\mathcal{R}(a,y), \mathcal{R}(b,x)\} \quad (\text{Since } \mathcal{R} \text{ is a fuzzy biorder}) \\
&= \max\{1-\mathcal{R}(a,y), 1-\mathcal{R}(b,x)\} \\
&= \max\{\mathcal{R}^d(y,a), \mathcal{R}^d(x,b)\},
\end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore $\mathcal{R}^d$ is a fuzzy biorder between X and A.

Conversely, assume that $\mathcal{R}^d$ is a fuzzy biorder between X and A. So, we have

$$\min\{\mathcal{R}^d(x,a), \mathcal{R}^d(y,b)\} \leq \max\{\mathcal{R}^d(x,b), \mathcal{R}^d(y,a)\}, \tag{7.8}$$

for each $a, b \in A$ and $x, y \in X$.

Now,

$$\begin{aligned} \min\{\mathcal{R}(a,x), \mathcal{R}(b,y)\} &= \min\{1-\mathcal{R}^d(x,a), 1-\mathcal{R}^d(y,b)\} \\ &= 1-\max\{\mathcal{R}^d(x,a), \mathcal{R}^d(y,b)\} \\ &\leq 1-\min\{\mathcal{R}^d(x,b), \mathcal{R}^d(y,a)\} \quad \text{(Since } \mathcal{R}^d \text{ is a fuzzy biorder)} \\ &= \max\{1-\mathcal{R}^d(x,b), 1-\mathcal{R}^d(y,a)\} \\ &= \max\{\mathcal{R}(b,x), \mathcal{R}(a,y)\}, \end{aligned}$$

for each $a, b \in A$ and $x, y \in X$. Therefore $\mathcal{R}$ is a fuzzy biorder between A and X. □

Union and intersection of fuzzy biorders need not be a fuzzy biorder. This is exhibited through the following examples.

Example 7.2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations between A and X, where $A = X = \{a, b\}$ and are given as follows:*

$\mathcal{R}_1$	a	b
a	0.7	0.7
b	0.3	0.8

and

$\mathcal{R}_2$	a	b
a	0.6	0.3
b	0.6	0.8

.

It is easy to verify that $\mathcal{R}_1$ and $\mathcal{R}_2$ both are fuzzy biorder between A and X, but $\mathcal{R}_1 \cap \mathcal{R}_2$ which is given as follows:

$\mathcal{R}_1 \cap \mathcal{R}_2$	a	b
a	0.6	0.3
b	0.3	0.8

is not a fuzzy biorder between A *and* X *as*

$$\min\{\mathcal{R}(a,a),\mathcal{R}(b,b)\} = 0.6 > \max\{\mathcal{R}(a,b),\mathcal{R}(b,a)\} = 0.3,$$

where $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$.

Example 7.3. *Let* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be two fuzzy relations between* A *and* X*, where* $A = X = \{a, b\}$ *and are given as follows:*

$\mathcal{R}_1$	a	b
a	*0.7*	*0.5*
b	*0.4*	*0.4*

and

$\mathcal{R}_2$	a	b
a	*0.6*	*0.6*
b	*0.3*	*0.8*

.

It is easy to verify that $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *both are fuzzy biorder between* A *and* X*, but* $\mathcal{R}_1 \cup \mathcal{R}_2$ *which is given as follows:*

$\mathcal{R}_1 \cup \mathcal{R}_2$	a	b
a	*0.7*	*0.6*
b	*0.4*	*0.8*

is not a fuzzy biorder between A *and* X *as*

$$\min\{\mathcal{R}(a,a),\mathcal{R}(b,b)\} = 0.7 > \max\{\mathcal{R}(a,b),\mathcal{R}(b,a)\} = 0.6,$$

where $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.

Definition 7.17. [61, 97] Let T be a left continuous t-norm. Then the *residual implication operator* I_T associated with T is defined as $I_T(x,y) = \sup\{z \in [0,1] : T(x,z) \leq y\}$, for each $(x,y) \in [0,1] \times [0,1]$. For example,

1. If $T = T_P$, then the residual implication operator I_{T_P} associated with T_P is given by

$$I_{T_P}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0,1] \times [0,1]$.

2. If $T = T_L$, then the residual implication operator I_{T_L} associated with T_L is given by

$$I_{T_L}(x, y) = \min\{1, 1 - x + y\} = \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{otherwise,} \end{cases}$$

for each $(x, y) \in [0,1] \times [0,1]$.

Definition 7.18. [34] Let $\mathcal{R}$ be a binary relation between A and X and for any $a \in A$ and $x \in X$, $a\mathcal{R} = \{y \in X : a\mathcal{R}y\}$ and $\mathcal{R}x = \{c \in A : c\mathcal{R}x\}$. Then the binary relations $\mathcal{R}_A$ on A and $\mathcal{R}_X$ on X are defined as follows:

$$a\mathcal{R}_A b \text{ if } b\mathcal{R} \subseteq a\mathcal{R}$$
$$x\mathcal{R}_X y \text{ if } \mathcal{R}x \subseteq \mathcal{R}y.$$

Now, we prove the following:

Proposition 7.19. *Let $\mathcal{R}$ be a fuzzy biorder on A. Then*

1. *If $\mathcal{R}$ is reflexive on A, then $\mathcal{R}$ is strongly S_M- complete and negatively S_M- transitive on A.*

2. *If $\mathcal{R}$ is irreflexive on A, then $\mathcal{R}$ is T_M-asymmetric and T_M-transitive on A.*

Proof. 1. Since T_M has no zero divisors, it follows from Theorem 3 in [32] that $\mathcal{R}$ is strongly S_M- complete on A. Now we prove that $\mathcal{R}$ is negatively S_M- transitive as follows:

Since $\min\{\mathcal{R}(a,b), \mathcal{R}(c,c)\} \leq \max\{\mathcal{R}(a,c), \mathcal{R}(c,b)\}$, for each $a, b, c \in A$ and $\mathcal{R}(a', a') = 1$, for each $a' \in A$, so $\mathcal{R}(a,b) \leq \max\{\mathcal{R}(a,c), \mathcal{R}(c,b)\}$, for each $a, b, c \in A$.

2. We show that $\min\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} = 0$, for each $a, b \in A$. Since

$$\min\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} \leq \max\{\mathcal{R}(b,b), \mathcal{R}(a,a)\},$$

for each $a, b \in A$ and by the irreflexiviy of $\mathcal{R}$, $\mathcal{R}(a', a') = 0$, for each $a' \in A$, so we get $\min\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} \leq 0$ and hence $\min\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} = 0$, for each $a, b \in A$. Now it follows from Lemma 5.1(i) in [33] that $\mathcal{R}$ is T_M- transitive on A.

□

Definition 7.20. [34] A binary relation $\mathcal{R}$ between A and X is said to be *representable* with respect to $\leq$(resp.,$<$) if there exist two mappings $u : A \to \mathbb{R}$ and $v : X \to \mathbb{R}$ such that $(a, x) \in \mathcal{R}$ iff $u(a) \leq v(x)$(resp., $u(a) < v(x)$), for each $a \in A$ and $x \in X$.

In the case of biorders, we have the following result related to its representability.

Proposition 7.21. *[34] Let $\mathcal{R}$ be a binary relation between A and X, where A and X both are countable. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder.*
2. *$\mathcal{R}$ is representable with respect to $\leq$.*
3. *$\mathcal{R}$ is representable with respect to $<$.*

Now we prove the fuzzy analogue of this proposition.

Proposition 7.22. *Let $\mathcal{R}$ be a fuzzy relation between A and X, where A and X both are countable. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder.*
2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is representable with respect to $\leq$.*
3. *Each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is representable with respect to $<$.*

Proof. (1) ⇒ (2) Let $\mathcal{R}$ be a fuzzy biorder between A and X, where A and X both are countable. Then in view of Proposition 7.15, each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ is a biorder between A and X, where A and X both are countable. Therefore from the Proposition 7.21, we get a representation of $\mathcal{R}_\alpha$ with respect to $\leq$, for each $\alpha \in [0, 1]$.

(2) ⇒ (3) Let each $\mathcal{R}_\alpha$, $\alpha \in [0, 1]$ be representable with respect to $\leq$. Then from

the Proposition 7.21, each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $<$.
(3) $\Rightarrow$ (1) Let each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ be representable with respect to $<$. Then from the Proposition 7.21, each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is a biorder between A and X. Finally from the Proposition 7.15, we get $\mathcal{R}$ is a fuzzy biorder. □

Definition 7.23. [34] Let $\mathcal{R}$ be a binary relation between A and X. A subset $M^* \subseteq A \cup X$ is said to be *widely dense* if for each $a \in A$ and $x \in X$, $a\mathcal{R}^c x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in A$, $m^*\mathcal{R}^c x$ and $m^*\mathcal{R}_A a$ or $m^* \in X$, $x\mathcal{R}_X m^*$ and $a\mathcal{R}^c m^*$.

Proposition 7.24. *[34] Let $\mathcal{R}$ be a binary relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder with a countable widely dense subset M^*.*
2. *$\mathcal{R}$ is representable with respect to $\leq$.*

Now we give the fuzzy analogue of the this proposition.

Proposition 7.25. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder and for each $\alpha \in [0,1]$, there exists a countable subset $M^*_\alpha \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a(\mathcal{R}_\alpha)^c x$ implies that there exists an element $m^* \in M^*_\alpha$ such that either $m^* \in A$, $m^*(\mathcal{R}_\alpha)^c x$ and $m^*(\mathcal{R}_\alpha)_A a$ or $m^* \in X$, $x(\mathcal{R}_\alpha)_X m^*$ and $a(\mathcal{R}_\alpha)^c m^*$.*
2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $\leq$.*

The proof follows from Proposition 7.15 and 7.24.

Definition 7.26. [34] Let $\mathcal{R}$ be a binary relation between A and X. A subset M^* of $A \cup X$ is said to be *strictly dense* if for all $a \in A$ and $x \in X$, $a\mathcal{R}x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}m^*$ and $m^*\mathcal{R}_X x$ or $m^* \in A$, $a\mathcal{R}_A m^*$ and $m^*\mathcal{R}x$.

Proposition 7.27. *[34] Let $\mathcal{R}$ be a binary relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a biorder with a countably strictly dense subset M^*.*

2. *$\mathcal{R}$ is representable with respect to $<$.*

Now we give the fuzzy analogue of this proposition.

Proposition 7.28. *Let $\mathcal{R}$ be a fuzzy relation between A and X. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is a fuzzy biorder and for each $\alpha \in [0,1]$, there exists a countable subset $M^*_\alpha \subseteq A \cup X$ such that for each $a \in A$ and $x \in X$, $a\mathcal{R}_\alpha x$ implies that there exists an element $m^* \in M^*$ such that either $m^* \in X$, $a\mathcal{R}_\alpha m^*$ and $m^*(\mathcal{R}_\alpha)_X x$ or $m^* \in A$, $a(\mathcal{R}_\alpha)_A m^*$ and $m^*\mathcal{R}_\alpha x$.*

2. *Each $\mathcal{R}_\alpha$, $\alpha \in [0,1]$ is representable with respect to $<$.*

The proof follows from Proposition 7.15 and 7.27.

Definition 7.29. [14] A binary relation $\mathcal{R}$ on A is said to be an *interval order* if:

1. it is asymmetric i.e, if $(a,b) \in \mathcal{R}$, then $(b,a) \notin \mathcal{R}$;

2. it satisfies the Ferrers property i.e, $(a,b) \in \mathcal{R}$ and $(c,d) \in \mathcal{R} \Rightarrow (a,d) \in \mathcal{R}$ or $(c,b) \in \mathcal{R}$.

Now we prove the following result which is a corollary of Proposition 7.15:

Corollary 7.30. *Let $\mathcal{R}$ be a fuzzy relation on A. Then $\mathcal{R}$ is an irreflexive fuzzy biorder on A iff for each $\alpha \in (0,1]$, $\mathcal{R}_\alpha$ is an interval order on A.*

The proof follows from Proposition 7.15, Proposition 7.19 and the facts that a fuzzy relation is T_M-asymmetric if and only if its $\alpha-$cuts are asymmetric for every $\alpha \in (0,1]$ and every T_M-asymmetric fuzzy relation is irreflexive.

Proposition 7.31. *[34] Let $\mathcal{R}$ be a binary relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an interval order with a countable widely dense subset M^*.*

2. *There exist two mappings $u : A \to \mathbb{R}$ and $r : A \to \mathbb{R}^+_0$ such that for $a,b \in A$, $a\mathcal{R}b \Leftrightarrow u(a) + r(a) \leq u(b)$.*

Proposition 7.32. *[34] Let $\mathcal{R}$ be a binary relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an interval order with a countable strictly dense subset M^*.*
2. *There exist two mappings $u : A \to \mathbb{R}$ and $r : A \to \mathbb{R}^+$ such that for all $a, b \in A$, $a\mathcal{R}b$ iff $u(a) + r(a) < u(b)$.*

We now give the following two propositions:

Proposition 7.33. *Let $\mathcal{R}$ be a fuzzy relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable widely dense subset M^*_α.*
2. *For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \to \mathbb{R}$ and $r_\alpha : A \to \mathbb{R}^+_0$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) \leq u_\alpha(b)$.*

The proof follows from Corollary 7.30 and Proposition 7.31.

Proposition 7.34. *Let $\mathcal{R}$ be a fuzzy relation on A. Then the following statements are equivalent:*

1. *$\mathcal{R}$ is an irreflexive fuzzy biorder such that for each $\alpha \in (0, 1]$, there exists a countable strictly dense subset M^*_α.*
2. *For each $\alpha \in (0, 1]$, there exist two mappings $u_\alpha : A \to \mathbb{R}$ and $r_\alpha : A \to \mathbb{R}^+$ such that for $a, b \in A$, $a\mathcal{R}_\alpha b \Leftrightarrow u_\alpha(a) + r_\alpha(a) < u_\alpha(b)$.*

The proof follows from Corollary 7.30 and Proposition 7.32.

7.3 Representability of fuzzy weak orders using the residual implication operator

Definition 7.35. [98] Let T be a left continuous t-norm. A fuzzy relation $\mathcal{R}$ on a finite set A is called *$T-$representable*, if there exists a mapping $f : A \to [0, 1]$ such that $\mathcal{R}(a,b) = \mathcal{R}_{T,f}(a,b)$, for each $(a,b) \in A \times A$, where $\mathcal{R}_{T,f}$ is the fuzzy relation defined by $\mathcal{R}_{T,f}(a,b) = I_T(f(b), f(a))$, for each $(a,b) \in A \times A$ and I_T denotes the residual implication operator associated with T .

Theorem 7.36. *[97] A $T-$representable fuzzy relation on a finite set A is a fuzzy weak order on A with respect to T.*

Theorem 7.37. *[98] Let $\mathcal{R}$ be a fuzzy relation on a finite set A and T be a left continuous t-norm. Then $\mathcal{R}$ is a fuzzy quasi order with respect to T iff it is intersection of a finite family of fuzzy weak orders with respect to T.*

Remark 7.38. Let $\mathcal{R}$ be a $T-$transitive fuzzy relation on A. Then the following property holds for every $a, b, c \in A$:

$$(\mathcal{R}(a,b) = 1 \text{ and } \mathcal{R}(b,c) = 1) \Rightarrow (\mathcal{R}(a,c) = 1 \text{ and } \mathcal{R}(c,a) \leq \mathcal{R}(c,b)).$$

The proof is trivial.

Baets et al.[11] had proved the following result in case of T_M. These authors have also given an example to show that the converse of the following result does not hold good in that case.

Proposition 7.39. *[11] If a fuzzy relation $\mathcal{R}$ on A is negatively S_M-transitive, then its strict part $P_\mathcal{R}$ is T_M-transitive.*

Here we show that none of the implications hold good in case of T_P, through counter examples.

Example 7.4. *Let $\mathcal{R}$ be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:*

$\mathcal{R}$	a	b	c
a	0	0.84	0.6
b	0.8	0	0.7
c	0.7	0.6	0

Then its strict part $P_\mathcal{R}$ is as follows:

$P_\mathcal{R}$	a	b	c
a	0	0.84	0
b	0	0	0.7
c	0.7	0	0

Now it is easy to verify that $\mathcal{R}$ is negatively S_P-transitive but $P_{\mathcal{R}}$ is not T_P-transitive as $P_{\mathcal{R}}(a,b).P_{\mathcal{R}}(b,c) = 0.84 \times 0.7 = 0.58 > P_{\mathcal{R}}(a,c) = 0$.

Example 7.5. *Let $\mathcal{R}$ be a fuzzy relation on $A = \{a, b, c\}$ whose matrix representation is as follows:*

$\mathcal{R}$	a	b	c
a	1	0.5	0.9
b	1	1	0.6
c	1	1	1

Then its strict part $P_{\mathcal{R}}$ is as follows:

$P_{\mathcal{R}}$	a	b	c
a	0	0	0
b	1	0	0
c	1	1	0

Now it is easy to verify that $P_{\mathcal{R}}$ is T_P-transitive but $\mathcal{R}$ is not negatively S_P-transitive as $\mathcal{R}(a,b) + \mathcal{R}(b,c) - \mathcal{R}(a,b).\mathcal{R}(b,c) = 0.5 + 0.6 - 0.3 = 0.8 < \mathcal{R}(a,c) = 0.9$.

Proposition 7.40. *If $\mathcal{R}$ is a fuzzy relation on A which is strongly S_M-complete. It satisfies, for each $a, b, c \in A$*

$$P_{\mathcal{R}}(a,b) = 1 \text{ and } P_{\mathcal{R}}(b,c) = 1 \Rightarrow \mathcal{R}(c,a) \leq S(\mathcal{R}(c,b), \mathcal{R}(b,a)),$$

if and only if $\mathcal{R}$ is negatively $S-$transitive.

Proof. To show that $\mathcal{R}$ is negatively $S-$transitive, we have to show that

$$S(\mathcal{R}(a,b), \mathcal{R}(b,c)) \geq \mathcal{R}(a,c) \tag{7.9}$$

for each $a, b, c \in A$. Assume the contrary. Let

$$\mathcal{R}(a,c) > S(\mathcal{R}(a,b), \mathcal{R}(b,c)) \tag{7.10}$$

for some $a, b, c \in A$. Then from (7.10) and since $\mathcal{R}$ is strongly S_M-complete, we get $\mathcal{R}(b,a) = 1 > \mathcal{R}(a,b)$ and $\mathcal{R}(c,b) = 1 > \mathcal{R}(b,c)$. This implies that $P_{\mathcal{R}}(b,a) = 1$

and $P_{\mathcal{R}}(c,b) = 1$. So by our assumption, we must have

$$\mathcal{R}(a,c) \leq S(\mathcal{R}(a,b), \mathcal{R}(b,c)),$$

which contradicts (7.10).

Conversely, if $\mathcal{R}$ is negatively $S-$transitive, then by its definition itself: $\mathcal{R}(c,a) \leq S(\mathcal{R}(c,b), \mathcal{R}(b,a)$, for each $a,b,c \in A$. □

Proposition 7.41. *The union $\mathcal{R}$ of any finite family $\{\mathcal{R}_i\}_{i=1}^{n}$ of fuzzy weak orders on A with respect to T is a fuzzy quasi-transitive relation on A with respect to T.*

Proof. Obviously, $\mathcal{R}$ is strongly S_M-complete. Let $a,b,c \in A$ be such that $P_{\mathcal{R}}(a,b) = 1$ and $P_{\mathcal{R}}(b,c) = 1$. Now $P_{\mathcal{R}}(a,b) = 1$ implies that $1 > \mathcal{R}(b,a)$ and $P_{\mathcal{R}}(b,c) = 1$ implies that $1 > \mathcal{R}(c,b)$. Then for each $i \in \{1,2,...,n\}$, $\mathcal{R}_i(b,a) < 1$ and since $\mathcal{R}_i$ is strongly S_M-complete so $\mathcal{R}_i(a,b) = 1$. Similarly, $\mathcal{R}(c,b) < 1$ implies that $\mathcal{R}_i(b,c) = 1$. Since each $\mathcal{R}_i$ is a fuzzy weak order with respect to T, so by Remark 7.38, we have

$$\mathcal{R}_i(c,a) \leq \mathcal{R}_i(c,b), \quad \text{for each } i \in \{1,2,...,n\} \tag{7.11}$$

Now, $\mathcal{R}(c,a) = \max_i \mathcal{R}_i(c,a) = \mathcal{R}_t(c,a)$, for some $t \in \{1,2,...,n\}$, then

$$\begin{aligned} S(\mathcal{R}(c,b), \mathcal{R}(b,a)) &\geq S_M(\mathcal{R}(c,b), \mathcal{R}(b,a)) \\ &\geq \mathcal{R}(c,b) \\ &\geq \mathcal{R}_t(c,b) \\ &\geq \mathcal{R}_t(c,a) = \mathcal{R}(c,a) \end{aligned}$$

Therefore, by Proposition 7.40, $\mathcal{R}$ is negatively $S-$ transitive. Thus, $\mathcal{R} = \bigcup_i \mathcal{R}_i$ is strongly S_M-complete as well as negatively $S-$transitive and hence it is a fuzzy quasi-transitive relation with respect to T. □

Characterizations for fuzzy weak orders with respect to T_M and T_P on a finite set which are T_M-representable and T_P-representable have been respectively obtained by Baets et al.[11] and Sali et al.[98]. In the following theorem, we have obtained a characterization for fuzzy weak orders with respect to T_L on a finite set which are T_L representable.

Theorem 7.42. *A fuzzy weak order $\mathcal{R}$ with respect to T_L on a finite set A is T_L-representable if and only if*

$$\mathcal{R}(a,b) < 1 \text{ and } \mathcal{R}(b,c) < 1 \Rightarrow \mathcal{R}(a,c) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \quad (7.12)$$

holds for each $a, b, c \in A$.

To prove the above theorem, we need to prove the following lemma, the proof of which is on the similar lines as that of Lemma 7 in [98].

Lemma 7.43. *Let $\mathcal{R}$ be a reflexive fuzzy relation on a finite set A satisfying (7.12). Then there exists $c \in A$ such that $\mathcal{R}(c,a) = 1$ for each $a \in A$.*

Proof. Assume the contrary, i.e, for each $c \in A$, there exists $a_c \in A$ such that $\mathcal{R}(c, a_c) < 1$. Now define an oriented graph $\vec{G} = (V, E)$, where $V = A$ and there is an arc from a to b iff $\mathcal{R}(a,b) < 1$. By our assumption the out-degree of each node is atleast one, so there is a directed cycle C in $\vec{G}$, which is obtained by taking connected nodes in the cyclic order. Let the nodes of C be $\{a_1, a_2, ..., a_n\}$. Then by using (7.12), we have

$$\begin{aligned} 1 = n - (n-1) &> \mathcal{R}(a_1,a_2) + \mathcal{R}(a_2,a_3) + ... + \mathcal{R}(a_{n-1},a_n) + \mathcal{R}(a_n,a_1) - (n-1) \\ &= \mathcal{R}(a_1,a_1) \end{aligned}$$

which contradicts the reflexivity of $\mathcal{R}$. □

Proof of the theorem. Let $\mathcal{R}$ be a fuzzy weak order with respect to T_L which is T_L-representable. Then for each $a, b \in A$, $\mathcal{R}(a,b) = \mathcal{R}_{T_L,f}(a,b) = I_{T_L}(f(b), f(a))$ for some mapping $f : A \to [0,1]$. If $\mathcal{R}(a,b) < 1$ and $\mathcal{R}(b,c) < 1$, for some $a, b, c \in A$, then $\mathcal{R}(a,b) = 1 - f(b) + f(a)$ and $\mathcal{R}(b,c) = 1 - f(c) + f(b)$ such that $f(b) > f(a)$ and $f(c) > f(b)$. This implies that $f(c) > f(a)$ and hence $\mathcal{R}(a,c) = 1 - f(c) + f(a) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$.

Conversely, let $\mathcal{R}$ be a fuzzy weak order with respect to T_L satisfying (7.12). Then $\mathcal{R}$ is reflexive and so by the previous lemma, there exists $c \in A$ such that $\mathcal{R}(c,a) = 1$, for each $a \in A$. Now define the mapping $f : A \to [0,1]$ by $f(a) =$

$\mathcal{R}(a,c)$, for each $a \in A$. We show that

$$\mathcal{R}(a,b) = \mathcal{R}_{T_L,f}(a,b) = I_{T_L}(f(b),f(a)) = \begin{cases} 1, & \text{if } f(b) \leq f(a) \\ 1 - f(b) + f(a), & \text{otherwise,} \end{cases}$$

To prove this we need to consider the following cases:

Case 1: If $f(a) \geq f(b)$ (i.e, $\mathcal{R}(a,c) \geq \mathcal{R}(b,c)$). In this case $I_{T_L}(f(b),f(a)) = 1$, so we have to show that $\mathcal{R}(a,b) = 1$. Assume the contrary that $\mathcal{R}(a,b) < 1$. If $\mathcal{R}(b,c) < 1$, then by (7.12),

$$\begin{aligned} \mathcal{R}(a,c) &= \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \\ &\leq \mathcal{R}(a,b) + \mathcal{R}(a,c) - 1 \\ &< \mathcal{R}(a,c) \quad (\text{since } \mathcal{R}(a,b) < 1) \end{aligned}$$

which is a contradiction. Next, if $\mathcal{R}(b,c) = 1$, then $\mathcal{R}(a,c) = 1$ and in view of the previous lemma, $\mathcal{R}(c,b) = 1$. So by the T_L-transitivity of $\mathcal{R}$, we have

$$\begin{aligned} T_L(\mathcal{R}(a,c),\mathcal{R}(c,b)) &\leq \mathcal{R}(a,b) \\ 1 = \max\{0, \mathcal{R}(a,c) + \mathcal{R}(c,b) - 1\} &\leq \mathcal{R}(a,b) < 1 \end{aligned}$$

which is again a contradiction. Hence in this case we are done.

Case 2: If $f(a) < f(b)$ (i.e., $\mathcal{R}(a,c) < \mathcal{R}(b,c)$). In this case $I_{T_L}(f(b),f(a)) = 1 - f(b) + f(a)$, so we have to show that $\mathcal{R}(a,b) = 1 - f(b) + f(a)$. Let $\mathcal{R}(b,c) < 1$. Now by the T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,b),\mathcal{R}(b,c)) = \max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} \leq \mathcal{R}(a,c)$. If $\max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$, then $\mathcal{R}(a,b) + \mathcal{R}(b,c) - 1 \leq \mathcal{R}(a,c) < \mathcal{R}(b,c)$ which implies that $\mathcal{R}(a,b) < 1$. If $\max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} = 0$, then $\mathcal{R}(a,b) \leq 1 - \mathcal{R}(b,c)$ and hence $0 < \mathcal{R}(a,b) < 1$(since $0 < \mathcal{R}(b,c) < 1$). Now by using (7.12), we have $\mathcal{R}(a,c) = \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1$ which implies that $\mathcal{R}(a,b) = 1 - \mathcal{R}(b,c) + \mathcal{R}(a,c) = 1 - f(b) + f(a)$. Next, if $\mathcal{R}(b,c) = 1$, then by the T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,b),\mathcal{R}(b,c)) = \max\{0, \mathcal{R}(a,b) + \mathcal{R}(b,c) - 1\} \leq \mathcal{R}(a,c)$ which implies that $\mathcal{R}(a,b) \leq \mathcal{R}(a,c)$. Again by using T_L-transitivity of $\mathcal{R}$, we have $T_L(\mathcal{R}(a,c),\mathcal{R}(c,b)) = \max\{0, \mathcal{R}(a,c) + \mathcal{R}(c,b) - 1\} \leq \mathcal{R}(a,b)$ which implies that $\mathcal{R}(a,c) \leq \mathcal{R}(a,b)$, since $\mathcal{R}(c,b) = 1$ using the previous lemma. Hence $\mathcal{R}(a,b) = \mathcal{R}(a,c) = 1 - \mathcal{R}(b,c) + \mathcal{R}(a,c) = 1 - f(b) + f(a)$. This proves the theorem.

7.4 Conclusion

In this chapter, representability of fuzzy biorders in terms of their $\alpha-$cuts and fuzzy weak orders using residual implication operators, have been studied. Further, we have shown that union of a finite family of fuzzy weak orders with respect to a t-norm T is fuzzy quasi-transitive with respect to T and counter examples have been produced to show that unions and intersections of fuzzy biorders need not be fuzzy biorder. In the last theorem, we have also obtained a characterization for a T_L-representable fuzzy weak orders.