Chapter 5

Symmetrically Global Pseudo-Differential Operators involving the Weinstein transform

5.1 Introduction

Symmetrically global pseudo-differential operators are the generalization of pseudo-differential operators in which the symbol satisfies similar decay estimates due to differentiation with respect to x - variable and ξ - variable. These operators are also examples of operators on non-compact manifolds. From [11, 48, 81], with the help of aforesaid operators, the compact embedding theorem for the Sobolev space, the Cauchy problem for SG - hyperbolic equations with constant multiplicities can be discussed. Exploiting the Fourier transform technique, many important observations of SG pseudo-differential operators were made by many authors and they got interesting results. In this connection, Egorov and Schulze [15, p. 238], Coriasco and Rodino [11], Nicola [47], Nicola and Rodino [48], Cappiello and Rodino [5], examined various properties of SG pseudo-differential operators by considering the same transform technique. Wong [81, p. 167], Dasgupta and Wong [14], discussed the characterization of SG pseudo-differential operators on $L^p(\mathbb{R}^n)$ - space.

For the Hankel transform concern, Pathak and Pandey [54] investigated minimal-maximal pseudo-differential operators associated with the Bessel operator and obtained many interesting results.

Motivated by the aforesaid works, our main objective of this chapter is to study the boundedness and compactness results for the symmetrically global pseudo-differential operators on Sobolev space $H_\alpha^{r,s,p}$ of order (r, s). With the help of the aforesaid results various properties of minimal-maximal operators for symmetrically global pseudo-differential operators involving the Weinstein transform are discussed.

The organizations of the present chapter are the following:

Section 5.1, is introductory which describes motivations regarding symmetrically global pseudo-differential operators. In Section 5.2, the Sobolev space of order (r, s) on $L_\alpha^p(\mathbb{R}_+^{n+1})$ - space is defined, boundedness and compactness properties associated with the Sobolev space $H_\alpha^{r,s,p}$ are obtained by exploiting the theory of the Weinstein transform. In the last Section, definitions and properties of minimal-maximal pseudo-differential operators are discussed associated with the Weinstein transform.

5.2 Symmetrically Global Pseudo-Differential Operators

In this section, symmetrically global pseudo-differential operator is defined and its various properties are discussed.

Definition 5.2.1. The symbol class S^{m_1,m_2} is the set of all functions $\sigma : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \to \mathbb{C}$, $m_1, m_2 \in \mathbb{R}$, such that $\forall \beta, \gamma \in \mathbb{N}_0^{n+1}$, there exists a constant $C_{\beta,\gamma} > 0$ depending only on β and γ, satisfying

$$\left| D_\xi^\beta D_x^\gamma \sigma(x,\xi) \right| \leq C_{\beta,\gamma}(1 + \|\xi\|^2)^{\frac{m_1 - |\beta|}{2}}(1 + \|x\|^2)^{\frac{m_2 - |\gamma|}{2}}. \tag{5.2.1}$$

A function $\sigma \in S^{m_1,m_2}$ is called symmetrically global symbol of order (m_1, m_2).

Lemma 5.2.2. *Let $m_1 \in \mathbb{R}$ and $m_2 \leq 0$. Then every symbol σ in S^{m_1,m_2} is in S^{m_1}.*

Proof. If $\sigma \in S^{m_1,m_2}$ and $q = \frac{m_2 - |\gamma|}{2}$, $m_2 \leq 0$, then (5.2.1) reduces to

$$\left| D_\xi^\beta D_x^\gamma \sigma(x,\xi) \right| \leq C_{\beta,\gamma}(1 + \|\xi\|^2)^{\frac{m_1 - |\beta|}{2}}. \tag{5.2.2}$$

This shows that $\sigma \in S^{m_1}$ for $m_1 \in \mathbb{R}$. □

Definition 5.2.3. Let $\sigma : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \to \mathbb{C}$ be a symbol. Then the pseudo-differential operator T_σ associated with symbol $\sigma \in S^{m_1,m_2}$ is defined by

$$(T_\sigma u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi), \; u \in S_*(\mathbb{R}_+^{n+1}). \tag{5.2.3}$$

Theorem 5.2.4. *Let σ be a symbol. Then the pseudo-differential operator T_σ maps the Schwartz space $S_*(\mathbb{R}_+^{n+1})$ into itself.*

Proof. The proof of the theorem is given in Chapter 2, Theorem 2.2.6. □

Theorem 5.2.5. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in S^0$. Then for $1 < p < \infty$, the pseudo-differential operator $T_\sigma : L_\alpha^p(\mathbb{R}_+^{n+1}) \to L_\alpha^p(\mathbb{R}_+^{n+1})$ is bounded.*

Proof. The proof can be seen in Chapter 4, Theorem 4.2.5. □

Theorem 5.2.6. *Let $\sigma(x,\xi)$ be a symbol in $S^{0,0}$. Then for $1<p<\infty$, the pseudo-differential operator $T_\sigma : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is bounded.*

Proof. From Theorem 5.2.5, the pseudo-differential operator $T_\sigma : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is bounded for $1<p<\infty$. If we take $\sigma \in S^{0,0}$, then from Lemma 5.2.2, we get $\sigma \in S^0$ for $S^{0,0} \subseteq S^0$. Therefore, from Theorem 5.2.5, the required result is proved. □

5.3 Sobolev Spaces

In this section, the Sobolev space of order (r,s) on $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - space is defined and boundedness and compactness properties discussed.

Definition 5.3.1. Let $r,s \in \mathbb{R}$. Then the Bessel potential of order (r,s) for $u \in S'_*(\mathbb{R}^{n+1}_+)$ is defined by

$$\begin{aligned}(J_{r,s,\alpha}u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{-r/2}(1+\|x\|^2)^{-s/2} \\ &\quad \times (\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi) \\ &= \big(T_{\sigma_{r,s}}u\big)(x),\end{aligned} \tag{5.3.1}$$

where

$$\sigma_{r,s}(x,\xi) = (1+\|\xi\|^2)^{-r/2}(1+\|x\|^2)^{-s/2} \in S^{-r,-s}.$$

Theorem 5.3.2. *Let $J_{-r,0,\alpha}$ and $J_{0,-s,\alpha}$ be the Bessel potential operators of order $(-r,0)$ and $(0,-s)$ respectiveley. Then the product $J_{0,-s,\alpha}J_{-r,0,\alpha}$ is a symmetrically global pseudo-differential operator of order $(-r,-s)$.*

Proof. Let $u \in S'_*(\mathbb{R}_+^{n+1})$. Then from (5.3.1), we have

$$(J_{0,-s,\alpha}J_{-r,0,\alpha}u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|x\|^2)^{s/2}\big(\mathcal{F}_\alpha(J_{-r,0,\alpha}u)\big)(\xi) \times d\mu_\alpha(\xi). \tag{5.3.2}$$

Using (5.3.1), we have

$$\begin{aligned}(J_{-r,0,\alpha}u)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{r/2}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= {\mathcal{F}_\alpha}^{-1}\big[(1+\|\xi\|^2)^{r/2}(\mathcal{F}_\alpha u)\big](x).\end{aligned}$$

By the property of Weinstein transform, it gives

$$\big(\mathcal{F}_\alpha(J_{-r,0,\alpha}u)\big)(\xi) = (1+\|\xi\|^2)^{r/2}(\mathcal{F}_\alpha u)(\xi). \tag{5.3.3}$$

From (5.3.2) and (5.3.3), we obtain

$$\begin{aligned}(J_{0,-s,\alpha}J_{-r,0,\alpha}u)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|x\|^2)^{s/2}(1+\|\xi\|^2)^{r/2}\\ &\quad\times(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= (J_{-r,-s,\alpha}u)(x), \qquad \forall u \in S'_*(\mathbb{R}_+^{n+1}).\end{aligned}$$

Hence,

$$J_{0,-s,\alpha}J_{-r,0,\alpha} = J_{-r,-s,\alpha}.$$

□

From Theorem 5.3.2, we have

$$J^{-1}_{r,s,\alpha} = J_{-r,-s,\alpha} = J_{0,-s,\alpha}J_{-r,0,\alpha}. \tag{5.3.4}$$

Lemma 5.3.3. *Let $u \in S'_*(\mathbb{R}^{n+1}_+)$. Then we have*

(i) $J_{0,0,\alpha}u = u$.

(ii) $J_{r,0,\alpha}J_{r',0,\alpha}u = J_{r+r',0,\alpha}u$.

Proof. (i) Taking $r = 0, s = 0$ in (5.3.1), we get

$$\begin{aligned}(J_{0,0,\alpha}u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= \mathcal{F}_\alpha{}^{-1}\big(\mathcal{F}_\alpha u\big)(x), \quad \forall u \in S'_*(\mathbb{R}^{n+1}_+)\\ &= u(x).\end{aligned}$$

(ii) From the Definition 5.3.1, we have

$$(J_{r,0,\alpha}J_{r',0,\alpha}u)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{-r/2}\big(\mathcal{F}_\alpha(J_{r',0,\alpha}u)\big)(\xi)d\mu_\alpha(\xi).$$

In view of (5.3.3), we can find

$$\begin{aligned}(J_{r,0,\alpha}J_{r',0,\alpha}u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{-r/2}(1+\|\xi\|^2)^{-r'/2}(\mathcal{F}_\alpha u)(\xi)\\ &\quad\times d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{-(r+r')/2}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= (J_{r+r',0,\alpha}u)(x).\end{aligned}$$

□

Definition 5.3.4. Let $r, s \in \mathbb{R}$ and $1 < p < \infty$. Then the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev space of order r, s is defined by

$$H^{r,s,p}_\alpha = \big\{u \in S'_*(\mathbb{R}^{n+1}_+) : J_{-r,-s,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)\big\}. \tag{5.3.5}$$

The space $H^{r,s,p}_\alpha$ forms a Banach space with the follwing norm:

$$\begin{aligned} \|u\|_{H^{r,s,p}_\alpha} &= \|J_{-r,-s,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \\ &= \Big(\int_{\mathbb{R}^{n+1}_+} \big|(J_{-r,-s,\alpha}u)(x)\big|^p d\mu_\alpha(x)\Big)^{1/p}. \end{aligned} \tag{5.3.6}$$

Remark. For $r = 0, s = 0$ the norm (5.3.6) will be

$$\|u\|_{H^{0,0,p}_\alpha} = \Big(\int_{\mathbb{R}^{n+1}_+} \big|(J_{0,0,\alpha}u)(x)\big|^p d\mu_\alpha(x)\Big)^{1/p}.$$

Hence, from Lemma 5.3.3, we get

$$H^{0,0,p}_\alpha = L^p_\alpha(\mathbb{R}^{n+1}_+).$$

Theorem 5.3.5. *Let $r, s \in \mathbb{R}$ and $1 < p < \infty$, then the Bessel potential operator $J_{-r,-s,\alpha} : H^{r,s,p}_\alpha \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is a surjective and isometry.*

Proof. Let $u \in H^{r,s,p}_\alpha$. Then from (5.3.6), we have

$$\|u\|_{H^{r,s,p}_\alpha} = \|J_{-r,-s,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

From above it follows that $J_{-r,-s,\alpha} : H_\alpha^{r,s,p} \to L_\alpha^p(\mathbb{R}_+^{n+1})$ is an isometry.

For each $v \in L_\alpha^p(\mathbb{R}_+^{n+1})$, choose $u = J_{-r,-s,\alpha}^{-1} v$, then

$$J_{-r,-s,\alpha} u = (J_{-r,-s,\alpha} J_{-r,-s,\alpha}^{-1}) v = v.$$

It finds that

$$\|u\|_{H_\alpha^{r,s,p}} = \|J_{-r,-s,\alpha} u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} = \|v\|_{L_\alpha^p(\mathbb{R}_+^{n+1})},$$

Therefore, $u \in H_\alpha^{r,s,p}$. Hence, $J_{-r,-s,\alpha} : H_\alpha^{r,s,p} \to L_\alpha^p(\mathbb{R}_+^{n+1})$ is an onto isometry. □

Theorem 5.3.6. *Let $\sigma(x,\xi)$ be a symbol in S^{m_1,m_2}, $m_1, m_2 \in \mathbb{R}$. Then for $r, s \in \mathbb{R}$ and $1 < p < \infty$, the pseudo-differential operator $T_\sigma : H_\alpha^{r,s,p} \to H_\alpha^{r-m_1,s-m_2,p}$ is a bounded linear operator.*

Proof. Consider the following pseudo-differential operators are given

$$J_{-r,-s,\alpha} : H_\alpha^{r,s,p} \to H_\alpha^{0,0,p},$$

$$T_\sigma J_{m_1,m_2,\alpha} : H_\alpha^{0,0,p} \to H_\alpha^{0,0,p},$$

and

$$J_{r-m_1,s-m_2,\alpha} : H_\alpha^{0,0,p} \to H_\alpha^{r-m_1,s-m_2,p}.$$

From Theorem 5.3.5, the operators $J_{-r,-s,\alpha} : H_\alpha^{r,s,p} \to H_\alpha^{0,0,p}$ and $J_{r-m_1,s-m_2,\alpha} : H_\alpha^{0,0,p} \to H_\alpha^{r-m_1,s-m_2,p}$ are bounded linear operators. By Theorem 5.2.6, $T_\sigma J_{m_1,m_2,\alpha} : H_\alpha^{0,0,p} \to H_\alpha^{0,0,p}$ is the bounded linear operator. Hence, $T_\sigma : H_\alpha^{r,s,p} \to H_\alpha^{r-m_1,s-m_2,p}$ is a bounded linear operator. □

Proposition 5.3.7. *Let $s > 0$ and $x \in \mathbb{R}_+^{n+1}$. We define*

$$K_s(x) = \frac{1}{2^{s/2}\Gamma(\frac{s}{2})} \int_0^\infty e^{-t/2} e^{-\|x\|^2/(2t)} t^{-(2\alpha+n-s+2)/2} \frac{dt}{t}, \tag{5.3.7}$$

then

(i) $K_s \in L^p_\alpha(\mathbb{R}^{n+1}_+), \quad \forall s \in (0, \infty),$

(ii) $(\mathcal{F}_\alpha K_s)(\xi) = (1 + \|\xi\|^2)^{-s/2} \quad \forall \xi \in \mathbb{R}^{n+1}_+.$

Proof. Proof of (i) can be done by using similar technique [81, p. 89]. To prove (ii) we take $\phi \in S_*(\mathbb{R}^{n+1}_+)$, then by (1.4.5), we have

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha K_s)(\xi)\phi(\xi)d\mu_\alpha(\xi) = \int_{\mathbb{R}^{n+1}_+} K_s(\xi)(\mathcal{F}_\alpha \phi)(\xi)d\mu_\alpha(\xi).$$

Applying (5.3.7), then right hand side of above expression becomes

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha K_s)(\xi)\phi(\xi)d\mu_\alpha(\xi) = \frac{1}{2^{s/2}\Gamma(\frac{s}{2})} \int_{\mathbb{R}^{n+1}_+} \Big(\int_0^\infty e^{-t/2} e^{-\|\xi\|^2/(2t)} t^{-(2\alpha+n-s+2)/2} \frac{dt}{t} \Big) \times (\mathcal{F}_\alpha \phi)(\xi) d\mu_\alpha(\xi).$$

In view of Fubini's theorem, we get

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha K_s)(\xi)\phi(\xi)d\mu_\alpha(\xi) = \frac{1}{2^{s/2}\Gamma(\frac{s}{2})} \int_0^\infty e^{-t/2} t^{-(2\alpha+n-s+2)/2} \times \Big(\int_{\mathbb{R}^{n+1}_+} e^{-\|\xi\|^2/(2t)} (\mathcal{F}_\alpha \phi)(\xi) d\mu_\alpha(\xi) \Big) \frac{dt}{t}. \tag{5.3.8}$$

Now, from [42, p. 595] we have

$$\mathcal{F}_\alpha \big(e^{-t\|x\|^2}\big)(\xi) = \frac{1}{(2t)^{\alpha+\frac{n}{2}+1}} e^{-\frac{\|\xi\|^2}{4t}}, \quad \forall t > 0. \tag{5.3.9}$$

Using (1.4.5) and (5.3.9), (5.3.8) becomes

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha K_s)(\xi)\phi(\xi)d\mu_\alpha(\xi) = \frac{1}{2^{s/2}\Gamma(\frac{s}{2})}\int_0^\infty e^{-t/2}t^{-(2\alpha+n-s+2)/2}t^{\alpha+\frac{n}{2}+1} \times \Big(\int_{\mathbb{R}^{n+1}_+}\phi(\xi)e^{-\frac{t\|\xi\|^2}{2}}d\mu_\alpha(\xi)\Big)\frac{dt}{t}$$
$$= \frac{1}{2^{s/2}\Gamma(\frac{s}{2})}\int_{\mathbb{R}^{n+1}_+}\phi(\xi)\Big(\int_0^\infty e^{-t(1+\|\xi\|^2)/2}t^{\frac{s}{2}-1}dt\Big) \times d\mu_\alpha(\xi).$$

By the property of Gamma function, we obtain

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha K_s)(\xi)\phi(\xi)d\mu_\alpha(\xi) = \int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{-s/2}\phi(\xi)d\mu_\alpha(\xi).$$

Finally, we get

$$(\mathcal{F}_\alpha K_s)(\xi) = (1+\|\xi\|^2)^{-s/2}.$$

□

Proposition 5.3.8. *Let $s \geq 0$ and $1 \leq p < \infty$. Then*

$$\|J_{s,0,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad \forall\ u \in L^p_\alpha(\mathbb{R}^{n+1}_+). \tag{5.3.10}$$

Proof. Let $\phi \in S_*(\mathbb{R}^{n+1}_+)$. Then from (5.3.1), we have

$$(J_{s,0,\alpha}\phi)(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{-s/2}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi).$$

Taking (1.4.2), we have

$$(J_{s,0,\alpha}\phi)(x) = \mathcal{F}_\alpha{}^{-1}\big[(1+\|\xi\|^2)^{-s/2}(\mathcal{F}_\alpha\phi)\big](x).$$

Therefore, by (1.4.1) we find

$$\mathcal{F}_\alpha(J_{s,0,\alpha}\phi)(\xi) = (1+\|\xi\|^2)^{-s/2}(\mathcal{F}_\alpha\phi)(\xi). \tag{5.3.11}$$

From (1.4.18), we have

$$\mathcal{F}_\alpha(K_s *_w \phi)(\xi) = (\mathcal{F}_\alpha K_s)(\xi)(\mathcal{F}_\alpha\phi)(\xi).$$

By Proposition 5.3.7, above obtained

$$\mathcal{F}_\alpha(K_s *_w \phi)(\xi) = (1+\|\xi\|^2)^{-s/2}(\mathcal{F}_\alpha\phi)(\xi). \tag{5.3.12}$$

Comparing (5.3.11) and (5.3.12), we get

$$\mathcal{F}_\alpha(J_{s,0,\alpha}\phi)(\xi) = \mathcal{F}_\alpha(K_s *_w \phi)(\xi).$$

Using (1.4.6), last expression gives

$$(J_{s,0,\alpha}\phi)(x) = (K_s *_w \phi)(x).$$

Therefore,

$$\|J_{s,0,\alpha}\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} = \|K_s *_w \phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)},$$

by using (1.4.17),

$$\|J_{s,0,\alpha}\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|K_s\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}\|\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

From Proposition 5.3.7, $K_s \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, then we can find a positive constant C such that

$$\|J_{s,0,\alpha}\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le C\|\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{5.3.13}$$

Since $S_*(\mathbb{R}^{n+1}_+)$ is dense in $L^p_\alpha(\mathbb{R}^{n+1}_+)$, Therefore, from (5.3.13) it follows that the linear operator $J_{s,0,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is bounded and satisfying (5.3.10). □

Lemma 5.3.9. *Let $r, s \ge 0$ and $1 < p < \infty$. Then $H^{r,s,p}_\alpha \subseteq L^p_\alpha(\mathbb{R}^{n+1}_+)$ and*

$$\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le \|u\|_{H^{r,s,p}_\alpha}, \quad \forall\ u \in H^{r,s,p}_\alpha. \tag{5.3.14}$$

Proof. Let $u \in H^{r,s,p}_\alpha$. Then from Definition 5.3.4 and (5.3.5), we get $J_{-r,-s,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)$. From (5.3.1), we have

$$\begin{aligned}(J_{-r,-s,\alpha}u)(x) &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{r/2}(1+\|x\|^2)^{s/2}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= (1+\|x\|^2)^{s/2}\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})(1+\|\xi\|^2)^{r/2}(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= (1+\|x\|^2)^{s/2}(J_{-r,0,\alpha}u)(x),\end{aligned}$$

where $J_{-r,0,\alpha}u$ is the pseudo-differential operator associated with the symbol $\sigma(\xi) = (1+\|\xi\|^2)^{r/2}, r \ge 0$.

Therefore,

$$J_{-r,0,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+), \qquad \forall\ u \in H^{r,s,p}_\alpha.$$

By Proposition 5.3.8, we have

$$\|J_{r,0,\alpha}v\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \le \|v\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}, \quad \forall\ v \in L^p_\alpha(\mathbb{R}^{n+1}_+).$$

If we take $v = J_{-r,0,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, then from the last inequality, we get

$$\|J_{r,0,\alpha}(J_{-r,0,\alpha}u)\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|J_{-r,0,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Utilizing Lemma 5.3.3, above yields

$$\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|J_{-r,0,\alpha}u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}.$$

Taking (5.3.6), we get

$$\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|u\|_{H^{r,0,p}_\alpha}. \tag{5.3.15}$$

Since

$$\|u\|_{H^{r,0,p}_\alpha} \leq \|u\|_{H^{r,s,p}_\alpha}, \ \forall s \geq 0.$$

Therefore, (5.3.15) becomes

$$\|u\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} \leq \|u\|_{H^{r,s,p}_\alpha}, \qquad \forall r, s \geq 0.$$

Hence,

$$H^{r,s,p}_\alpha \subseteq L^p_\alpha(\mathbb{R}^{n+1}_+), \quad 1 < p < \infty.$$

□

Theorem 5.3.10. $\big($***Sobolev embedding theorem***$\big)$ *Let $r_1, r_2, s_1, s_2 \in \mathbb{R}$ be such that $r_1 \leq r_2$ and $s_1 \leq s_2$. Then for $1 < p < \infty$, $H^{r_2,s_2,p}_\alpha \subseteq H^{r_1,s_1,p}_\alpha$ and the inclusion map*

$$i : H^{r_2,s_2,p}_\alpha \hookrightarrow H^{r_1,s_1,p}_\alpha,$$

is bounded.

Proof. Assume that $r_1 \leq r_2$ and $s_1 \leq s_2$ and let $u \in H_\alpha^{r_2,s_2,p}$. Then by (5.3.6), we obtain

$$\|u\|_{H_\alpha^{r_1,s_1,p}} = \|J_{-r_1,-s_1,\alpha}u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})}.$$

From Lemma 5.3.9, we get

$$\|u\|_{H_\alpha^{r_1,s_1,p}} \leq \|J_{-r_1,-s_1,\alpha}u\|_{H_\alpha^{r_2-r_1,s_2-s_1,p}}.$$

In view of Theorem 5.3.6, the last expression yields

$$\|u\|_{H_\alpha^{r_1,s_1,p}} \leq \|u\|_{H_\alpha^{r_2,s_2,p}}, \quad \forall u \in H_\alpha^{r_2,s_2,p}.$$

It follows that $H_\alpha^{r_2,s_2,p} \subseteq H_\alpha^{r_1,s_1,p}$ and the inclusion map

$$i : H_\alpha^{r_2,s_2,p} \hookrightarrow H_\alpha^{r_1,s_1,p},$$

is bounded. □

Taking the concept of Grushin[8], we define the following symbol class:

Definition 5.3.11. The symbol class S_0^m is the set of all functions $\sigma : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \to \mathbb{C}$, $m \in \mathbb{R}$, such that $\forall \beta, \gamma \in \mathbb{N}_0^{n+1}$, there exists a bounded function $C_{\beta,\gamma}(x)$ *on* $\mathbb{R}_+^{n+1}$ satisfying

$$\left| D_\xi^\beta D_x^\gamma \sigma(x,\xi) \right| \leq C_{\beta,\gamma}(x)(1 + \|\xi\|)^{m-|\beta|}, \tag{5.3.16}$$

and

$$\lim_{\|x\|\to\infty} C_{\beta,\gamma}(x) = 0, \quad for\ |\gamma| \neq 0.$$

For $\sigma \in S_0^m$, the pseudo-differential operator T_σ is defined by (5.2.3).

Theorem 5.3.12. *Let $\sigma(x,\xi)$ be a symbol in S_0^m, $m \in \mathbb{R}$. Then for $s \in \mathbb{R}$ and $1 < p < \infty$, the pseudo-differential operator $T_\sigma : H_\alpha^{s+m,p} \to H_\alpha^{s-\epsilon,p}$ is a compact operator for every $\epsilon > 0$.*

Proof. Let $\sigma(x,\xi)$ be a symbol in S_0^m, $m \in \mathbb{R}$ and $\sigma_k(x,\xi)$ be the function which is defined on $\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}$ by

$$\sigma_k(x,\xi) = \phi\Big(\frac{x}{k}\Big)\sigma(x,\xi), \quad \forall k \in \mathbb{N}, \tag{5.3.17}$$

where $\phi \in C_0^\infty(\mathbb{R}_+^{n+1})$ such that $\phi(x) = 1,\ for\ \|x\| \leq 1$.

Therefore, from (2.2.5) we have

$$\begin{aligned}(T_{\sigma_k}u)(x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma_k(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\phi\Big(\frac{x}{k}\Big)\sigma(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= \phi\Big(\frac{x}{k}\Big)(T_\sigma u)(x)\\ &= \phi_k(x)(T_\sigma u)(x), \quad \forall u \in S_*(\mathbb{R}_+^{n+1}).\end{aligned}$$

From above we can write

$$T_{\sigma_k} = \phi_k T_\sigma, \tag{5.3.18}$$

where

$$\phi_k(x) = \phi\Big(\frac{x}{k}\Big), \quad k \in \mathbb{N}.$$

By Theorem 5.3.6 and Theorem 5.3.10, the linear operator $T_\sigma : H^{s+m,p}_\alpha \to H^{s,p}_\alpha$ is bounded and the linear operator $\phi_k : H^{s,p}_\alpha \to H^{s-\epsilon,p}_\alpha$ is compact. Therefore, it follows that the linear operator $T_{\sigma_k} : H^{s+m,p}_\alpha \to H^{s-\epsilon,p}_\alpha$ is compact. In view of (5.3.18), we find that $T_{\sigma_k} \to T_\sigma \quad as\ k \to \infty$. This proves that $T_\sigma : H^{s+m,p}_\alpha \to H^{s-\epsilon,p}_\alpha$ is a compact operator. □

Corollary 5.3.13. *Let $\alpha > -\frac{1}{2}$ and $\epsilon > 0$. Then the pseudo-differential operator $J_{\epsilon,\epsilon,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is a compact operator for $1 < p < \infty$.*

Proof. If $\sigma(x,\xi) = (1+\|\xi\|^2)^{-\epsilon/2}(1+\|x\|^2)^{-\epsilon/2}$ and $J_{\epsilon,\epsilon,\alpha} = T_\sigma$ then by taking $s = \epsilon$, $m = -s$ in Theorem 5.3.12 we find that $H^{0,p}_\alpha = L^p_\alpha(\mathbb{R}^{n+1}_+)$, we get the required result. □

Theorem 5.3.14. *Let $r_1, r_2, s_1, s_2 \in \mathbb{R}$ be such that $r_1 < r_2$ and $s_1 < s_2$. Then for $1 < p < \infty$ the inclusion map*

$$i : H^{r_2,s_2,p}_\alpha \hookrightarrow H^{r_1,s_1,p}_\alpha,$$

is a compact operator.

Proof. Choose $\epsilon > 0$ such that

$$r_2 - r_1 - \epsilon > 0 \text{ and } s_2 - s_1 - \epsilon > 0.$$

Consider the following linear operators:

$$J^{-1}_{\epsilon,\epsilon,\alpha} J_{-r_1,-s_1,\alpha} : H^{r_2,s_2,p}_\alpha \to H^{r_2-r_1-\epsilon,s_2-s_1-\epsilon,p}_\alpha,$$

$$i : H^{r_2-r_1-\epsilon,s_2-s_1-\epsilon,p}_\alpha \hookrightarrow L^p_\alpha(\mathbb{R}^{n+1}_+),$$

$$J_{\epsilon,\epsilon,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+),$$

and

$$J_{r_1,s_1,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to H^{r_1,s_1,p}_\alpha.$$

Using (5.3.5), the operator $J_{r_1,s_1,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to H^{r_1,s_1,p}_\alpha$ is bounded. From Theorem 5.3.5 and Theorem 5.3.6 the operators $i : H^{r_2-r_1-\epsilon,s_2-s_1-\epsilon,p}_\alpha \hookrightarrow L^p_\alpha(\mathbb{R}^{n+1}_+)$ and $J^{-1}_{\epsilon,\epsilon,\alpha}J_{-r_1,-s_1,\alpha} : H^{r_2,s_2,p}_\alpha \to H^{r_2-r_1-\epsilon,s_2-s_1-\epsilon,p}_\alpha$ all are bounded. By Corollary 5.3.13, the operator $J_{\epsilon,\epsilon,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is compact. Hence, the operator

$$i : H^{r_2,s_2,p}_\alpha \hookrightarrow H^{r_1,s_1,p}_\alpha,$$

defined by

$$i = J_{r_1,s_1,\alpha}J_{\epsilon,\epsilon,\alpha}iJ^{-1}_{\epsilon,\epsilon,\alpha}J_{-r_1,-s_1,\alpha},$$

is compact. □

Theorem 5.3.15. *Let $r, s \in \mathbb{R}$ and $\alpha > -\frac{1}{2}$. Then the Schwartz space $S_*(\mathbb{R}^{n+1}_+)$ is dense in $H^{r,s,p}_\alpha(\mathbb{R}^{n+1}_+)$ for $1 < p < \infty$.*

Proof. Let $u \in H^{r,s,p}_\alpha$. Then from (5.3.5), we have $J_{-r,-s,\alpha}u \in L^p_\alpha(\mathbb{R}^{n+1}_+)$. By the density of $S_*(\mathbb{R}^{n+1}_+)$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$, we can find find a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $S_*(\mathbb{R}^{n+1}_+)$ such that $\phi_k \to J_{-r,-s,\alpha}u$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. Therefore, $u \in H^{r,s,p}_\alpha$. Let $f_k = J^{-1}_{-r,-s,\alpha}\phi_k$, for all $\phi_k \in S_*(\mathbb{R}^{n+1}_+)$, then by Theorem 5.2.4, we get $f_k \in S_*(\mathbb{R}^{n+1}_+)$.

Now, from (5.3.6), we have

$$\begin{aligned}
\|f_k - u\|_{H_\alpha^{r,s,p}} &= \|J_{-r,-s,\alpha}(f_k - u)\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} \\
&= \|J_{-r,-s,\alpha} f_k - J_{-r,-s,\alpha} u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} \\
&= \|J_{-r,-s,\alpha} J_{-r,-s,\alpha}^{-1} \phi_k - J_{-r,-s,\alpha} u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} \\
&= \|\phi_k - J_{-r,-s,\alpha} u\|_{L_\alpha^p(\mathbb{R}_+^{n+1})} \\
&\to 0, \quad as\ k \to \infty.
\end{aligned}$$

Hence, $f_k \to u$ in $H_\alpha^{r,s,p}$ as $k \to \infty$. □

5.4 Minimal and Maximal Operators

In this section, minimal-maximal pseudo-differential operators are made by exploiting the Weinstein transform technique and its various properties discussed.

Definition 5.4.1. Let σ be a symbol in S^{m_1,m_2} and T_σ be the pseudo-differential operator. Suppose there exists an operator $T_\sigma^* : S_*(\mathbb{R}_+^{n+1}) \to S_*(\mathbb{R}_+^{n+1})$ such that

$$\langle T_\sigma u, v\rangle = \langle u, T_\sigma^* v\rangle, \quad for\ all\ u, v \in S_*(\mathbb{R}_+^{n+1}). \tag{5.4.1}$$

Then T_σ^* is called a formal adjoint of T_σ.

From [81, p. 95], some useful definitions which are used in Chapter are given below:

Definition 5.4.2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be complex Banach spaces. Then a linear operator T from dense subspace of X, denoted by $\mathcal{D}(T)$ into Y is said to be closed if for any sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}(T)$ such that $\phi_k \to \phi$ in X and $T(\phi_k) \to y$ in Y as $k \to \infty$, we have $T(\phi) = y$.

Definition 5.4.3. Let T be the linear operator with domain $\mathcal{D}(T)$. Then it is said to be closable if for any sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}(T)$ such that $\phi_k \to 0$ in X and $T(\phi_k) \to y$ in Y as $k \to \infty$, we have $y = 0$.

Definition 5.4.4. Let T and S be linear operator from X into Y with domain $\mathcal{D}(T)$ and $\mathcal{D}(S)$ respectively. Then S is said to be an extension of T if $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S = T$ on $\mathcal{D}(T)$.

Proposition 5.4.5. *Let T be linear operator from X into Y with domain $\mathcal{D}(T)$. Then T has a closed extension if and only if T is closable.*

Proof. Utilizing [81, p. 95] we can proof the above theorem. □

Proposition 5.4.6. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. Then the pseudo-differential operator T_σ defined by (2.2.5) is closable.*

Proof. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence in $S_*(\mathbb{R}^{n+1}_+)$ such that $\phi_k \to 0$ in $S_*(\mathbb{R}^{n+1}_+)$ and $T(\phi_k) \to \phi$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. Then, by (5.4.1) for any ψ in $S_*(\mathbb{R}^{n+1}_+)$, we have

$$\langle T_\sigma \phi_k, \psi\rangle = \langle \phi_k, T^*_\sigma \psi\rangle, \quad \forall k \in \mathbb{N}. \tag{5.4.2}$$

Since $\phi_k \to 0$ and $T(\phi_k) \to \phi$ as $k \to \infty$, Therefore, from (5.4.2), we have

$$\langle \phi, \psi\rangle = \langle 0, T^*_\sigma \psi\rangle, \quad as\ k \to \infty.$$

Thus,

$$\langle \phi, \psi\rangle = 0, \quad for\ all\ \psi \in S_*(\mathbb{R}^{n+1}_+).$$

Since $S_*(\mathbb{R}^{n+1}_+)$ is dense in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ by Theorem 1.3.3. Therefore, $\phi = 0$. Hence, by Definition 5.4.3, T_σ is closable. □

From the aforesaid proposition we describes the proposition which is helpful to define the minimal pseudo-differential operators, maximal pseudo-differential operators and its various properties.

Proposition 5.4.7. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. Then the operator $T_{\sigma,0}$ is the smallest closed extension of T_σ, that is if $\mathcal{B}$ be any closed extension of T_σ, then it is also a closed extension of $T_{\sigma,0}$.*

Proof. The proof of the proposition can be obtained from [81, p. 96]. □

Definition 5.4.8. Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$ and T_σ be the symmetrically global pseudo-differential operator defined in (2.2.5). Then the smallest closed extension of T_σ is called minimal pseudo-differential operators and denoted by $T_{\sigma,0}$.

Definition 5.4.9. The domain $\mathcal{D}(T_{\sigma,0})$ of the operator $T_{\sigma,0}$ consists all the functions u in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ for which there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $S_*(\mathbb{R}^{n+1}_+)$ such that $\phi_k \to u$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ and $T_\sigma(\phi_k) \to f$, for some $f \in L^p_\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. Since the limit f is independent of the choice $\{\phi_k\}_{k\in\mathbb{N}}$ in $S_*(\mathbb{R}^{n+1}_+)$, so we can define

$$T_{\sigma,0}u = f. \tag{5.4.3}$$

Definition 5.4.10. Let u and f be functions in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ and $\mathcal{D}(T_{\sigma,1})$ be the domain of the operator $T_{\sigma,1}$. Then $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f$ if and only if

$$\langle u, T^*_\sigma \varphi\rangle = \langle f, \varphi\rangle, \quad \varphi \in S_*(\mathbb{R}^{n+1}_+). \tag{5.4.4}$$

Definition 5.4.11. Let σ be a symbol in S^{m_1,m_2}, $m_1, m_2 \in \mathbb{R}$. Then for all $u \in S'_*(\mathbb{R}^{n+1}_+)$, we define the linear functional $T_\sigma : S_*(\mathbb{R}^{n+1}_+) \to \mathbb{C}$ by

$$(T_\sigma u)(\varphi) = u(\overline{T^*_\sigma \overline{\varphi}}), \quad \textit{for all } \varphi \in S_*(\mathbb{R}^{n+1}_+). \tag{5.4.5}$$

Proposition 5.4.12. *Let* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. *Then we have*

$$T_{\sigma,1}u = T_\sigma u, \quad u \in \mathcal{D}(T_{\sigma,1}), \tag{5.4.6}$$

in the distribution sense.

Proof. Let $u \in \mathcal{D}(T_{\sigma,1})$. Then, by (5.4.4), we have

$$\langle u, T_\sigma^* \varphi \rangle = \langle T_{\sigma,1}u, \varphi \rangle, \quad \varphi \in S_*(\mathbb{R}_+^{n+1}).$$

Since u and $T_{\sigma,1}u$ are distributions in $S'_*(\mathbb{R}_+^{n+1})$, then using (5.4.5), we obtain

$$(T_{\sigma,1}u)(\overline{\varphi}) = u(\overline{T_\sigma^* \varphi}), \quad \varphi \in S_*(\mathbb{R}_+^{n+1}), \tag{5.4.7}$$

and

$$(T_\sigma u)(\overline{\varphi}) = u(\overline{T_\sigma^* \varphi}), \quad \varphi \in S_*(\mathbb{R}_+^{n+1}). \tag{5.4.8}$$

With the help of (5.4.7) and (5.4.8), we find

$$T_{\sigma,1}u = T_\sigma u, \quad \forall u \in \mathcal{D}(T_{\sigma,1}),$$

in the distribution sense. □

Proposition 5.4.13. *Let* $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. *Then the operator* $T_{\sigma,1}$ *is an extension of* $T_{\sigma,0}$.

Proof. Let $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Then, by Definition 5.4.9, there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $S_*(\mathbb{R}_+^{n+1})$ such that $\phi_k \to u$ in $L^p_\alpha(\mathbb{R}_+^{n+1})$ and $T_\sigma(\phi_k) \to f$, for some $f \in L^p_\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. By (5.4.1) for any ψ in $S_*(\mathbb{R}^{n+1}_+)$, we have

$$\langle T_\sigma \phi_k, \psi \rangle = \langle \phi_k, T^*_\sigma \psi \rangle, \quad \forall k \in \mathbb{N}.$$

As $k \to \infty$, from above we obtain

$$\langle f, \psi \rangle = \langle u, T^*_\sigma \psi \rangle.$$

Hence, from Definition 5.4.10, we have $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$. This implies that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$. □

Proposition 5.4.14. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. Then $T_{\sigma,1}$ is a closed linear operator from $L^p_\alpha(\mathbb{R}^{n+1}_+)$ into $L^p_\alpha(\mathbb{R}^{n+1}_+)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $S_*(\mathbb{R}^{n+1}_+)$.*

Proof. From (5.4.4), we find that $S_*(\mathbb{R}^{n+1}_+) \subseteq \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}$ is linear operator. Now, we need to show $T_{\sigma,1}$ is closed. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $\mathcal{D}(T_{\sigma,1})$ such that $\phi_k \to u$ in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ and $T_{\sigma,1}(\phi_k) \to f$, for some $u, f \in L^p_\alpha(\mathbb{R}^{n+1}_+)$ as $k \to \infty$. Using (5.4.1) for any φ in $S_*(\mathbb{R}^{n+1}_+)$, we have

$$\langle T_{\sigma,1} \phi_k, \varphi \rangle = \langle \phi_k, T^*_\sigma \varphi \rangle, \quad \forall k \in \mathbb{N}.$$

As $k \to \infty$, from above we obtain

$$\langle f, \varphi \rangle = \langle u, T^*_\sigma \varphi \rangle.$$

Therefore, by Definition 5.4.10, we get $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$. Hence, $T_{\sigma,1}$ is closed. □

With the help of [81, p. 98, Exercise 13.2], we define the following definition:

Definition 5.4.15. Let $1 < p < \infty$, p' be the conjugate of p and $T_{\sigma,1}$ from $L^p_\alpha(\mathbb{R}^{n+1}_+)$ into $L^p_\alpha(\mathbb{R}^{n+1}_+)$ be the linear operator with dense domain $\mathcal{D}(T_{\sigma,1})$. Then the operator $T^t_{\sigma,1}$ from $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ into $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ is defined as follows:
$\mathcal{D}(T^t_{\sigma,1})$ is the set of all functionals u in $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ for which there exists a functional f in $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ such that

$$\langle u, T_{\sigma,1}v\rangle = \langle f, v\rangle, \quad for\ all\ v \in \mathcal{D}(T_{\sigma,1}). \tag{5.4.9}$$

Then $T^t_{\sigma,1}$ is called a true adjoint of $T_{\sigma,1}$.

Proposition 5.4.16. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. Then $S_*(\mathbb{R}^{n+1}_+) \subseteq \mathcal{D}(T^t_{\sigma,1})$ and $T^t_{\sigma,1}u = T^*_\sigma u$.*

Proof. From Proposition 5.4.14, $T_{\sigma,1}$ is a closed linear operator from $L^p_\alpha(\mathbb{R}^{n+1}_+)$ into $L^p_\alpha(\mathbb{R}^{n+1}_+)$, for $1 < p < \infty$, with domain $\mathcal{D}(T_{\sigma,1})$ containing $S_*(\mathbb{R}^{n+1}_+)$. Therefore, from [81, p. 97], $T^t_{\sigma,1}$ is a closed linear operator from $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$ into $L^{p'}_\alpha(\mathbb{R}^{n+1}_+)$, where p' is the conjugate of p. Let $u \in S_*(\mathbb{R}^{n+1}_+)$. Then by (5.4.9), we obtain

$$\langle u, T_{\sigma,1}v\rangle = \langle T^*_\sigma u, v\rangle, \quad for\ all\ v \in \mathcal{D}(T_{\sigma,1}). \tag{5.4.10}$$

Hence, in view of Definition 5.4.15, we get $u \in \mathcal{D}(T_{\sigma,1})$. By definition of $T^t_{\sigma,1}$, we have $u \in \mathcal{D}(T^t_{\sigma,1})$ and $T^t_{\sigma,1}u = T^*_\sigma u$. ☐

Proposition 5.4.17. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$. Then the operator $T_{\sigma,1}$ is the largest closed extension of T_σ.*

Proof. From [81, p. 102], the proof of the proposition can be obtained. ☐

Definition 5.4.18. Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 \in \mathbb{R}$ and T_σ be the symmetrically global pseudo-differential operator defined in (2.2.5). Then the largest closed extension of T_σ is called maximal pseudo-differential operator and denoted by $T_{\sigma,1}$.

In view of definitions and propositions, the weak solution of the symmetrically global pseudo-differential equations are discussed by exploiting the theory of the Weinstein transform.

Definition 5.4.19. Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$ and f be a function in $L^p_\alpha(\mathbb{R}^{n+1}_+)$. A function u in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ is called a weak solution of the symmetrically global pseudo-differential equation $T_\sigma u = f$ *on* $\mathbb{R}^{n+1}_+$ if

$$\langle u, T^*_\sigma \varphi\rangle = \langle f, \varphi\rangle, \quad \varphi \in S_*(\mathbb{R}^{n+1}_+).$$

Proposition 5.4.20. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$ and f be a function in $L^p_\alpha(\mathbb{R}^{n+1}_+)$. Then a function u in $L^p_\alpha(\mathbb{R}^{n+1}_+)$ is a weak solution of the symmetrically global pseudo-differential equation*

$$T_\sigma u = f \quad on\ \mathbb{R}^{n+1}_+, \tag{5.4.11}$$

if and only if

$$u \in \mathcal{D}(T_{\sigma,1})\ and\ T_{\sigma,1} u = f. \tag{5.4.12}$$

Proof. We take (5.4.11) has a weak solution u in $L^p_\alpha(\mathbb{R}^{n+1}_+)$. Then by Definition 5.4.19, we have

$$\langle u, T^*_\sigma \varphi\rangle = \langle f, \varphi\rangle, \quad \varphi \in S_*(\mathbb{R}^{n+1}_+).$$

In view of Definition 5.4.10 and [81, p. 99], it follows that

$$u \in \mathcal{D}(T_{\sigma,1})\ and\ T_{\sigma,1} u = f.$$

Conversely assume that (5.4.12) holds. Then by using the Definitions 5.4.10 and 5.4.19, we get the required result. □

5.5 Conclusion

Employing the theory of the Fourier transform technique Kohn-Nirenberg [32], Hörmander [26], Grushin [22], Kumano-go [34], Taylor [70], Wong [81], others made a proper foundation for the theory of pseudo-differential operators and Coriasco-Rodino [11], Nicola [47], Nicola-Rodino [48], Cappiello-Rodino [5] and Dasgupta-Wong [14] gave adequate development regarding the theory of SG pseudo-differential operators. The importance of the aforesaid theory is that they construct the proper smell of pseudo-differential operators related to the non-compact manifolds. Motivated by the above works, the symmetrically global pseudo-differential operator is introduced and found many properties in Sobolev space $H_\alpha^{r,s,p}$ by exploiting theory of the Weinstein transform. Using the aforesaid theory, various properties of minimal-maximal operators for symmetrically global pseudo-differential operators are obtained. This theory will make the future development in the theory of pseudo-differential operators.

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