Chapter 5

Delayed Output Feedback based Leader-Follower and Leaderless Consensus Control of Uncertain Multiagent Systems

A distributed artificially delayed output feedback-based leader-follower and leaderless consensus control for the uncertain general linear multiagent systems (MASs) with an arbitrary relative degree is presented in this chapter. Based on the measured relative output and their delayed information with respect to neighbors, two distributed controllers are designed, which relaxes the requirement of all relative states information of agents. Through the Lyapunov-Krasovskii functional, linear matrix inequalities (LMIs) are formulated, and these LMIs are feasible for arbitrary small delays selected by the user. Delay-dependent sufficient conditions are provided to guarantee the asymptotic convergence of nominal error systems and ultimately uniform boundedness in the case of a perturbed error system, using the input to state stability (ISS) theory. Finally, the efficacy of the proposed method is illustrated through numerical examples.

5.1 Introduction

The research on MASs has broadly developed over the past few decades due to its extensive application in various fields, i.e., unmanned ariel vehicles (UAV), the formation of spacecraft, mobile robot networks, etc. [50–52]. The consensus problem of MASs is a hot research topic aiming to accomplish a common goal for all the agents. The consensus problems are classified into leader-follower and leaderless consensus problems. The consensus problem is a leader-follower consensus problem if the leader agent decides the final consensus value [54, 55, 61], otherwise known as a leaderless consensus problem [62–64]. For linear and nonlinear MASs, the leaderless and leader-follower consensus control is discussed in [69] and [70], respectively.

The consensus problem of first-order systems and second-order systems are extensively studied in the literature [73–75]. Consensus control of linear MASs, including first-order and the second-order, as a particular case, is also considered. With the assumption of all relative states as measurable, the consensus problem of general linear MASs based on the algebraic Riccati equation is reported in [76,77] and by employing the LMI method in [78]. Utilizing static output information, the consensus problem of general linear MASs with an assumption on the system dynamics is investigated in [79,80]. An assumption on system dynamics is relaxed, and a dynamic output feedback controller is designed under a uniform connected directed graph in [81]. A distributed observer-based consensus control for general linear MASs with a directed spanning tree is investigated in [82,83]. Recently, output feedback-based leader-follower consensus control for nonlinear time-delay MASs using dynamic compensators to estimate the agent's unknown states are reported in [84]. It is noteworthy that, when all the states are not measurable, distributed observer-based consensus control is significantly examined. But, at the price of adding observer dynamics into the system dynamics, and delivers more computation.

Therefore, to relax the requirement of estimation of unknown states using observer, many authors recommended a artificially delayed output feedback control for some class of systems [14, 17, 167]. The construction of artificially delayed output feedback control requires the output and their delayed information. Thus, artificially delayed output feedback control has advantages over observer-based control in terms of relaxing the unknown state estimation. In the work of [6, 18], an artificial delayed output feedback controller is proposed for the second-order and high-order integrator system by employing a Lyapunov-Krasovskii functional. A similar approach is exploited for the second-order MASs under undirected graph using the frequency-domain method in [20] and Lyapunov stability approach in [86]. Next, the same theory for consensus control of second-order MASs with a directed graph is studied in [87]. More recently, cooperative tracking control for the relative degree two general linear MASs based on the corresponding output information and its delayed information is examined in [88]. Despite the abovementioned results, there are many other consensus problems where excellent contributions can be performed, such as bipartite consensus, arbitrary-time consensus, output feedback consensus, consensus using time scale theory, consensus under switching communication topologies, etc.

In the various applications of MASs, all states information is needed in order to achieve the final goal. But in practice, all the states are not available for measurement; only output is available. Therefore, distributed observer-based consensus control is significantly examined in the literature. But, due to the addition of observer dynamics into the system dynamics, more computation is required. Therefore, a new technique is required for the MASs.

The contribution of the chapter is as follows:

(i) An artificially delayed output feedback consensus control protocol is designed.

(ii) The consensus is achieved for the directed network topology with a directed spanning tree.

(iii) The results are valid for general linear MASs with an arbitrary relative degree.

(iv) The Lyapunov-Krasovskii functional is constructed, and delay-dependent sufficient conditions are provided.

(v) Robust stability analysis is presented with the help of ISS.

(vi) The novelty and significance of the proposed method is provided based on the comparative results.

The distributed controllers are implemented, which utilizes agent's relative output and their artificial delayed information, such that the consensus is accomplished for the uncertain general linear MASs. Using ISS theory, it is confirmed that the asymptotic convergence is achieved for the nominal error system, and ultimately uniform boundedness is obtained in the case of the perturbed error system. Apart from the contributions, there are some challenges of the studied problem as follows: (i) Selection of artificial delay h such that the LMI becomes feasible. (ii) Selection of positive definite matrices P, S, R_α to calculate the controller gain matrix K.

The outline of the chapter is as follows. Section 5.2 formulates the problem statement. Section 5.3 provides the leader-follower consensus problem. Section 5.4 describes the leaderless consensus problem. Simulation results of the both leader-follower and leadeless problem are presented in Section 5.5. Finally, Section 5.6 summarised the chapter.

5.1.1 Graph Theory

A directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = 1, \cdots, N$ is the set of vertices corresponding to each agent and $\mathcal{E} = \{(i,j)|i,j \in \mathcal{V}\}$ is the set of edges between two different agents. An edge $(i,j) \in \mathcal{E}$ means that agent j can obtain information from agent i but not necessarily vice versa. For agent $i \in \mathcal{V}, j$ is named a neighbor of i if $(i,j) \in \mathcal{E}$ The neighbors of agent i are denoted by $N_i = \{j|(i,j) \in \mathcal{E}\}$. A directed graph $\mathcal{G}$ has a spanning tree if there is an agent named root, such that there is a directed path from the root to every other agent in the graph. The digraph $\mathcal{G}$ is strongly connected if there is a directed path between any two distinct nodes.

The leaderless consensus problem of MASs are defined by a directed graph $\mathcal{G}$ of N agents. The adjacency matrix of $\mathcal{G}$ is $A = [a_{ij}] \in \mathbb{R}^{N\times N}$. The entries a_{ij} are defined as $a_{ii} = 0, a_{ij} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. Define the in-degree of agent i as $d_i = \sum_{j=1}^{N} a_{ij}$ and the in-degree matrix as $D = \text{diag}\{di\} \in \mathbb{R}^{N\times N}$. The Laplacian matrix $\mathcal{L}$ of directed graph $\mathcal{G}$ is defined by $\mathcal{L} = D - A$.

In the leader-follower consensus problem, the MASs involves a leader and N followers. A directed graph $\mathcal{G}$ describes the intercommunication graph between the N agents. The leader named agent 0 can only convey its information to the connected follower agents and cannot obtain information from any follower agents. Assume $\mathcal{L}$ is the Laplacian matrix of graph $\mathcal{G}$. Define modified Laplacian matrix $\mathcal{H} = \mathcal{L}+\mathcal{B}$, where $\mathcal{B} = \text{diag}\left\{b_1 \quad b_2 \quad \cdots \quad b_N\right\}$ with $b_i = 1$ if agent i can obtain information from leader and $b_i = 0$, otherwise.

Next, consider the following unforced system

$$\dot{z}(t) = f(z(t), z_h(t), d(t)), \tag{5.1}$$

where $z(t) \in \mathbb{R}^n$ and $z_h(t) \in \mathbb{R}^n$ is the system state, where $z_h(t) := z(h+t), -h \leq t \leq 0$, $d(t) \in \mathbb{R}^m$ is the disturbance signal, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differential function.

Definition 4 *[42] The system* (5.1) *is said to be uniform globally ISS if there exists a* $\mathcal{KL}$ *function* β *and a* $\mathcal{K}$ *function* γ *such that, for any initial time* t_0 *and initial state*

$z_{t_0} = \phi \in \mathbb{W}$ and any measurable, locally essentially bounded input $d(t)$, the solution $z(t, t_0, \phi)$ exists for all $t \leq t_0$ and moreover if holds

$$|z(t, t_0, \phi)| \leq \max(\beta(\|\phi\|_{\mathbb{W}}, t - t_0), \gamma(|d_{[t_0,t]}|_{\infty})). \tag{5.2}$$

5.2 Problem Formulation

Consider a group of N agents interacting with neighbors via a network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with following dynamics

$$\begin{cases} \dot{z}_i(t) & = Az_i(t) + B(u_i(t) + d_i(t)), \\ y_i(t) & = Cz_i(t) = z_{i,1}(t), \end{cases} \tag{5.3}$$

where $z_i(t) \in \mathbb{R}^n, u_i(t) \in \mathbb{R}^m, y_i(t) \in \mathbb{R}^m$ and $d_i(t) \in \mathbb{R}^m$ are the state, control input, measured output and disturbance vectors of the i-th agent, respectively. Here, the disturbances are considered as unknown but the upper bound is assumed to be known. The matrices $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n \times n}$, and $B = \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \in \mathbb{R}^{n \times m}$ are known with $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $B_2 \in \mathbb{R}^{m \times m}$. The matrix A may or may not be Hurwitz, and it is assumed that the pair (A, B) is stabilizable, and the pair (A, C) is detectable.

Next, the objective is to design an artificially delayed output feedback consensus control for the leader-follower and leaderless MASs.

For a leaderless consensus problem, there are N agents and $i = 1, 2, \cdots, N$. For the leader-follower consensus problem, there are N followers with $i = 1, 2, \cdots, N$ and a leader agent with $i = 0$. The agent 0 is the leader agent with $u_0(t) = d_0(t) \equiv 0$. It is assumed that the leader agent states are only accessible by a part of follower agents in $\mathcal{G}$, and the information from the follower agent to a leader agent cannot be obtained.

5.3 Leader-Follower Consensus Problem

In this section, we have presented the output feedback based leader-follower consensus problem of MASs (5.3) with N agents using measured output and its delayed version.

5.3.1 Assumption and Lemmas

Assumption 2 *The directed communication graph $\mathcal{G}$ has a spanning tree and at least one follower agent can interact with the leader agent.*

Lemma 8 *[89, 90] Assume directed graph $\mathcal{G}$ has a spanning tree. Suppose*

$$p = \begin{bmatrix} p_1, \cdots, p_n \end{bmatrix}^T = \mathcal{H}^{-T} \mathbf{1}_N, \quad \mathcal{H} := \mathcal{L} + \mathcal{B},$$
$$P = diag(p_i),$$
$$Q = P\mathcal{H} + \mathcal{H}^T P.$$

Then, $P > 0$ and $Q > 0$. For notational brevity, define $\varpi_0 = \lambda_{min}(Q)$.

Lemma 9 *[89] Under Assumption (2), all eigenvalues of $\mathcal{H}$ have positive real parts.*

5.3.2 Distributed Controller Design

Consider the neighborhood output measurement error of agent i as follows

$$e_i(t) = \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)) + b_i(z_{0,1}(t) - z_{i,1}(t)). \tag{5.4}$$

Thus, the distributed delayed consensus control with $h_0 = 0$ is proposed for each follower i as

$$u_i(t) = \sum_{\alpha=0}^{n-1} K_\alpha e_i(t - h_\alpha), \qquad i = 1, \cdots, N, \tag{5.5}$$

where $K_\alpha = \begin{bmatrix} K_0, \cdots, K_{n-1} \end{bmatrix} \in \mathbb{R}^{m \times nN}$ denotes the coupling strength gain matrices, and $0 = h_0 < h_1 < \cdots < h_{n-1}$ is a known artificial delay.

Define $u(t) = \begin{bmatrix} u_1^T(t) & \cdots & u_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$, $y(t) = \begin{bmatrix} y_1^T(t) & \cdots & y_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$, and write the global form of (5.5) as

$$u(t) = -\sum_{\alpha=0}^{n-1} (\mathcal{H} \otimes I_m) K_\alpha \left\{ z_1(t - h_\alpha) - \underline{z_0}(t - h_\alpha) \right\} \tag{5.6}$$

where $\mathcal{H}$ is defined earlier, $z_1(t - h_\alpha) \in \mathbb{R}^{nN}$, and $\underline{z_0}(t - h_\alpha) = \mathbf{1}_N \otimes z_0(t - h_\alpha)$.

We represent the global state as $z(t) = \begin{bmatrix} z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{nN}$, $\underline{z_0}(t) = \mathbf{1}_N \otimes z_0(t)$. The global consensus error is defined as $\Psi(t) = z(t) - \underline{z_0}(t)$. The aforementioned agent dynamics (5.3) under control input (5.5) leads to the following global consensus error dynamics

$$\dot{\Psi}(t) = (I_N \otimes A)\Psi(t) - \sum_{\alpha=0}^{n-1}(\mathcal{H} \otimes B)K_\alpha \bar{z}_1(t-h_\alpha) + (I_N \otimes B)d(t), \tag{5.7}$$

where $\bar{z}_1(t-h_\alpha) = z_1(t-h_\alpha) - \underline{z_0}(t-h_\alpha)$ and $d(t) = \begin{bmatrix} d_1^T(t) & \cdots & d_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$..

By employing Taylor's expanson with the integral term of the remainder mentioned in [6], we have

$$\bar{z}_1(t-h_\alpha) = \sum_{\beta=0}^{n-1} \frac{1}{\beta!}(-h_\alpha)^\beta \bar{z}_{\beta+1}(t) + W_\alpha(t), \quad \alpha = 1, \cdots, n-1, \tag{5.8}$$

where

$$W_\alpha(t) = \frac{(-1)^n}{(n-1)!}\int_{t-h_\alpha}^{t} (s-t+h_\alpha)^{n-1} \bar{z}_1^n(s)ds, \tag{5.9}$$

and where $\bar{z}_1^n(s)$ is the n^{th} derivative of $\bar{z}_1(s)$. It is important to note that $W_\alpha(t) = O(h_\alpha^n)$.

It follows from (5.8) that

$$\sum_{\alpha=0}^{n-1} K_\alpha \bar{z}_1(t-h_\alpha) = \sum_{\beta=0}^{n-1} \overline{K_\beta}\Psi_{\beta+1}(t) + \sum_{\alpha=1}^{n-1} K_\alpha W_\alpha(t), \tag{5.10}$$

where

$$\begin{aligned} \overline{K_0} &= \sum_{\alpha=0}^{n-1} K_\alpha \quad \text{and,} \\ \overline{K_\beta} &= \frac{(-1)^\beta}{\beta!}\sum_{\alpha=1}^{n-1} h_\alpha^\beta K_\alpha, \qquad \beta = 1, \cdots, n-1. \end{aligned} \tag{5.11}$$

Based on (5.11) and coupling strength gain matrix $K = K_\alpha = \begin{bmatrix} K_0 & \cdots & K_{n-1} \end{bmatrix}$ one can obtain new coupling strength gain matrix $\overline{K} = \begin{bmatrix} \overline{K_0} & \overline{K_\beta} \end{bmatrix}$

$$\overline{K} = K\Delta$$

$$\Delta = \begin{bmatrix} I_m & 0 & 0 & \cdots & 0 \\ I_m & -h_1 I_m & \frac{h_1^2}{2} I_m & \cdots & \frac{(-h_1)^{n-1}}{(n-1)!} I_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I_m & -h_{n-1} I_m & \frac{h_{n-1}^2}{2} I_m & \cdots & \frac{(-h_{n-1})^{n-1}}{(n-1)!} I_m \end{bmatrix}. \tag{5.12}$$

where Δ is Vandermonde-type matrix and it is invertible due to the different values of delay.

The substitution of (5.10) into (5.7) leads to

$$\dot{\Psi}(t) = \left(I_N \otimes A - \mathcal{H} \otimes B\overline{K}\right)\Psi(t) - \sum_{\alpha=1}^{n-1}(\mathcal{H} \otimes B)K_\alpha W_\alpha(t) + (I_N \otimes B)d(t). \tag{5.13}$$

5.3.3 Stability Analysis

Theorem 3 *Consider the MASs (5.3) stabilized by the control input (5.5). Assume the network graph satisfies Assumption 2. Given* $K_\alpha \in \mathbb{R}^{m\times n}(\alpha = 0, \cdots, n-1)$ *and the known constant delays* $0 = h_0 < h_1 < \cdots < h_{n-1}$ *such that the matrix* $D = 2A - \frac{\varpi_0}{p_m}B\overline{K}$ *with* $\overline{K} = K\Delta$ *where* Δ *is defined in (5.12) and* $K = \begin{bmatrix} K_0 & \cdots & K_{n-1}\end{bmatrix}$ *is Hurwitz. For a given* $\sigma > 0$*, suppose there exist positive definite matrices* $P \in \mathbb{R}^{N\times N}$*,* $S \in \mathbb{R}^{n\times n}$ *and* $R_\alpha \in \mathbb{R}^{m\times m}(\alpha = 1, \cdots, n-1)$ *that satisfy the LMI (5.14)*

$$\begin{bmatrix} P\otimes(\sigma S + SD) & -P\mathcal{H}\otimes SB & \cdots & -P\mathcal{H}\otimes SB & P\otimes SB & \Pi^T M^T\tilde{R} \\ * & -(n!)^2 e^{-\sigma h_1}\overline{R_1} & \cdots & 0 & 0 & \Phi^T M^T \tilde{R} \\ * & * & \cdots & 0 & 0 & \Phi^T M^T\tilde{R} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & -(n!)^2 e^{-\sigma h_{n-1}}\overline{R_{n-1}} & 0 & \Phi^T M^T \tilde{R} \\ * & * & \cdots & * & -\gamma\Gamma & \Sigma^T M^T\tilde{R} \\ * & * & \cdots & * & * & -\tilde{R} \end{bmatrix} < 0 \tag{5.14}$$

where $\Pi = \left(I_N \otimes A - \mathcal{H}\otimes B\overline{K}\right)$*,* $M = I_N \otimes \begin{bmatrix} 0_{m\times(n-1)m} & I_m\end{bmatrix}$*,* $\tilde{R} = \sum_{\alpha=1}^{n-1} h_\alpha^{2n} K_\alpha^T \overline{R_\alpha} K_\alpha$*,* $\overline{R_\alpha} = I_N \otimes R_\alpha$*,* $\Phi = \mathcal{H}\otimes B$*,* $\Gamma = I_N \otimes S$*, and* $\Sigma = I_N \otimes B$*. Notation* $\varpi_0 = \lambda_{min}(Q)$*,* $p_M = \max\{p_1, \cdots, p_N\}$*, where* $Q, p_i,\ i = 1, \cdots, N$ *are defined in Lemma 8.*

Then, system (5.13) is ISS under external disturbance input.

Proof: Consider the following Lyapunov-Krasovskii functional

$$V(\Psi_t, \dot{\Psi}_t) = V_1(\Psi) + V_2(\Psi_t, \dot{\Psi}_t), \tag{5.15}$$

where

$$V_1(\Psi) = \Psi^T(t)(P\otimes S)\Psi(t), \tag{5.16}$$

$$V_2(\Psi_t, \dot{\Psi}_t) = \sum_{\alpha=1}^{n-1} h_\alpha^n \int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s - t + h_\alpha)^n \bar{z}_1^{nT}(s) \times K_\alpha^T \overline{R_\alpha} K_\alpha \bar{z}_1^n(s)ds, \tag{5.17}$$

where $\bar{z}_1(t) = z_1(t) - z_0(t)$ and $\bar{z}_1^n(t)$ is the n^{th} derivative of $\bar{z}_1(t)$.

The derivative of $V_1(t)$ along (5.13) leads to

$$\begin{aligned}\dot{V}_1(t) = 2\Psi^T(t)\left[P \otimes SA - P\mathcal{H} \otimes SB\overline{K}\right]\Psi(t) - 2\Psi^T(t)\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ + 2\Psi^T(t)(P \otimes SB)d(t),\end{aligned} \tag{5.18}$$

where $W_\alpha(t)$ is defined in (5.9).

Employing Lemma 8 yields

$$\begin{aligned}\dot{V}_1(\Psi) &= \Psi^T(t)\left[P \otimes 2SA - Q \otimes SB\overline{K}\right]\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ &\quad + 2\Psi^T(t)(P \otimes SB)d(t) \\ &\leq \Psi^T(t)\left[P \otimes 2SA - \varpi_0 I_N \otimes SB\overline{K}\right]\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ &\quad + 2\Psi^T(t)(P \otimes SB)d(t) \\ &\leq \Psi^T(t)\left[P \otimes \left(2SA - \frac{\varpi_0}{p_M}SB\overline{K}\right)\right]\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ &\quad + 2\Psi^T(t)(P \otimes SB)d(t) \\ &= \Psi^T(t)\left(P \otimes SD\right)\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ &\quad + 2\Psi^T(t)(P \otimes SB)d(t) \\ &= \Psi^T(t)\left[P \otimes (\sigma S + SD)\right]\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H} \otimes SB)K_\alpha W_\alpha(t) \\ &\quad + 2\Psi^T(t)(P \otimes SB)d(t) - \sigma V_1(t).\end{aligned} \tag{5.19}$$

By, employing the fact that $Q \geq \varpi_0 I_N$ and $p_M I_N \geq P$, one can obtain the first and second inequalities, respectively.

Now, take the derivative of $V_2(\Psi_t, \dot{\Psi}_t)$ one obtains

$$\begin{aligned}\dot{V}_2(\Psi_t, \dot{\Psi}_t) &= \sum_{\alpha=1}^{n-1} h_\alpha^{2n}\bar{z}_1^{nT}(t)K_\alpha^T\overline{R_\alpha}K_\alpha\bar{z}_1^n(t) - \sigma V_2(t) \\ &\quad - \sum_{\alpha=1}^{n-1} nh_\alpha^n\int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s-t+h_\alpha)^{n-1}\bar{z}_1^{nT}(s) \\ &\quad \times K_\alpha^T\overline{R_\alpha}K_\alpha\bar{z}_1^n(s)ds.\end{aligned} \tag{5.20}$$

From the Jensen's inequality (Lemma 6) for $W_\alpha(t)$ we have

$$
\begin{aligned}
&-nh_\alpha^n\int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s-t+h_\alpha)^{n-1}\bar{z}_1^{nT}(s)K_\alpha^T\overline{R_\alpha}K_\alpha\bar{z}_1^n(s)ds\\
&\le -(n!)^2e^{-\sigma h_\alpha}W_\alpha^T(t)K_\alpha^T\overline{R_\alpha}K_\alpha W_\alpha(t).
\end{aligned}\tag{5.21}
$$

The substitution of (5.21) into (5.20) leads to

$$
\begin{aligned}
\dot{V}_2(\Psi_t,\dot{\Psi}_t) &\le \sum_{\alpha=1}^{n-1} h_\alpha^{2n}\bar{z}_1^{nT}(t)K_\alpha^T\overline{R_\alpha}K_\alpha\bar{z}_1^n(t) - \sum_{\alpha=1}^{n-1}(n!)^2e^{-\sigma h_\alpha}W_\alpha^T(t)K_\alpha^T\overline{R_\alpha}K_\alpha W_\alpha(t)\\
&\quad -\sigma V_2(t)\\
&= \bar{z}_1^{nT}(t)\tilde{R}\bar{z}_1^n(t) - \sum_{\alpha=1}^{n-1}(n!)^2e^{-\sigma h_\alpha}W_\alpha^T(t)K_\alpha^T\overline{R_\alpha}K_\alpha W_\alpha(t) - \sigma V_2(t)\\
&= \dot{\Psi}^T(t)M^T\tilde{R}M\dot{\Psi}(t) - \sum_{\alpha=1}^{n-1}(n!)^2e^{-\sigma h_\alpha}W_\alpha^T(t)K_\alpha^T\overline{R_\alpha}K_\alpha W_\alpha(t) - \sigma V_2(t),
\end{aligned}\tag{5.22}
$$

where $\bar{z}_1^n(t) = M\dot{\Psi}(t)$.

By combining (5.19) with (5.22) and substitute it into $\dot{V}(\Psi_t,\dot{\Psi}_t)$ yields

$$
\begin{aligned}
\dot{V}(\Psi_t,\dot{\Psi}_t) &\le \Psi^T(t)\left[P\otimes(\sigma S+SD)\right]\Psi(t) - 2\Psi^T\sum_{\alpha=1}^{n-1}(P\mathcal{H}\otimes SB)K_\alpha W_\alpha(t)\\
&\quad + 2\Psi^T(t)(P\otimes SB)d(t) - \sum_{\alpha=1}^{n-1}(n!)^2e^{-\sigma h_\alpha}W_\alpha^T(t)K_\alpha^T\overline{R_\alpha}K_\alpha W_\alpha(t)\\
&\quad + \dot{\Psi}^T(t)M^T\tilde{R}M\dot{\Psi}(t) - \sigma V_1(t) - \sigma V_2(t).
\end{aligned}\tag{5.23}
$$

One can write inequality (5.23) as LMI (5.24)

$$
\begin{aligned}
&\dot{V}(\Psi_t,\dot{\Psi}_t) \le\\
&\eta(t)^T\begin{bmatrix}
P\otimes(\sigma S+SD) & -P\mathcal{H}\otimes SB & \cdots & -P\mathcal{H}\otimes SB & P\otimes SB\\
* & -(n!)^2e^{-\sigma h_1}\overline{R_1} & \cdots & 0 & 0\\
* & * & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
* & * & \cdots & -(n!)^2e^{-\sigma h_{n-1}}\overline{R_{n-1}} & 0\\
* & * & \cdots & * & -\gamma\Gamma
\end{bmatrix}\eta(t)\\
&\quad + \dot{\Psi}^T(t)M^T\tilde{R}M\dot{\Psi}(t) - \sigma V(t) + \gamma\Gamma d^2(t),
\end{aligned}\tag{5.24}
$$

where

$$\eta(t) = \mathrm{col}\left[\Psi(t) \quad K_1W_1(t) \quad \cdots \quad K_{n-1}W_{n-1}(t) \quad d(t)\right].$$

The substitution of (5.13) into (5.24) along with schur complement leads to LMI (5.14). The feasibilty of LMI (5.14) shows that

$$\dot{V}(\Psi_t, \dot{\Psi}_t) \leq -\sigma V(\Psi_t, \dot{\Psi}_t) + \gamma\Gamma d^2(t), \tag{5.25}$$

which implies that $V(\Psi_t, \dot{\Psi}_t)$ is an ISS Lyapunov-Krasovskii functional for (5.13). Therefore, system (5.13) satisfies the ISS property with respect to the external disturbance input $d(t)$. From the Lemma 4, if there exists $\mu_1, \mu_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|\Psi(t)|) \leq V(\Psi_t, \dot{\Psi}_t) \leq \alpha_2(\|\Psi_t\|_{\mathbb{W}})$ for all $\Psi_t \in \mathbb{W}^{1,\infty}_{[-h.0]}$ with $\mu_1(s) = \lambda_{\min}(P \otimes S)s^2$ and to determine μ_2 suppose

$$\begin{aligned}
V_2(\Psi_t, \dot{\Psi}_t) &\leq \sum_{\alpha=1}^{n-1} h_\alpha^n \int_{t-h_\alpha}^{t} (s-t+h_\alpha)^n \bar{z}_1^{nT}(s) K_\alpha^T \overline{R_\alpha} K_\alpha \bar{z}_1^n(s) ds \\
&\leq \sum_{\alpha=1}^{n-1} h_\alpha^{2n} \int_{t-h_\alpha}^{t} \bar{z}_1^{nT}(s) K_\alpha^T \overline{R_\alpha} K_\alpha \bar{z}_1^n(s) ds \\
&\leq \sum_{\alpha=1}^{n-1} h_\alpha^{2n+1} \sup_{s\in[t-h,t]} \left(\bar{z}_1^{nT}(s) K_\alpha^T \overline{R_\alpha} K_\alpha \bar{z}_1^n(s)\right) \\
&\leq h\|\dot{\Psi}_t^T M^T \tilde{R} M \dot{\Psi}_t\| \leq h\|\Psi_t^T M^T \tilde{R} M \Psi_t\|_{\mathbb{W}},
\end{aligned}$$

then, $\mu_2(S) = \lambda_{\max}(P \otimes S)s^2 + h\lambda_{\max}(M^T\tilde{R}M)s^2$. The proof is completed.

Remark 12 *In this paper, the agent dynamics consist of the general linear system with bounded disturbances. After the regularization of the system (5.3), as mentioned in section (5.2), the developed control law (5.5) can be immediately applied to the regularized system. The gain selection $\overline{K}$ performs the matrix D Hurwitz. It follows from (5.12) that applying gain $\overline{K}$ with an arbitrary small delay h, gain K can be efficiently calculated. The LMI (5.14) describes the robustness analysis of the leader-follower consensus problem.*

5.4 Leaderless Consensus Problem

In this section, we have presented the output feedback based leaderless consensus problem of MASs (5.3) with N agents $1, 2, \cdots, N$ using measured state and its delayed state.

5.4.1 Assumption and Lemmas

Assumption 3 *The directed communication graph $\mathcal{G}$ has a spanning tree.*

Lemma 10 *Consider the graph $\mathcal{G}$ has a directed spanning tree. Suppose $p = \begin{bmatrix} p_1, \cdots, p_n \end{bmatrix}^T > 0$ is a left eigenvector of the Laplacian matrix $\mathcal{L}$ with one eigenvalue as 0, i.e., $\mathcal{L}p = 0$. Define*

$$P = diag(p_i),$$
$$Q = P\mathcal{L} + \mathcal{L}^T P.$$

Then, $P > 0$ and $Q \geq 0$. For notational brevity, define $\lambda_0 = \lambda_2(Q)$ and $\lambda_2(Q)$ is the second minimum eigenvalue of matrix Q.

5.4.2 Distributed Controller Design

Consider the neighborhood output measurement error of agent i as follows

$$\varepsilon_i(t) = \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)). \tag{5.26}$$

Thus, the distributed delayed consensus control with $h_0 = 0$ is proposed for each agent i as

$$u_i(t) = \sum_{\alpha=0}^{n-1} \kappa_\alpha \varepsilon_i(t - h_\alpha), \qquad i = 1, \cdots, N, \tag{5.27}$$

where $\kappa_\alpha = \begin{bmatrix} \kappa_0, \cdots, \kappa_{n-1} \end{bmatrix} \in \mathbb{R}^{m \times nN}$ denotes the coupling strength, and $0 = h_0 < h_1 < \cdots < h_{n-1}$ is a known artificial delay.

Define $u(t) = \begin{bmatrix} u_1^T(t) & \cdots & u_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$, $y(t) = \begin{bmatrix} y_1^T(t) & \cdots & y_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$, and write the global form of (5.27) as follows

$$u(t) = -\sum_{\alpha=0}^{n-1} (\mathcal{L} \otimes I_m)\kappa_\alpha z_1(t - h_\alpha) \tag{5.28}$$

where $\mathcal{L}$ is defined earlier and $z_1(t - h_\alpha) \in \mathbb{R}^{nN}$.

We represent the global state as $z(t) = \begin{bmatrix} z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{nN}$. The aforementioned agent dynamics (5.3) with control input (5.27) leads to the following global state dynamics

$$\dot{z}(t) = (I_N \otimes A)z(t) - \sum_{\alpha=0}^{n-1} (\mathcal{L} \otimes B)\kappa_\alpha z_1(t - h_\alpha) + (I_N \otimes B)d(t), \tag{5.29}$$

where $d(t) = \begin{bmatrix} d_1^T(t) & \cdots & d_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{mN}$.

Next, employing Taylor's expansion with the integral term of the remainder mentioned in [6] yields

$$z_1(t - h_\alpha) = \sum_{\beta=0}^{n-1} \frac{1}{\beta!} (-h_\alpha)^\beta \, z_{\beta+1}(t) + W_\alpha(t) \qquad \alpha = 1, \cdots, n-1, \tag{5.30}$$

where

$$W_\alpha(t) = \frac{(-1)^n}{(n-1)!} \int_{t-h_\alpha}^{t} (s - t + h_\alpha)^{n-1} \, z_1^n(s) ds, \tag{5.31}$$

and where $z_1^n(s)$ is the n^{th} derivative of $z_1(s)$. It is important to note that $W_\alpha(t) = O(h_\alpha^n)$.

From (5.30), we have

$$\sum_{\alpha=0}^{n-1} \kappa_\alpha z_1(t - h_\alpha) = \sum_{\beta=0}^{n-1} \overline{\kappa_\beta} z_{\beta+1}(t) + \sum_{\alpha=1}^{n-1} \kappa_\alpha W_\alpha(t), \tag{5.32}$$

where

$$\begin{aligned} \overline{\kappa_0} &= \sum_{\alpha=0}^{n-1} \kappa_\alpha \quad \text{and,} \\ \overline{\kappa_\beta} &= \frac{(-1)^\beta}{\beta!} \sum_{\alpha=1}^{n-1} h_\alpha^\beta \kappa_\alpha, \qquad \beta = 1, \cdots, n-1. \end{aligned} \tag{5.33}$$

Based on (5.33) and coupling strength gain matrix $\kappa = \kappa_\alpha = \begin{bmatrix} \kappa_0 & \cdots & \kappa_{n-1} \end{bmatrix}$ one can obtain new coupling strength gain matrix $\overline{\kappa} = \begin{bmatrix} \overline{\kappa_0} & \overline{\kappa_\beta} \end{bmatrix}$

$$\overline{\kappa} = \kappa \Delta$$

$$\Delta = \begin{bmatrix} I_m & 0 & 0 & \cdots & 0 \\ I_m & -h_1 I_m & \frac{h_1^2}{2} I_m & \cdots & \frac{(-h_1)^{n-1}}{(n-1)!} I_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I_m & -h_{n-1} I_m & \frac{h_{n-1}^2}{2} I_m & \cdots & \frac{(-h_{n-1})^{n-1}}{(n-1)!} I_m \end{bmatrix}. \tag{5.34}$$

The Vandermonde-type matrix Δ is invertible due to the different values of delay.

The substitution of (5.32) into (5.29) leads to

$$\dot{z}(t) = (I_N \otimes A - \mathcal{L} \otimes B\overline{\kappa}) \, z(t) - \sum_{\alpha=1}^{n-1} (\mathcal{L} \otimes B) \kappa_\alpha W_\alpha(t) + (I_N \otimes B) d(t). \tag{5.35}$$

5.4.3 Stability Analysis

Theorem 4 *Consider the MASs (5.3) achieved consensus by the control input (5.27). Assume the network graph satisfies Assumption 3. Given* $\kappa_\alpha \in \mathbb{R}^{m\times n}(\alpha = 0, \cdots, n-1)$ *and the known constant delays* $0 = h_0 < h_1 < \cdots < h_{n-1}$ *such that the matrix* $D = 2A - \frac{\lambda_0}{p_m}B\overline{\kappa}$ *with* $\overline{\kappa} = \kappa\Delta$ *where* Δ *is defined in (5.34) and* $\kappa = \begin{bmatrix}\kappa_0 & \cdots & \kappa_{n-1}\end{bmatrix}$ *is Hurwitz for the consensus. For a given* $\sigma > 0$*, suppose there exist positive definite matrices* $P \in \mathbb{R}^{N\times N},\ S \in \mathbb{R}^{n\times n}$ *and* $R_\alpha \in \mathbb{R}^{m\times m}(\alpha = 1, \cdots, n-1)$ *that satisfy LMI (5.36).*

$$\begin{bmatrix} P\otimes(\sigma S + SD) & -P\mathcal{L}\otimes SB & \cdots & -P\mathcal{L}\otimes SB & P\otimes SB & \Pi^T M^T \tilde{R} \\ * & -(n!)^2 e^{-\sigma h_1}\overline{R_1} & \cdots & 0 & 0 & \Phi^T M^T \tilde{R} \\ * & * & \cdots & 0 & 0 & \Phi^T M^T \tilde{R} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & -(n!)^2 e^{-\sigma h_{n-1}}\overline{R_{n-1}} & 0 & \Phi^T M^T \tilde{R} \\ * & * & \cdots & * & -\gamma\Gamma & \Sigma^T M^T \tilde{R} \\ * & * & \cdots & * & * & -\tilde{R} \end{bmatrix} < 0 \tag{5.36}$$

where $\Pi = (I_N \otimes A - \mathcal{L}\otimes B\overline{\kappa})$, $M = I_N \otimes \begin{bmatrix} 0_{m\times(n-1)m} & I_m \end{bmatrix}$, $\tilde{R} = \sum_{\alpha=1}^{n-1} h_\alpha^{2n}\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha$, $\overline{R_\alpha} = I_N \otimes R_\alpha$, $\Phi = \mathcal{L}\otimes B$, $\Gamma = I_N \otimes S$, *and* $\Sigma = I_N \otimes B$*. Notation* $\lambda_0 = \lambda_2(Q)$ *(second minimum eigenvalue of Q matrix),* $p_M = \max\{p_1, \cdots, p_N\}$*, where* $Q, p_i,\ i = 1, \cdots, N$ *are defined in Lemma 10.*

Then, system (5.35) is achieved exact consensus under external disturbance input.

Proof: Consider the following Lyapunov-Krasovskii functional

$$V(z_t, \dot{z}_t) = V_1(z) + V_2(z_t, \dot{z}_t), \tag{5.37}$$

where

$$V_1(z) = z^T(t)(P\otimes S)z(t), \tag{5.38}$$

$$V_2(z_t, \dot{z}_t) = \sum_{\alpha=1}^{n-1} h_\alpha^n \int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s - t + h_\alpha)^n z_1^{nT}(s)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha z_1^n(s)ds, \tag{5.39}$$

where $z_1^n(t)$ is the n^{th} derivative of $z_1(t)$.

The derivative of $V_1(t)$ along (5.35) leads to

$$\dot{V}_1(t) = 2z^T(t)\left[P \otimes SA - P\mathcal{L} \otimes SB\overline{\kappa}\right] z(t) - 2z^T(t)\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ + 2z^T(t)(P \otimes SB)d(t), \tag{5.40}$$

where $W_\alpha(t)$ is defined in (5.31).

Employing Lemma 10 yields

$$\begin{aligned} \dot{V}_1(z) &= z^T(t)\left[P \otimes 2SA - Q \otimes SB\overline{\kappa}\right] z(t) - 2z^T\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &\quad + 2z^T(t)(P \otimes SB)d(t) \\ &\leq z^T(t)\left[P \otimes 2SA - \lambda_0 I_N \otimes SB\overline{\kappa}\right] z(t) - 2z^T\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &\quad + 2z^T(t)(P \otimes SB)d(t) \\ &\leq z^T(t)\left[P \otimes \left(2SA - \frac{\lambda_0}{p_M}SB\overline{\kappa}\right)\right] z(t) - 2z^T\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &\quad + 2z^T(t)(P \otimes SB)d(t) \\ &= z^T(t)\left(P \otimes SD\right) z(t) - 2z^T\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &\quad + 2z^T(t)(P \otimes SB)d(t) \\ &= z^T(t)\left[P \otimes (\sigma S + SD)\right] z(t) - 2z^T\sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &\quad + 2z^T(t)(P \otimes SB)d(t) - \sigma V_1(t). \end{aligned} \tag{5.41}$$

By, employing the fact that $Q \geq \lambda_0 I_N$ and $p_M I_N \geq P$, one can obtain the first and second inequalities, respectively.

Now, take the derivative of $V_2(z_t, \dot{z}_t)$ we get

$$\begin{aligned} \dot{V}_2(z_t, \dot{z}_t) = &\sum_{\alpha=1}^{n-1} h_\alpha^{2n} z_1^{nT}(t)\kappa_\alpha^T\overline{R_\alpha}\kappa_\alpha z_1^n(t) - \sigma V_2(t) \\ &- \sum_{\alpha=1}^{n-1} n h_\alpha^n \int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s - t + h_\alpha)^{n-1} z_1^{nT}(s) \\ &\times \kappa_\alpha^T\overline{R_\alpha}\kappa_\alpha z_1^n(s)ds. \end{aligned} \tag{5.42}$$

Based on the Jensen's inequality (Lemma 6) for $W_\alpha(t)$ we have

$$\begin{aligned} &- nh_\alpha^n \int_{t-h_\alpha}^{t} e^{\sigma(s-t)}(s-t+h_\alpha)^{n-1} z_1^{nT}(s)\kappa_\alpha^T \overline{R_\alpha} \kappa_\alpha z_1^n(s)ds \\ &\quad \leq -(n!)^2 e^{-\sigma h_\alpha} W_\alpha^T(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha W_\alpha(t). \end{aligned} \tag{5.43}$$

The substitution of (5.43) into (5.42) leads to

$$\begin{aligned} \dot{V}_2(z_t, \dot{z}_t) &\leq \sum_{\alpha=1}^{n-1} h_\alpha^{2n} z_1^{nT}(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha z_1^n(t) - \sigma V_2(t) - \sum_{\alpha=1}^{n-1}(n!)^2 e^{-\sigma h_\alpha} W_\alpha^T(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha W_\alpha(t) \\ &= z_1^{nT}(t)\tilde{R} z_1^n(t) - \sigma V_2(t) - \sum_{\alpha=1}^{n-1}(n!)^2 e^{-\sigma h_\alpha} W_\alpha^T(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha W_\alpha(t) \\ &= \dot{z}^T(t) M^T \tilde{R} M \dot{z}(t) - \sigma V_2(t) - \sum_{\alpha=1}^{n-1}(n!)^2 e^{-\sigma h_\alpha} W_\alpha^T(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha W_\alpha(t), \end{aligned} \tag{5.44}$$

where $z_1^n(t) = M\dot{z}(t)$.

By combining (5.41) with (5.44) and substitute it into $\dot{V}(z_t, \dot{z}_t)$ yields

$$\begin{aligned} \dot{V}(z_t, \dot{z}_t) \leq z^T(t)\left[P \otimes (\sigma S + SD)\right] z(t) &- 2z^T \sum_{\alpha=1}^{n-1}(P\mathcal{L} \otimes SB)\kappa_\alpha W_\alpha(t) \\ &+ \dot{z}^T(t) M^T \tilde{R} M \dot{z}(t) - \sum_{\alpha=1}^{n-1}(n!)^2 e^{-\sigma h_\alpha} W_\alpha^T(t)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha W_\alpha(t) \\ &+ 2z^T(t)(P \otimes SB)d(t) - \sigma V_1(t) - \sigma V_2(t). \end{aligned} \tag{5.45}$$

One can write inequality (5.45) as LMI (5.46)

$$\begin{aligned} \dot{V}(z_t, \dot{z}_t) \leq \eta(t)^T &\begin{bmatrix} P \otimes (\sigma S + SD) & -P\mathcal{L} \otimes SB & \cdots & -P\mathcal{L} \otimes SB & P \otimes SB \\ * & -(n!)^2 e^{-\sigma h_1}\overline{R_1} & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & -(n!)^2 e^{-\sigma h_{n-1}}\overline{R_{n-1}} & 0 \\ * & * & \cdots & * & -\gamma\Gamma \end{bmatrix} \eta(t) \\ &+ \dot{z}^T(t) M^T \tilde{R} M \dot{z}(t) - \sigma V(t) + \gamma\Gamma d^2(t), \end{aligned} \tag{5.46}$$

where $\eta(t) = \text{col}\begin{bmatrix} z(t) & \kappa_1 W_1(t) & \cdots & \kappa_{n-1} W_{n-1}(t) & d(t) \end{bmatrix}$.

Next, substitution of (5.35) into (5.46) along with schur complement leads to the LMI (5.36). The feasibilty of LMI (5.36) shows that

$$\dot{V}(z_t, \dot{z}_t) \leq -\sigma V(z_t, \dot{z}_t) + \gamma\Gamma d^2(t). \tag{5.47}$$

It follows from (5.47) that $V(z_t, \dot{z}_t)$ is an ISS Lyapunov-Krasovskii functional for (5.35). Therefore, system (5.35) satisfies the ISS property with respect to the external disturbance input $d(t)$. Based on the Lemma 4, if there exists $\mu_1, \mu_2 \in \kappa_\infty$ such that $\alpha_1(|z(t)|) \leq V(z_t, \dot{z}_t) \leq \alpha_2(\|z_t\|_{\mathbb{W}})$ for all $z_t \in \mathbb{W}^{1,\infty}_{[-h.0]}$ with $\mu_1(s) = \lambda_{\min}(P \otimes S)s^2$ and to determine μ_2 suppose

$$
\begin{aligned}
&V_2(z_t, \dot{z}_t) \leq \\
&\sum_{\alpha=1}^{n-1} h_\alpha^n \int_{t-h_\alpha}^{t} (s-t+h_\alpha)^n z_1^{nT}(s)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha z_1^n(s)ds \leq \sum_{\alpha=1}^{n-1} h_\alpha^{2n} \int_{t-h_\alpha}^{t} z_1^{nT}(s)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha z_1^n(s)ds \\
&\leq \sum_{\alpha=1}^{n-1} h_\alpha^{2n+1} \sup_{s\in[t-h,t]} \left(z_1^{nT}(s)\kappa_\alpha^T \overline{R_\alpha}\kappa_\alpha z_1^n(s)\right) \leq h\|\dot{z}_t^T M^T \tilde{R} M \dot{z}_t\| \leq h\|z_t^T M^T \tilde{R} M z_t\|_{\mathbb{W}},
\end{aligned}
$$

then, $\mu_2(S) = \lambda_{\max}(P \otimes S)s^2 + h\lambda_{\max}(M^T\tilde{R}M)s^2$. The proof is completed.

Remark 13 *In this chapter, the agent dynamics consists of the general linear system with bounded disturbances. After the regularization of the system (5.3), the developed control law (5.27) can be immediately applied to the regularized system. The gain selection* $\overline{\kappa}$ *ensures that the matrix* D *is Hurwitz. It follows from (5.34) that using the gain* $\overline{\kappa}$ *along with small delay* h*, gain* κ *can be efficiently calculated. The LMI (5.36) describes the robustness analysis of the leaderless consensus problem.*

5.5 Simulation Results

Example 1: Consider a network of second-order systems, i.e., the agent dynamics in (5.3) are given by

$$
z_i(t) = \begin{bmatrix} z_{i,1}(t) \\ z_{i,2}(t) \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0.1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$
$$
C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; d_i(t) = 4\sin(5t).
$$

An artificially delayed output feedback control law (5.5) is

$$
\begin{aligned}
u_i(t) &= K_0 e_i(t-h_0) + K_1 e_i(t-h_1), \quad i = 1, \cdots, 5. \qquad (5.48)\\
e_i(t) &= \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)) + b_i(z_{0,1}(t) - z_{i,1}(t)).
\end{aligned}
$$

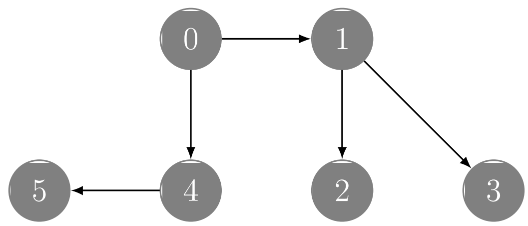

Figure 5.1: Communication topology among agents for leader-follower consensus

with design parameters $K = \begin{bmatrix} K_0 & K_1 \end{bmatrix} = \begin{bmatrix} 5.4659 & -3.8697 \end{bmatrix}$ and an artificial delay $h_0 = 0,\ h = h_1 = 0.5$. One can calculate the control gain K form $\overline{K}$ as mentioned in (5.12) by selecting the appropriate value of an artificial delay h. The gain $\overline{K} = \begin{bmatrix} 1.5962 & 1.9348 \end{bmatrix}$ can be chosen such that the matrix D becomes Hurwitz as stated in Theorem 3. The initial values of agents are $z_0(0) = 0.3 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_1(0) = 0.4 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_2(0) = 0.5 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_3(0) = 0.6 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_4(0) = -0.4 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_5(0) = -0.5 \begin{bmatrix} 1 & 1 \end{bmatrix}$. Assume that the communication graph is given by Fig 5.1. The corresponding modified Laplacian matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

with eigenvalues $1,\ 1,\ 1,\ 1,\ 1$.

In addition, to validate the effectiveness of the proposed method, a comparison is made with static output feedback control ((5.48) with $h_0 = h_1 = 0$) and observer-based control [168]. The static output feedback control for system (5.3) is defined as

$$u_i(t) = (K_0 + K_1)e_i(t), \quad i = 1, \cdots, 5. \tag{5.49}$$

$$e_i(t) = \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)) + b_i(z_{0,1}(t) - z_{i,1}(t)),$$

where K_0 and K_1 are same as (5.48). The observer-based control for system (5.3) is defined as

$$\dot{\zeta} = F\zeta_i + Gy_i + TBu_i$$

$$u_i = cKQ_1 \sum_{j=1}^{N} a_{ij}(y_i - y_j) + b_i(y_i - y_0)$$

$$+ cKQ_2 \sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j) + b_i(\zeta_i - \zeta_0), \ i = 1, \cdots, 5. \quad (5.50)$$

The observer and controller parameters for the (5.50) is $c = 2,\ F = -2,\ G = -1, T = \begin{bmatrix} -0.5128 & 0.2564 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 \\ 3.9 \end{bmatrix}$ and $K = \begin{bmatrix} -4.7719 & -3.6266 \end{bmatrix}$. The detailed calculation of parameters value of (5.50) can be found in [168].

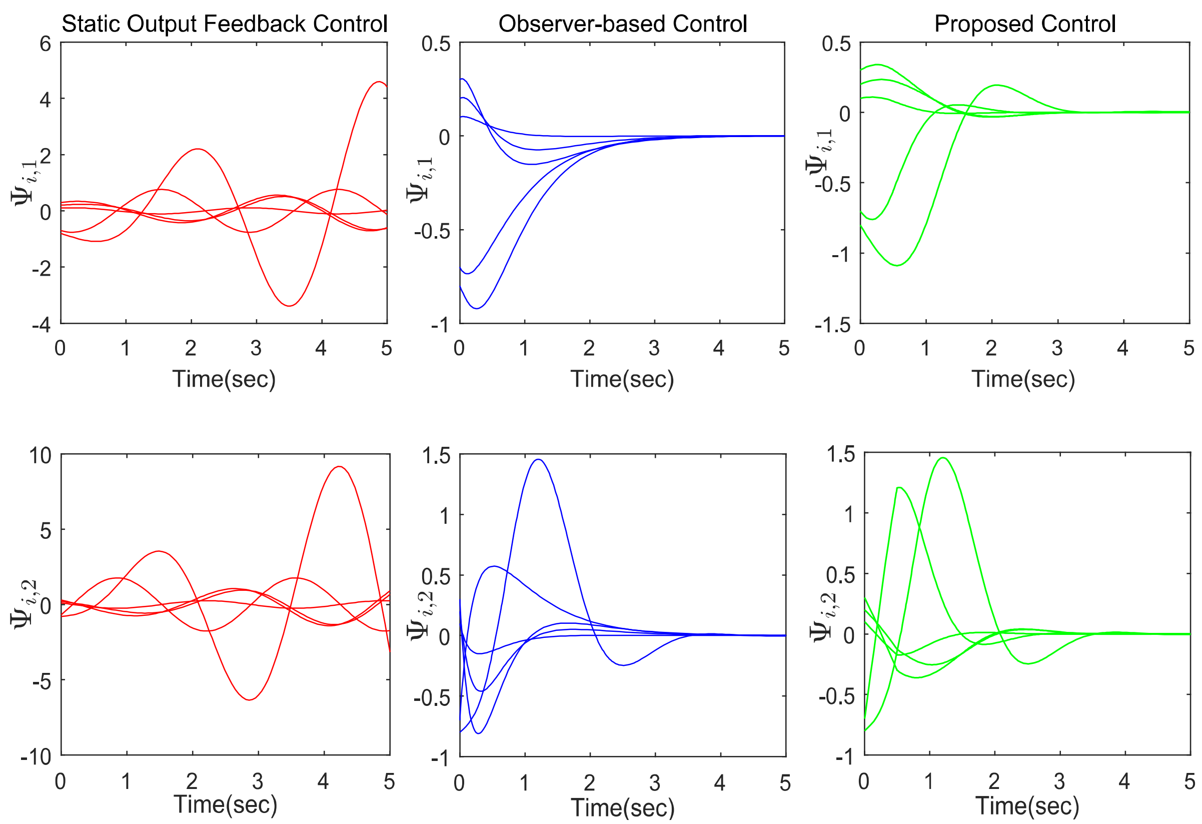

Figure 5.2: Tracking errors obtained using static output feedback control (left in red), observer-based control (middle in blue), and proposed control (right in green) in the case of the nominal system. Here $\Psi_{i,k}$ denotes the k^{th} component of $\Psi_i,\ i = 1, \cdots, 5$.

Fig. 5.2 and 5.3 show the tracking errors obtained for the nominal and perturbed system, respectively, using different control methods. The left portion (red) of Fig. 5.2 depicts the tracking errors obtained using a static output feedback control is diverging, which

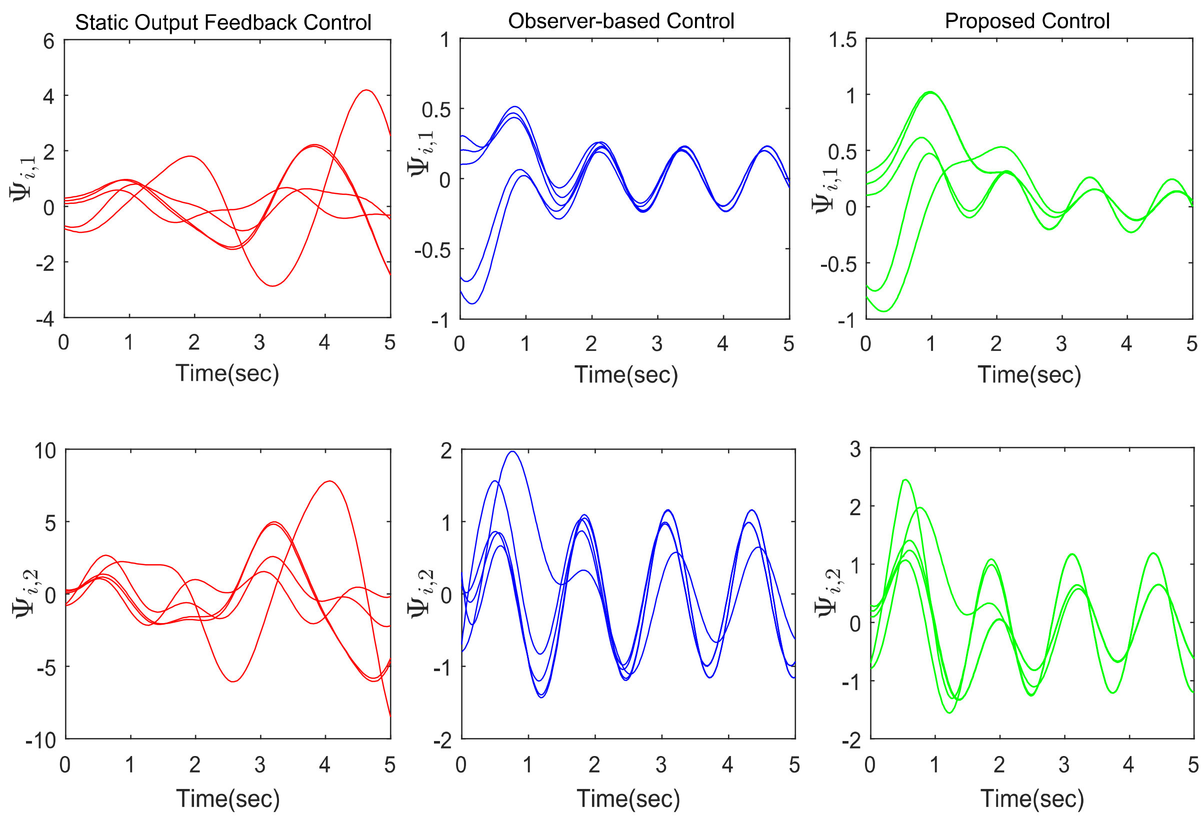

Figure 5.3: Tracking errors obtained using static output feedback control (left in red), observer-based control (middle in blue), and proposed control (right in green) in the case of a perturbed system. Here $\Psi_{i,k}$ denotes the k^{th} component of Ψ_i, $i = 1, \cdots, 5$.

shows static output feedback is not sufficient to stabilize the closed-loop system. The middle portion (blue) of Fig. 5.2 represents the tracking errors obtained using observer-based control (5.50) is converging to zero. The right portion (green) of Fig. 5.2 describes the tracking errors obtained with proposed control is converging to zero. In Fig. 5.3, all parameters are the same as those above only except for the unknown bounded disturbances. Next, the left portion (red) of Fig. 5.3 illustrates the tracking error obtained using the static output feedback control is diverging. The middle (blue) and the right (green) portion of Fig. 5.3 depicts the ultimately uniform boundedness of the tracking errors, respectively.

Example 2. Consider a network of second-order integrators, i.e., the agent dynamics in (5.3) are given by

$$z_i(t) = \begin{bmatrix} z_{i,1}(t) \\ z_{i,2}(t) \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ d_i(t) = 4\sin(5t).$$

An artificially delayed output feedback control (5.27) is

$$u_i(t) = \kappa_0 \varepsilon_i(t - h_0) + \kappa_1 \varepsilon_i(t - h_1), \quad i = 1, \cdots, 6. \tag{5.51}$$
$$\varepsilon_i(t) = \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)).$$

with design parameters $\kappa = \begin{bmatrix} \kappa_0 & \kappa_1 \end{bmatrix} = \begin{bmatrix} 22.5490 & -21.3904 \end{bmatrix}$ and an artificial delay $h_0 = 0,\ h_1 = h = 0.5$. One can calculate the control gain κ form $\overline{\kappa}$ as mentioned in (5.34) by selecting the appropriate value of an artificial delay h. The gain $\overline{\kappa} = \begin{bmatrix} 1.1586 & 2.1390 \end{bmatrix}$ can be chosen such that the matrix D becomes Hurwitz as stated in Theorem 4. The initial values of agents are $z_1(0) = 0.4 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_2(0) = 0.5 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_3(0) = 0.6 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_4(0) = -0.4 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_5(0) = -0.5 \begin{bmatrix} 1 & 1 \end{bmatrix}$, $z_6(0) = -0.6 \begin{bmatrix} 1 & 1 \end{bmatrix}$. Assume that the communication graph is given by Fig 5.4. The corresponding Laplacian matrix $\mathcal{L}$ is

$$\mathcal{L} = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

whose nonzero eigenvalues are $1,\ 1.3376 \pm 0.5623i,\ 2,\ 3.3247$.

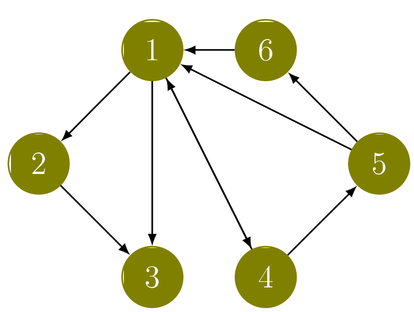

Figure 5.4: Communication topology among agents for leaderless consensus

In addition, to validate the effectiveness of the proposed method, a comparison is made with static output feedback control ((5.51) with $h_0 = h_1 = 0$) and observer-based control [168]. The static output feedback control for system (5.3) is defined as

$$u_i(t) = (\kappa_0 + \kappa_1)\varepsilon_i(t), \quad i = 1, \cdots, 6. \tag{5.52}$$
$$\varepsilon_i(t) = \sum_{j=1}^{N} a_{ij}(z_{j,1}(t) - z_{i,1}(t)),$$

where κ_0 and κ_1 are the same as (5.51). The observer-based control for system (5.3) is defined as

$$\begin{aligned} \dot{\zeta} &= F\zeta_i + Gy_i + TBu_i \\ u_i &= c\kappa Q_1 \sum_{j=1}^{N} a_{ij}(y_i - y_j) + c\kappa Q_2 \sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j), \\ &\qquad\qquad i = 1, \cdots, 6. \end{aligned} \tag{5.53}$$

The observer and controller parameters for the (5.53) is $c = 1,\ F = -2,\ G = -1, T = \begin{bmatrix} -0.5 & 0.25 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ and $\kappa = \begin{bmatrix} -5.0141 & -3.7372 \end{bmatrix}$. The detailed calculation of parameters value of (5.53) can be found in [168].

Fig. 5.5 and 5.6 show the trajectories of agents obtained for the nominal and perturbed system, respectively, using different control methods. The left portion (red) of Fig. 5.5 depicts the trajectories of agents obtained using static output feedback control is diverging, which shows static output feedback is not sufficient to stabilize the closed-loop system. The middle portion (blue) of Fig. 5.5 represents the consensus of agents trajectories obtained using observer-based control (5.53) is converging to zero. The right portion (green) of Fig. 5.5 describes the consensus of agents trajectories obtained with proposed control is converging to zero. In Fig. 5.6, all parameters are the same as those above only except for the unknown bounded disturbances. Next, the left portion (red) of Fig. 5.6 illustrates the tracking error obtained using the static output feedback control is diverging. The middle (blue) and the right (green) portion of Fig. 5.6 illustrates the ultimately uniform boundedness of the agent's trajectories.

It is noteworthy that the simulation results obtained through the static output feedback are not as desired, which implies that static output feedback control cannot provide the desired results. The simulation results obtained using observer-based control (5.50)

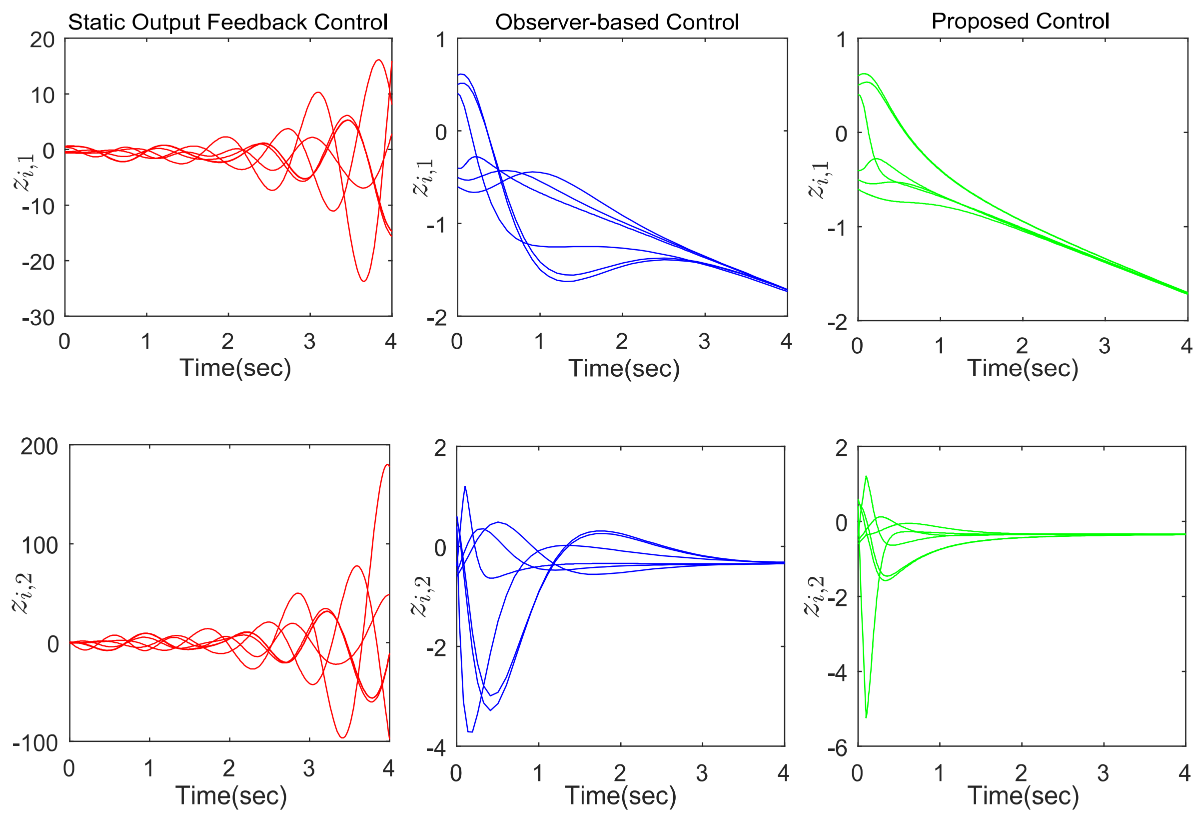

Figure 5.5: Trajectories of the agents were obtained using static output feedback control (left in red), observer-based control (middle in blue), and proposed control (right in green) in the case of a nominal system. Here $z_{i,k}$ denotes the k^{th} component of z_i, $i = 1, \cdots, 6$.

and (5.53) achieved the desired results but at the cost of adding observer dynamics into the system dynamics. One has to calculate the observer and controller design parameters both, which lead to computation complexities. In the proposed method, one has to calculate only controller gain for some small delay h, which is simple and straightforward. Therefore, it can be concluded that the proposed control is better and effective.

5.6 Summary

An artificially delayed output feedback-based leaderless and leader-follower consensus control for general uncertain linear MASs is examined in this paper. This controller requires knowledge of the the agent's relative output and their delayed information with respect to neighbors. Through the Lyapunov-Krasovskii functional LMIs are formulated, and these LMIs are always feasible for arbitrary small delays selected by the user. Moreover, robustness concerning the matched disturbances is investigated with ISS and ensured

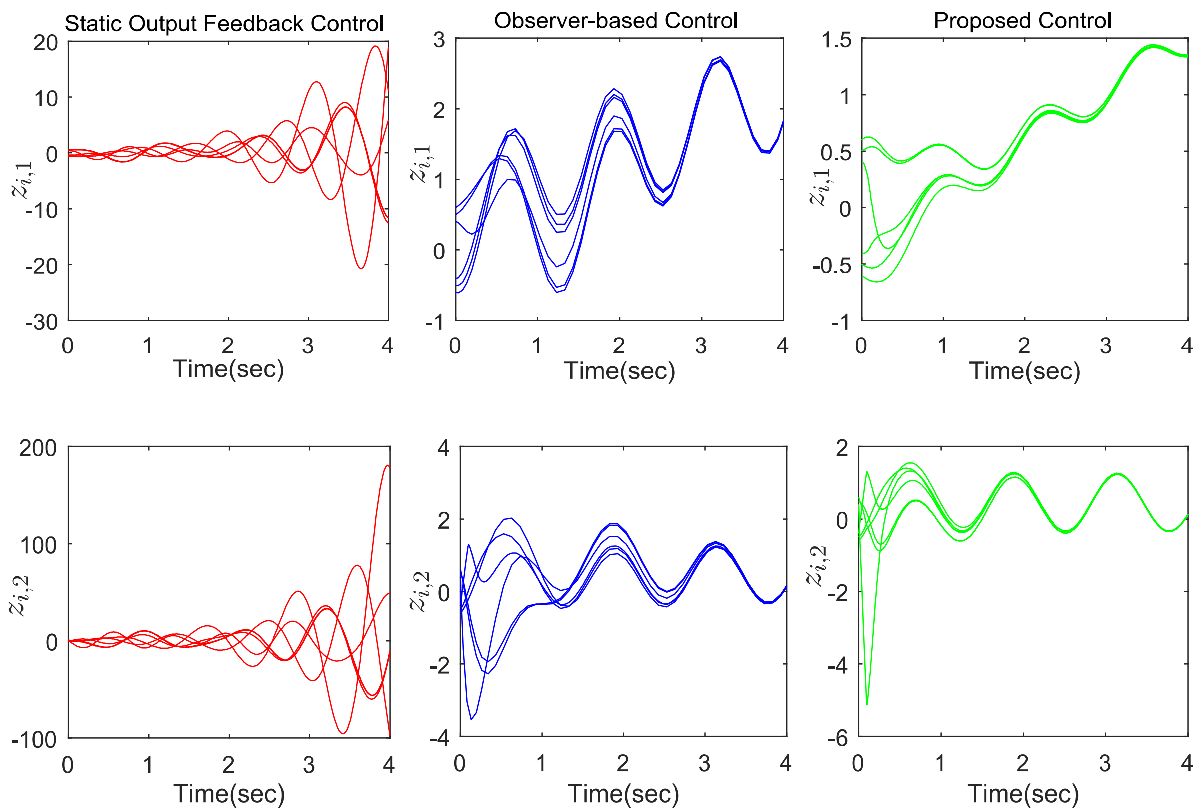

Figure 5.6: Trajectories of the agents with disturbances obtained using static output feedback control (left in red), observer-based control (middle in blue), and proposed control (right in green) in the case of a perturbed system. Here $z_{i,k}$ denotes the k^{th} component of z_i, $i = 1, \cdots, 6$.

asymptotic convergence of the error systems in the disturbances-free case and ultimately uniform boundedness of the error systems in the case of the disturbances. The simulation results demonstrated the efficacy of the proposed method.