Chapter 7

Conclusion and Future Scope

This thesis focused on the study of cyclic codes and some of its generalizations like constacyclic, skew cyclic, and skew constacyclic codes over some finite non-chain rings. Moreover, it explored their applications in the construction of quantum and LCD codes. The overall conclusion of the thesis can be summarized as follows:

- Chapter 1 served as the foundation of the thesis, introducing essential concepts and definitions from algebra, classical coding theory, and the theory of quantum error-correction. This chapter laid the groundwork for the subsequent chapters by exploring key properties and results related to various algebraic structures, algebraic codes, and quantum error-correcting codes.

- In Chapter 2, we investigated cyclic codes over the non-chain ring

$$\mathcal{S} = \mathbb{F}_q[\mathsf{u}, \mathsf{v}, \mathsf{w}]/\langle \mathsf{u}^3 - \mathsf{u}, \mathsf{v}^2 - \mathsf{v}, \mathsf{w}^2 - \mathsf{w}, \mathsf{uv}, \mathsf{vu}, \mathsf{uw}, \mathsf{wu}, \mathsf{vw} - \mathsf{wv}\rangle.$$

  Some key structural properties of these cyclic codes and their duals were explored. Furthermore, the chapter examined the construction of quantum and

LCD codes from cyclic codes over $\mathcal{S}$. Consequently, several new and better codes were obtained contributing to the code database [9].

- Chapter 3 focused on the study skew cyclic codes over the non-chain ring

$$\mathcal{R} = \mathbb{F}_q[u_1, u_2, \ldots, u_r]/\langle u_i^3 - u_i, u_i u_j - u_j u_i \rangle_{i,j=1}^{r}.$$

The structural properties of skew cyclic codes over this ring and their duals were thoroughly examined. Additionally, methods for constructing quantum and LCD codes from skew cyclic codes were discussed, leading to the development of many new codes with improved parameters.

- Chapter 4 investigated skew constacyclic codes over a general class of non-chain rings

$$\mathcal{T} = \mathbb{F}_q[u_1, u_2, \ldots, u_r]/\langle f_i(u_i), u_i u_j - u_j u_i \rangle_{i,j=1}^{r}.$$

An explicit form for the generator matrix of the Gray image of linear codes over $\mathcal{T}$ was obtained. Several key results on the structural properties of skew constacyclic codes over $\mathcal{T}$ were established. It is proved that the Gray image of skew constacyclic codes over $\mathcal{T}$ belongs to a new class of codes which is a particular case of skew multi-twisted codes and a generalization of skew quasi-twisted codes. The findings of this chapter were critical for the construction of quantum and LCD codes from skew constacyclic codes, expanding the applicability of these codes.

- Building upon the results from Chapter 4, Chapter 5 delved into the construction of quantum and LCD codes from skew constacyclic codes over $\mathcal{T}$. The chapter explored the Euclidean and Hermitian duals of these codes and introduced methods for constructing quantum codes from them. Several new

and improved quantum codes were constructed, thereby contributing significantly to the quantum code database [9]. Moreover, numerous BKLCs, MDS and AMDS codes as Gray images of skew constacyclic LCD codes over $\mathcal{T}$ are obtained.

- Chapter 6 focused on the construction of entanglement-assisted quantum error-correcting codes (EAQECCs) from constacyclic codes over $\mathcal{T}$. It was proven that, under a polynomial Gray map, the image of constacyclic codes over $\mathcal{T}$ is a cyclic code. A method for constructing EAQECCs from these codes was developed, resulting in the construction of several MDS EAQECCs.

Conclusion

This thesis contributed to the study of several classes of algebraic codes like cyclic, constacyclic, skew cyclic, and skew constacyclic codes over some finite non-chain rings. The research not only advanced the theory of error-correcting codes in classical communication systems but also paved the way for the construction of quantum error-correcting codes, which are vital for realizing fault-tolerant quantum computing. The exploration of quantum codes, LCD codes, and the development of entanglement-assisted quantum error-correcting codes (EAQECCs), offered promising avenues for future research. The findings provided a solid foundation for continued studies in coding theory over finite rings, with significant implications for the development of quantum technologies.

Future scope

I believe that the findings of this thesis will not only enhance the understanding of cyclic codes and their generalizations over finite non-chain rings but also inspire the readers to pursue further research in this direction. Looking ahead, some promising directions for future work are worth mentioning:

- There are some generalizations of cyclic codes like double cyclic codes, multi-twisted codes, and skew multitwisted codes. These codes are not much explored over rings. Instead of studying them over different non-chain rings, we can study them over a general class of non-chain rings $\mathcal{T}$.
- One notable advantage of generalizing cyclic codes is the potential to construct codes with desired parameters within the generalized class, which may not have been possible with the original cyclic codes. Investigating the dual-containing conditions for multitwisted and skew multitwisted codes can further facilitate their application in constructing quantum codes. These broader classes of codes have the potential to yield quantum codes with improved parameters.
- Another direction for research could involve identifying the conditions under which these generalized codes over finite fields can be classified as LCD codes. This investigation could be further extended to derive similar conditions for these codes over finite rings, broadening the applicability of the results. Such an extension would not only deepen our understanding of LCD codes in various algebraic structures but also potentially lead to new constructions of LCD codes with desirable properties for practical applications in error correction and cryptography.

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