Chapter 3

Global quasi-synchronization of complex-valued recurrent neural networks with time-varying delay and interaction terms

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This chapter is concerned about the effects of interaction term on global quasi-synchronization of complex valued recurrent neural networks with time varying delay. We have considered two non-identical Hopfield complex valued recurrent neural networks with time-varying delays as master and response systems. In the practical application of quasi-synchronization, the error system does not always tend to zero along time, but it varies within a small bound. The optimal synchronization error bound has been discussed by using fundamental calculus. The chapter is ended with a numerical example to verify the obtained results.

3.1 Some preliminaries and Problem formulation

Assumption 2. *Let us consider* $z = z_1 + iz_2$, *where* $z_1, z_2 \in R$. $f_k(z)$, $g_k(z)$ *and* $h_k(z)$ *are defined as*

$f_k(z) = f_k^R(z_1) + if_k^I(z_2)$, $g_k(z) = g_k^R(z_1) + ig_k^I(z_2)$ *and* $h_k(z) = h_k^R(z_1) + ih_k^I(z_2)$,

where $k = 1, 2, ..., n$, *and* $f_k^R(.), f_k^I(.), g_k^R(.), g_k^I(.), h_k^R(.), h_k^I(.) : R \to R$ *satisfy the Lipschitz conditions given as*

$$\|f_k^R(\nu) - f_k^R(\eta)\|_p \leq r_k\|\nu - \eta\|_p,$$

$$\|f_k^I(\nu) - f_k^I(\eta)\|_p \leq s_k\|\nu - \eta\|_p,$$

$$\|g_k^R(\nu) - g_k^R(\eta)\|_p \leq m_k\|\nu - \eta\|_p,$$

$$\|g_k^I(\nu) - g_k^I(\eta)\|_p \leq n_k\|\nu - \eta\|_p,$$

$$\|h_k^R(\nu) - h_k^R(\eta)\|_p \leq p_k\|\nu - \eta\|_p,$$

$$\|h_k^I(\nu) - h_k^I(\eta)\|_p \leq q_k\|\nu - \eta\|_p,$$

where $\|.\|_p$ *represents the vector norm,* $p = 1, 2, \infty$, *and* r_k, s_k, m_k, n_k, p_k, *and* q_k *are the Lipschitz constants,* ν *and* $\eta \in R^n$.

Lemma 3.1. *[104] Let us consider* $v(t) : [-\tau, +\infty) \to [0, +\infty)$ *be a continuous function and for all* $t > 0$, *it has*

$$D^+(v(t)) \leq -k_1 v(t) + k_2 \bar{v}(t) + k_3,$$

where $\bar{v}(t) \triangleq sup_{-\tau\leq s\leq 0} v(s)$, *if* $k_1 > k_2 > 0, k_3 > 0$,

then the inequality $v(t) \leq sup_{-\tau\leq s\leq 0} v(s)e^{-rt} + \frac{k_3}{r}$ *holds for* $t \geq t_0$, *where $D^+v(t)$ denotes the upper-right derivative which is defined as* $D^+v(t) = \overline{lim}_{h\to 0^+}\frac{v(t+h)-v(t)}{h}$, *where r is the unique solution of $r = k_1 - k_2e^{r\tau}$.*

3.1.1 Problem formulation

Let us assume the CVRNN with time-varying delays and interaction terms as the drive system as

$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t-\tau(t))) + \epsilon_1 Dh(\widetilde{w}(t)) + L(t), \tag{3.1.1}$$

with the initial condition, $\quad w(s) = \psi(s), s \in [t_0 - \tau, t_0]$,

where $\psi(s) = (\psi_1(s), \psi_2(s), .., \psi_n(s)) \in C^n$, in which real and imaginary parts of $\psi(s)$ are continuous on $[t_0 - \tau, t_0]$ and $w(t) = (w_1(t), w_2(t), ..., w_n(t))^T \in C^n$ represents state vector of n-neurons of the neural networks with time t, $C = diag(c_1, c_2, ..., c_n) \in R^{n\times n}$, $c_k > 0$ represents self-feedback connection weight matrix. $A = (a_{kj})_{n\times n}$ and $B = (b_{kj})_{n\times n}$, where $A, B \in C^{n\times n}$, represent without and with delayed connection weight matrix in the model (3.1.1) respectively. $D = (d_{kj})_{n\times n}$ denotes the interaction structures whose entries d_{kj} represent the intensity of the interaction from k in network w to j in network $\widetilde{w}$ and ϵ_1 is outer interaction strength. $L = (l_1, l_2, ..., l_n)^T$ denote external input vector, $f(w(t)) = (f_1(w_1(t)), f_2(w_2(t)), ..., f_n(w_n(t)))^T : C^n \to C^n$, $g(w(t-\tau(t))) = (g_1(w_1(t-\tau(t))), g_2(w_2(t-\tau(t))), ..., g_n(w_n(t-\tau(t))))^T : C^n \to C^n$ represent without and with time-varying delay activation functions, and $h(w(t)) = (h_1(w_1(t)), h_2(w_2(t)), ..., h_n(w_n(t)))^T : C^n \to C^n$ is interaction function between two networks, where $\tau(t)$ denotes transmission delay, which satisfies $0 \le \tau(t) \le \tau(\tau > 0)$, where τ is a known constant.

In view of Assumption 2, if we consider $w(t) = u(t) + iv(t)$, where $u(t), v(t) \in R^n$, then equation (3.1.1) can be rewritten as

$$
\begin{aligned}
\dot{u}(t) =& - Cu(t) + A^R f^R(u(t)) - A^I f^I(v(t)) + B^R g^R(u(t-\tau(t))) \\
& - B^I g^I(v(t-\tau(t))) + \epsilon_1 D^R h^R(\tilde{u}(t)) - \epsilon_1 D^I h^I(\tilde{v}(t)) + L^R(t), \\
\dot{v}(t) =& - Cv(t) + A^I f^R(u(t)) + A^R f^I(v(t)) + B^I g^R(u(t-\tau(t))) \\
& + B^R g^I(v(t-\tau(t))) + \epsilon_1 D^I h^R(\tilde{u}(t)) + \epsilon_1 D^R h^I(\tilde{v}(t)) + L^I(t).
\end{aligned} \tag{3.1.2}
$$

The initial conditions of equation (3.1.2) will be

$$
\begin{cases}
u(s) = \psi^R(s), \\
v(s) = \psi^I(s), -\tau \leq s \leq 0,
\end{cases}
$$

where $\psi^R(s)$ denotes the real part of $\psi(s)$, $\psi^I(s)$ denotes the imaginary part of $\psi(s)$ and $\|\psi^R\|_p = sup_{t_0-\tau\leq s\leq t_0}\|\psi^R(s)\|_p$ denotes the norm of the function $\psi^R \in C([t_0-\tau, t_0], R^n)$ and $\|\psi^I\|_p = sup_{t_0-\tau\leq s\leq t_0}\|\psi^I(s)\|_p$ is the norm of the function $\psi^I \in C([t_0-\tau, t_0], R^n)$. The initial functions $u(s)$ and $v(s)$ are continuous functions.

Now considering the corresponding response system as

$$
\dot{\tilde{w}}(t) = -C\tilde{w}(t) + Af(\tilde{w}(t)) + Bg(\tilde{w}(t-\tau(t))) + \epsilon_1 Dh(w(t)) + L(t) + M(t), \tag{3.1.3}
$$

where $M(t) = (M_1(t), M_2(t), ..., M_n(t))^T \in C^n$ denotes the coupling control, $\tilde{w}(t) = (\tilde{w}_1(t), \tilde{w}_2(t), ..., \tilde{w}_n(t))^T \in C^n$, $C, A, B, D, L(t), \epsilon_1$ and $\tau(t)$ have the same meanings as described in the system (3.1.1).

The initial condition of the system (3.1.3) is taken as
$\tilde{w}(s) = \tilde{\psi}(s), s \in [t_0-\tau, t_0]$,

where $\widetilde{\psi}(s) = (\widetilde{\psi}_1(s), \widetilde{\psi}_2(s), ..., \widetilde{\psi}_n(s)) \in C^n$, $\widetilde{\psi}^R(s)$ and $\widetilde{\psi}^I(s)$ are continuous functions on interval $[t_0 - \tau, t_0]$.

Remark 3.1.1. Here, $h(\widetilde{w}(t))$ and $h(w(t))$ represent the interaction function from network $w(\widetilde{w})$ to network $\widetilde{w}(w)$. There are lots of active forms between two networks, for instance, communicated by signals, special nodes or bidirectional actions. In Li et al.[105], the authors choose the interaction function of the form $(H - \frac{\partial f(x_i)}{\partial x_i})$ to realize outer synchronization, where $H \in R^{n\times n}$ is a constant Hurwitz matrix.

Now, separating equation (3.1.3) into real and imaginary parts, we get

$$\begin{aligned}\dot{\widetilde{u}}(t) =& - C\widetilde{u}(t) + A^R f^R(\widetilde{u}(t)) - A^I f^I(\widetilde{v}(t)) + B^R g^R(\widetilde{u}(t-\tau(t)))\\ &- B^I g^I(\widetilde{v}(t-\tau(t))) + \epsilon_1 D^R h^R(u(t)) - \epsilon_1 D^I h^I(v(t)) + L^R(t) + M^R(t),\\ \dot{\widetilde{v}}(t) =& - C\widetilde{v}(t) + A^I f^R(\widetilde{u}(t)) + A^R f^I(\widetilde{v}(t)) + B^I g^R(\widetilde{u}(t-\tau(t)))\\ &+ B^R g^I(\widetilde{v}(t-\tau(t))) + \epsilon_1 D^I h^R(u(t)) + \epsilon_1 D^R h^I(v(t)) + L^I(t) + M^I(t),\end{aligned} \tag{3.1.4}$$

with initial conditions as

$$\begin{cases}\widetilde{u}(s) = \widetilde{\psi}^R(s),\\ \widetilde{v}(s) = \widetilde{\psi}^I(s), -\tau \leq s \leq 0,\end{cases}$$

where $M^R(t), M^I(t)$ are the control input vectors and defined as

$$M^R(t) = \Omega e^R(t) + P sign(e^R(t)), \tag{3.1.5}$$

and

$$M^I(t) = \Omega e^I(t) + Q sign(e^I(t)), \tag{3.1.6}$$

where $M^R(t) = [M_1^R(t), M_2^R(t), ..., M_n^R(t)]^T, M^I(t) = [M_1^I(t), M_2^I(t), ..., M_n^I(t)]^T$, $e^R(t) = [e_1^R(t), e_2^R(t), ..., e_n^R(t)]^T, e^I(t) = [e_1^I(t), e_2^I(t), ..., e_n^I(t)]^T$

and $\Omega = \begin{pmatrix} \alpha_{11} \dots \alpha_{1n} \\ \vdots \ddots \vdots \\ \alpha_{n1} \dots \alpha_{nn} \end{pmatrix}$ stands for the controller gain matrix.

Let us define the error functions for synchronization of drive and response systems as $e^R(t) = \widetilde{u}(t) - u(t), e^I(t) = \widetilde{v}(t) - v(t)$. The drive-response systems will be synchronized if $e^R(t) \to 0$ and $e^I(t) \to 0$ as $t \to \infty$. From equations (3.1.5) and (3.1.6), we get the error systems as

$$\begin{aligned}
\dot{e}^R(t) =& - Ce^R(t) + A^R\widetilde{f}^R(e(t)) - A^I\widetilde{f}^I(e(t)) + B^R\widetilde{g}^R(e(t-\tau(t))) \\
& - B^I\widetilde{g}^I(e(t-\tau(t))) - \epsilon_1 D^R\widetilde{h}^R(e(t)) + \epsilon_1 D^I\widetilde{h}^I(e(t)) + \Omega e^R(t) + Psign(e^R(t)), \\
\dot{e}^I(t) =& - Ce^I(t) + A^R\widetilde{f}^I(e(t)) + A^I\widetilde{f}^R(e(t)) + B^R\widetilde{g}^I(e(t-\tau(t))) \\
& + B^I\widetilde{g}^R(e(t-\tau(t))) - \epsilon_1 D^R\widetilde{h}^I(e(t)) - \epsilon_1 D^I\widetilde{h}^R(e(t)) + \Omega e^I(t) + Qsign(e^I(t)),
\end{aligned} \tag{3.1.7}$$

where

$e^R(t) = (e_1^R(t), e_2^R(t)...e_n^R(t))^T \in R^n, e^I(t) = (e_1^I(t), e_2^I(t)...e_n^I(t))^T \in R^n$,

$\widetilde{f}^R(e(t)) = f^R(\widetilde{w}(t)) - f^R(w(t)), \widetilde{f}^I(e(t)) = f^I(\widetilde{w}(t)) - f^I(w(t))$,

$\widetilde{h}^R(e(t)) = h^R(\widetilde{w}(t)) - h^R(w(t)), \widetilde{h}^I(e(t)) = h^I(\widetilde{w}(t)) - h^I(w(t))$,

$\widetilde{g}^R(e(t-\tau(t))) = g^R(\widetilde{w}(t-\tau(t))) - g^R(w(t-\tau(t)))$,

$\widetilde{g}^I(e(t-\tau(t))) = g^I(\widetilde{w}(t-\tau(t))) - g^I(w(t-\tau(t)))$.

The quasi-synchronization can be achieved between the systems (3.1.2) and (3.1.4) by choosing of the controller according to global quasi-stability criteria.

3.2 Main results

Theorem 3.2. *If the controller gain matrix* Ω *satisfies the following inequality under Assumption 2 as*

$$0 < (m_k + n_k)(\|B^R\|_p + \|B^I\|_p) < \\ -\Big\{\mu_p(-C+\Omega) + (r_k+s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p)\Big\}, \tag{3.2.1}$$

where $p = 1, 2, \infty$, *then the considered drive CVRNNs (3.1.1) and response CVRNNs (3.1.3) with time-varying delay and interaction terms will be globally quasi-synchronized under the controller (3.1.5) and (3.1.6).*

Proof. Let us assume the Lyapunov function as

$$V_1(e(t)) = \|e^R(t)\|_p + \|e^I(t)\|_p.$$

Using the definition of upper-right Dini derivative and Taylor's formula with remainder of Peano, the upper-right Dini derivative of $V_1(e(t))$ with respect to time t is obtained as

$$\begin{aligned} D^+V_1(e(t)) =& \overline{lim}_{\epsilon\to 0^+} \frac{\|e^R(t+\epsilon)\|_p + \|e^I(t+\epsilon)\|_p - \|e^R(t)\|_p - \|e^I(t)\|_p}{\epsilon} \\ =& \overline{lim}_{\epsilon\to 0^+} \Big\{\|e^R(t) + \epsilon\dot{e}^R(t) + \mathcal{O}(\epsilon)\|_p + \|e^I(t) + \epsilon\dot{e}^I(t) + \mathcal{O}(\epsilon)\|_p \\ & - \|e^R(t)\|_p - \|e^I(t)\|_p\Big\}\Big/\epsilon \end{aligned}$$

$$
\begin{aligned}
=&\overline{lim}_{\epsilon\to 0^+}\bigg\{\|e^R(t)+\epsilon\bigg(-Ce^R(t)+A^R\widetilde{f}^R(e(t))-A^I\widetilde{f}^I(e(t))\\
&+B^R\widetilde{g}^R(e(t-\tau(t)))-B^I\widetilde{g}^I(e(t-\tau(t)))-\epsilon_1 D^R\widetilde{h}^R(e(t))+\epsilon_1 D^I\widetilde{h}^I(e(t))\\
&+\Omega e^R(t)+Psign(e^R(t))\bigg)+\mathcal{O}(\epsilon)\|_p-\|e^R(t)\|_p\bigg\}\bigg/\epsilon\\
&+\overline{lim}_{\epsilon\to 0^+}\bigg\{\|e^I(t)+\epsilon\bigg(-Ce^I(t)+A^R\widetilde{f}^I(e(t))+A^I\widetilde{f}^R(e(t))\\
&+B^R\widetilde{g}^I(e(t-\tau(t)))+B^I\widetilde{g}^R(e(t-\tau(t)))-\epsilon_1 D^R\widetilde{h}^I(e(t))\\
&-\epsilon_1 D^I\widetilde{h}^R(e(t))+\Omega e^I(t)+Qsign(e^I(t))\bigg)+\mathcal{O}(\epsilon)\|_p-\|e^I(t)\|_p\bigg\}\bigg/\epsilon\\
\le&\overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^R(t)\|_p+\|A^R\|_p\|\widetilde{f}^R(e(t))\|_p\\
&+\|A^I\|_p\|\widetilde{f}^I(e(t))\|_p+\|B^R\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p+\|B^I\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p\\
&+\epsilon_1\|D^R\|_p\|\widetilde{h}^R(e(t))\|_p+\epsilon_1\|D^I\|_p\|\widetilde{h}^I(e(t))\|_p+\|P\|_p\\
&+\overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^I(t)\|_p+\|A^R\|_p\|\widetilde{f}^I(e(t))\|_p\\
&+\|A^I\|_p\|\widetilde{f}^R(e(t))\|_p+\|B^R\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p+\|B^I\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p\\
&+\epsilon_1\|D^R\|_p\|\widetilde{h}^I(e(t))\|_p+\epsilon_1\|D^I\|_p\|\widetilde{h}^R(e(t))\|_p+\|Q\|_p. \qquad (3.2.2)
\end{aligned}
$$

From Assumption 2, we have

$$
\begin{aligned}
\|\widetilde{f}^R(e(t))\|_p &\le r_k\|e(t)\|_p\\
&= r_k\|e^R(t)+ie^I(t)\|_p\\
&\le r_k(\|e^R(t)\|_p+\|e^I(t)\|_p),\\
\|\widetilde{f}^I(e(t))\|_p &\le s_k\|e(t)\|_p\\
&= s_k\|e^R(t)+ie^I(t)\|_p\\
&\le s_k(\|e^R(t)\|_p+\|e^I(t)\|_p). \qquad (3.2.3)
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\|\widetilde{g}^R(e(t-\tau(t)))\|_p &\leq m_k\|e(t-\tau(t))\|_p = m_k\|e^R(t-\tau(t))+ie^I(t-\tau(t))\|_p \\
&\leq m_k(\|e^R(t-\tau(t))\|_p+\|e^I(t-\tau(t))\|_p), \\
\|\widetilde{g}^I(e(t-\tau(t)))\|_p &\leq n_k\|e(t-\tau(t))\|_p = n_k\|e^R(t-\tau(t))+ie^I(t-\tau(t))\|_p \\
&\leq n_k(\|e^R(t-\tau(t))\|_p+\|e^I(t-\tau(t))\|_p), \\
\|\widetilde{h}^R(e(t))\|_p &\leq p_k\|e(t)\|_p = p_k\|e^R(t)+ie^I(t)\|_p \\
&\leq p_k(\|e^R(t)\|_p+\|e^I(t)\|_p), \\
\|\widetilde{h}^I(e(t))\|_p &\leq q_k\|e(t)\|_p = q_k\|e^R(t)+ie^I(t)\|_p \\
&\leq q_k(\|e^R(t)\|_p+\|e^I(t)\|_p).
\end{aligned} \tag{3.2.4}$$

Using the inequalities (3.2.3) and (3.2.4) in the inequality (3.2.2), we get

$$\begin{aligned}
D^+V_1(e(t)) \leq& \overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^R(t)\|_p+\|A^R\|_p\|\widetilde{f}^R(e(t))\|_p \\
&+\|A^I\|_p\|\widetilde{f}^I(e(t))\|_p+\|B^R\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p+\|B^I\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p \\
&+\epsilon_1\|D^R\|_p\|\widetilde{h}^R(e(t))\|_p+\epsilon_1\|D^I\|_p\|\widetilde{h}^I(e(t))\|_p+\|P\|_p \\
&+\overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^I(t)\|_p+\|A^R\|_p\|\widetilde{f}^I(e(t))\|_p \\
&+\|A^I\|_p\|\widetilde{f}^R(e(t))\|_p+\|B^R\|_p\|\widetilde{g}^I(e(t-\tau(t)))\|_p+\|B^I\|_p\|\widetilde{g}^R(e(t-\tau(t)))\|_p \\
&+\epsilon_1\|D^R\|_p\|\widetilde{h}^I(e(t))\|_p+\epsilon_1\|D^I\|_p\|\widetilde{h}^R(e(t))\|_p+\|Q\|_p \\
\leq& \overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^R(t)\|_p+\|A^R\|_p r_k(\|e^R(t)\|_p+\|e^I(t)\|_p) \\
&+\|A^I\|_p s_k(\|e^R(t)\|_p+\|e^I(t)\|_p)+\|B^R\|_p m_k(\|e^R(t-\tau(t))\|_p \\
&+\|e^I(t-\tau(t))\|_p)+\|B^I\|_p n_k(\|e^R(t-\tau(t))\|_p+\|e^I(t-\tau(t))\|_p) \\
&+\epsilon_1\|D^R\|_p p_k(\|e^R(t)\|_p+\|e^I(t)\|_p)+\epsilon_1\|D^I\|_p q_k(\|e^R(t)\|_p+\|e^I(t)\|_p)+\|P\|_p
\end{aligned}$$

$$
\begin{aligned}
&+ \overline{lim}_{\epsilon\to 0^+}\frac{\|I+\epsilon(-C+\Omega)\|_p-1}{\epsilon}\|e^I(t)\|_p + \|A^R\|_p s_k(\|e^R(t)\|_p + \|e^I(t)\|_p)\\
&+ \|A^I\|_p r_k(\|e^R(t)\|_p + \|e^I(t)\|_p) + \|B^R\|_p n_k(\|e^R(t-\tau(t))\|_p\\
&+ \|e^I(t-\tau(t))\|_p) + \|B^I\|_p m_k(\|e^R(t-\tau(t))\|_p + \|e^I(t-\tau(t))\|_p)\\
&+ \epsilon_1\|D^I\|_p p_k(\|e^R(t)\|_p + \|e^I(t)\|_p) + \epsilon_1\|D^R\|_p q_k(\|e^R(t)\|_p + \|e^I(t)\|_p) + \|Q\|_p.
\end{aligned}
\tag{3.2.5}
$$

Using the definition (1.4.4) in equation(3.2.5), we obtain

$$
\begin{aligned}
D^+V_1(e(t)) \leq &\mu_p(-C+\Omega)\|e^R(t)\|_p + (r_k\|A^R\|_p + s_k\|A^I\|_p + \epsilon_1 p_k\|D^R\|_p + \epsilon_1 q_k\|D^I\|_p)\\
&\times(\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k\|B^R\|_p + n_k\|B^I\|_p)(\|e^R(t-\tau(t))\|_p\\
&+ \|e^I(t-\tau(t))\|_p) + \|P\|_p + \mu_p(-C+\Omega)\|e^I(t)\|_p + (s_k\|A^R\|_p + r_k\|A^I\|_p\\
&+ \epsilon_1 q_k\|D^R\|_p + \epsilon_1 p_k\|D^I\|_p)(\|e^R(t)\|_p + \|e^I(t)\|_p) + (n_k\|B^R\|_p + m_k\|B^I\|_p)\\
&\times(\|e^R(t-\tau(t))\|_p + \|e^I(t-\tau(t))\|_p) + \|Q\|_p\\
\leq &\mu_p(-C+\Omega)(\|e^R(t)\|_p + \|e^I(t)\|_p) + (r_k\|A^R\|_p + s_k\|A^I\|_p + \epsilon_1 p_k\|D^R\|_p\\
&+ \epsilon_1 q_k\|D^I\|_p + s_k\|A^R\|_p + r_k\|A^I\|_p + \epsilon_1 q_k\|D^R\|_p + \epsilon_1 p_k\|D^I\|_p)(\|e^R(t)\|_p\\
&+ \|e^I(t)\|_p) + (m_k\|B^R\|_p + n_k\|B^I\|_p + n_k\|B^R\|_p + m_k\|B^I\|_p)(\|e^R(t-\tau(t))\|_p\\
&+ \|e^I(t-\tau(t))\|_p) + (\|P\|_p + \|Q\|_p)\\
\leq &\mu_p(-C+\Omega)(\|e^R(t)\|_p + \|e^I(t)\|_p) + \Big\{(r_k+s_k)(\|A^R\|_p + \|A^I\|_p)\\
&+ \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p)\Big\}(\|e^R(t)\|_p + \|e^I(t)\|_p)\\
&+ (m_k+n_k)(\|B^R\|_p + \|B^I\|_p)(\|e^R(t-\tau(t))\|_p + \|e^I(t-\tau(t))\|_p)\\
&+ (\|P\|_p + \|Q\|_p)\\
\leq &\Big\{\mu_p(-C+\Omega) + (r_k+s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\times (\|e^R(t)\|_p + \|e^I(t)\|_p) + (m_k + n_k)(\|B^R\|_p + \|B^I\|_p) \\
&\times (\|e^R(t-\tau(t))\|_p + \|e^I(t-\tau(t))\|_p) + (\|P\|_p + \|Q\|_p) \\
\leq &\Big\{ \mu_p(-C+\Omega) + (r_k + s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p \\
&+ \|D^I\|_p) \Big\} V_1(e(t)) + (m_k+n_k)(\|B^R\|_p + \|B^I\|_p) V_1(e(t-\tau(t))) + (\|P\|_p + \|Q\|_p) \\
\leq &\Big\{ \mu_p(-C+\Omega) + (r_k + s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p) \Big\} V_1(e(t)) \\
&+ (m_k+n_k)(\|B^R\|_p + \|B^I\|_p) sup_{-\tau\leq s\leq 0} V_1(e(s)) + (\|P\|_p + \|Q\|_p). \\
&\leq k_1 V_1(e(t)) + k_2 sup_{-\tau\leq s\leq 0} V_1(e(s)) + k_3. \qquad (3.2.6)
\end{aligned}
$$

Let, $k_1 = -\Big\{ \mu_p(-C+\Omega) + (r_k+s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p) \Big\}$, $k_2 = (m_k+n_k)(\|B^R\|_p + \|B^I\|_p)$ and $k_3 = (\|P\|_p + \|Q\|_p)$.

Then from the condition (3.2.1), we have $0 < k_2 < k_1$ and using Lemma 3.1, we get

$$
V_1(e(t)) \leq \; sup_{-\tau\leq s\leq 0} V_1(e(s)) e^{-rt} + \frac{k_3}{r}, \qquad (3.2.7)
$$

where unique positive solution $r > 0$ is estimated as

$$
\begin{aligned}
r =& k_1 - k_2 e^{r\tau} \\
=& -\Big\{ \mu_p(-C+\Omega) + (r_k+s_k)(\|A^R\|_p + \|A^I\|_p) + \epsilon_1(p_k+q_k)(\|D^R\|_p + \|D^I\|_p) \Big\} \\
&- (m_k+n_k)(\|B^R\|_p + \|B^I\|_p) e^{r\tau}. \qquad (3.2.8)
\end{aligned}
$$

Therefore the error system e(t) converges exponentially to attractive region $\gamma = \{e(t) : \|e(t)\| \leq \frac{k_3}{r}\}$. Hence, we can conclude from Definition 1.5.2 that the required quasi-synchronization between CVRNNs (3.1.1) and (3.1.3) is obtained. The proof is completed. □

Corollary 3.3. *Under the Assumption 2, the drive system (3.1.1) and the response system (3.1.3) with time-varying delay and interaction terms will be globally quasi-synchronized, if the coupling matrix* Ω *satisfies the following condition*

$$\begin{aligned} 0 <&(m_k+n_k)(\|B^R\|_p+\|B^I\|_p) \\ &< -\Big\{\mu_p(-C)+\mu(\Omega)+(r_k+s_k)(\|A^R\|_p+\|A^I\|_p)+\epsilon_1(p_k+q_k) \\ &(\|D^R\|_p+\|D^I\|_p)\Big\}, \end{aligned} \tag{3.2.9}$$

where $p = 1, 2, \infty$.

Meanwhile, in this region $\|e(t)\| \leq \frac{k_3}{r}$, *the synchronization error converges exponentially, where* r *is the unique solution of* $r = k_1 - k_2 e^{r\tau}$.

Proof. Using Lemma 1.3, we get

$$\mu_p(-C+\Omega) \leq \mu_p(-C) + \mu_p(\Omega), \tag{3.2.10}$$

which gives

$$\begin{aligned} &-\{\mu_p(-C)+\mu_p(\Omega)+(r_k+s_k)(\|A^R\|_p+\|A^I\|_p)+\epsilon_1(p_k+q_k)(\|D^R\|_p+\|D^I\|_p)\} \\ &\quad < -\{\mu_p(-C+\Omega)+(r_k+s_k)(\|A^R\|_p+\|A^I\|_p)+\epsilon_1(p_k+q_k)(\|D^R\|_p+\|D^I\|_p)\}. \end{aligned}$$

Therefore from equation(3.2.9), we have

$$\begin{aligned} 0 <&(m_k+n_k)(\|B^R\|_p+\|B^I\|_p) \\ &< -\Big\{\mu_p(-C)+\mu_p(\Omega)+(r_k+s_k)(\|A^R\|_p+\|A^I\|_p)+\epsilon_1(p_k+q_k) \\ &(\|D^R\|_p+\|D^I\|_p)\Big\}. \end{aligned} \tag{3.2.11}$$

Hence, from the Theorem 3.2, we can conclude that drive-response systems (3.1.1) and (3.1.3) will be globally quasi-synchronized with the help of the controllers given in (3.1.5) and (3.1.6). □

3.3 Numerical Example

In this section, a numerical example is taken to validate the effectiveness and reliability of the theoretical results given in previous section.

Example 3.3.1. Let us assume CVRNNs with interaction and time-varying delay terms as the drive system as

$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t-\tau(t))) + \epsilon_1 Dh(\widetilde{w}(t)) + L(t), \tag{3.3.1}$$

with the following parameters

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 2+8i & -0.1-0.2i \\ -8-1.2i & 8+4i \end{pmatrix}, B = \begin{pmatrix} -1.2+1.2i & -0.1-0.2i \\ -0.2-0.2i & -1+i \end{pmatrix},$$

$$D = \begin{pmatrix} -1.1+0.2i & -0.1-0.3i \\ -0.2-0.3i & -3+0.5i \end{pmatrix}, \Omega = \begin{pmatrix} -64 & 10 \\ 15 & -60 \end{pmatrix}, P = \begin{pmatrix} -0.4 & 0.2 \\ 0.1 & -0.3 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.2 & -0.12 \\ -0.2 & 0.3 \end{pmatrix}, L(t) = \begin{pmatrix} e^{-3t^2} + ie^{-5t^2} \\ e^{-2t^2} + ie^{-3t^2} \end{pmatrix}, \tau(t) = 0.1cos(t).$$

The activation functions are taken as

$$f_p(w_p(t)) = \frac{|u_p+1|-|u_p-1|}{2} + i\frac{|v_p+1|-|v_p-1|}{2},$$
$$g_p(w_p(t)) = \frac{1}{1+e^{(-u_p+2v_p)}} + i\frac{1-e^{(-2u_p-v_p)}}{1+e^{(-2u_p-v_p)}},$$
$$h_p(w_p(t)) = \frac{1-e^{(-v_p)}}{1+e^{(-v_p)}} + i\frac{1}{1+e^{(-u_p)}}(p=1,2).$$

Figures 3.1 and 3.2 show the 3-D plots of the trajectories $u_1(t), v_1(t)$ and $u_2(t), v_2(t)$ with respect to time t of the system (3.3.1) with time-varying delay $\tau(t) = 0.1cos(t)$ and interaction strength $\epsilon_1 = 0.5$.

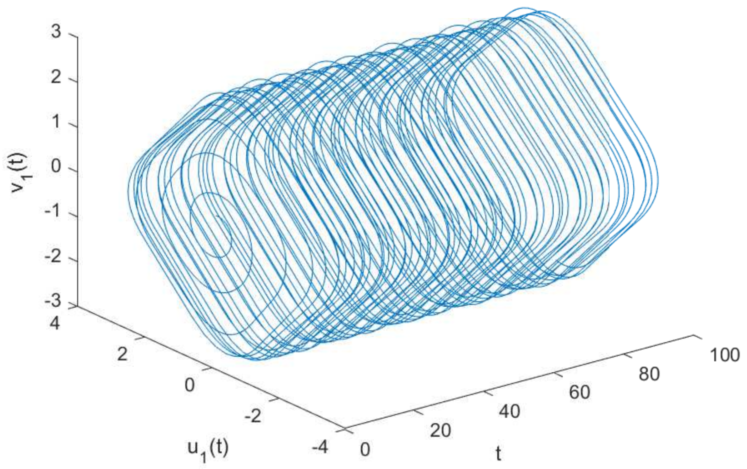

Figure 3.1: Plots of the state trajectories $u_1(t), v_1(t)$ with respect to time t in 3-dimensional space of the system (3.3.1) with time varying delay term $\tau(t) = 0.1cos(t)$.

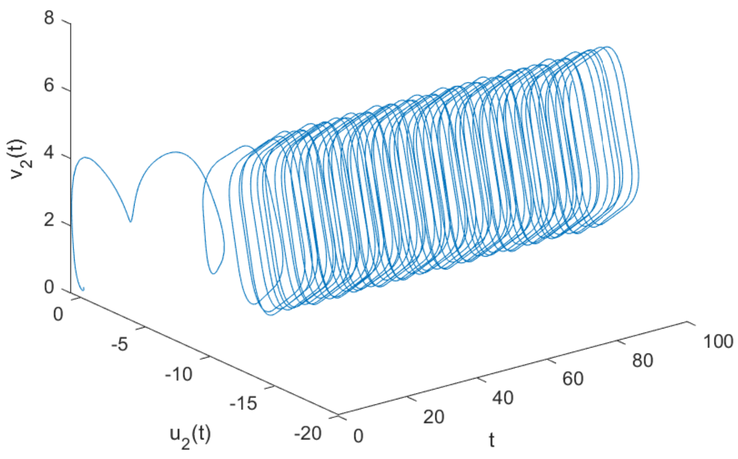

Figure 3.2: Plots of the state trajectories $u_2(t)$ and $v_2(t)$ with respect to time t in 3-dimensional space of the system (3.3.2) with time varying delay term $\tau(t) = 0.1cos(t)$.

The corresponding response system (3.1.3) with the same parameters is expressed as

$$\dot{\widetilde{w}}(t) = -C\widetilde{w}(t) + Af(\widetilde{w}(t)) + Bg(\widetilde{w}(t - \tau(t))) + \epsilon_1 Dh(w(t)) + L(t) + M(t), \tag{3.3.2}$$

where $M(t) = \Omega(\widetilde{w}(t) - w(t))$ is the coupling control matrix.

Let us assume p=2, and the system satisfies the Assumption 2. The system parameters are taken as $r_k = 1, s_k = 1, m_k = \sqrt{2}/2$, $n_k = \sqrt{2}$, $p_k = 0.5$ and $q_k = 0.25$. Taking the initial conditions of drive- response systems (3.3.1) and (3.3.2) as

$w_1(s) = -0.15 + 0.45i, w_2(s) = -0.25 + 0.35i,$

$\widetilde{w}_1(s) = -0.30 - 0.30i,\ \widetilde{w}_2(s) = 0.30 + 0.30i,$

Then, we can verify that $k_1 = 10.4201$, $k_2 = 5.5257$, i.e., $k_1 > k_2$, and error level $\delta = 0.3131$. Hence, all the conditions of the Theorem 3.2 hold. Therefore, in view of the theorem 3.2, the systems (3.3.1) and (3.3.2) will be quasi-synchronized with error bound $\delta = 0.3131$ under the defined controller functions given in equations (3.1.5) and (3.1.6).

Figures 3.3, 3.4, 3.5, and 3.6 confirm that the systems (3.3.1) and (3.3.2) are globally quasi-synchronized with time-varying delay terms and interaction terms. Figure 3.7 also confirms the global quasi-synchronization from the error systems with error bound = 0.3131.

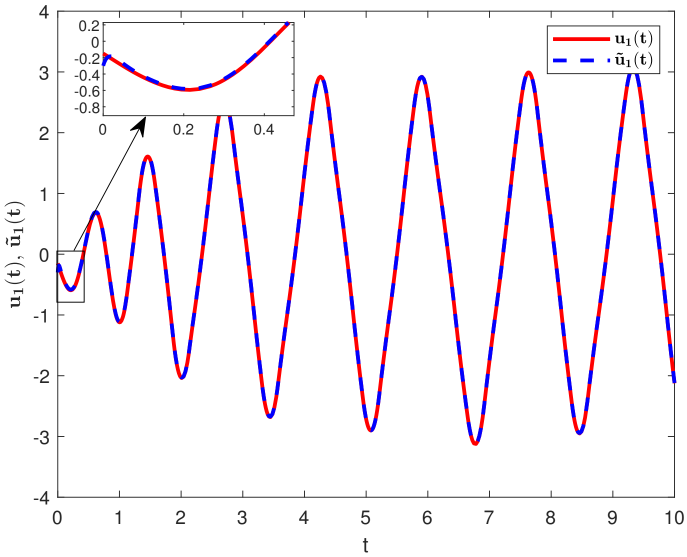

Figure 3.3: Plots of the state trajectories $u_1(t)$ and $\tilde{u}_1(t)$ with respect to time t, of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1cos(t)$.

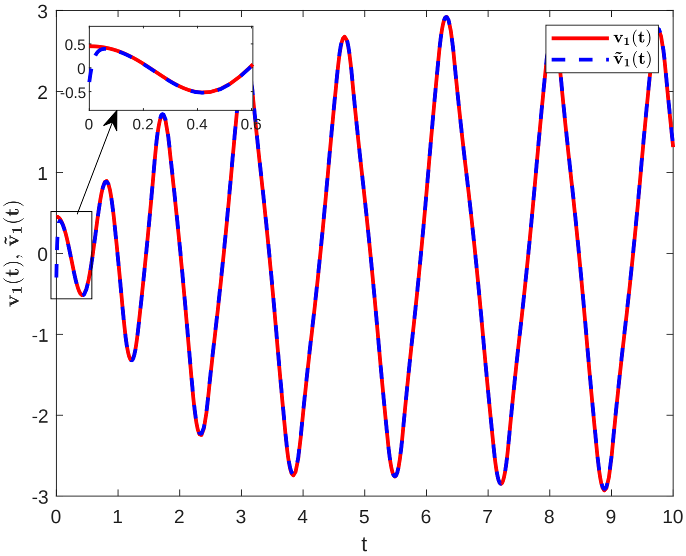

Figure 3.4: Plots of the state trajectories $v_1(t)$ and $\widetilde{v}_1(t)$ with respect to time t, of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1cos(t)$.

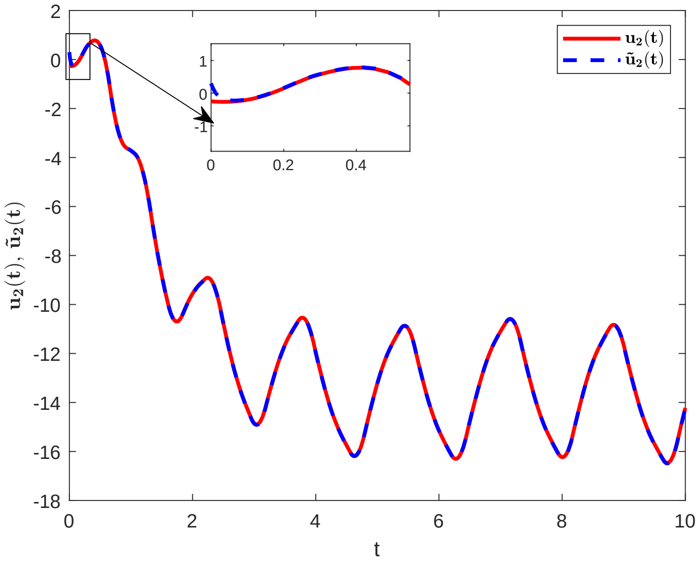

Figure 3.5: Plots of the state trajectories $u_2(t)$ and $\widetilde{u}_2(t)$ with respect to time t, of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1cos(t)$.

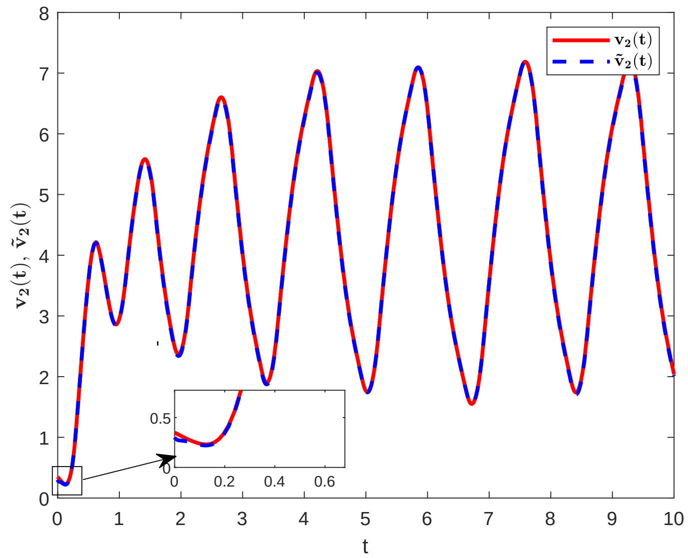

Figure 3.6: Plots of the state trajectories $v_2(t)$ and $\widetilde{v}_2(t)$ with respect to time t, of master system (3.3.1) and response system (3.3.2) with time varying delay term $\tau(t) = 0.1cos(t)$.

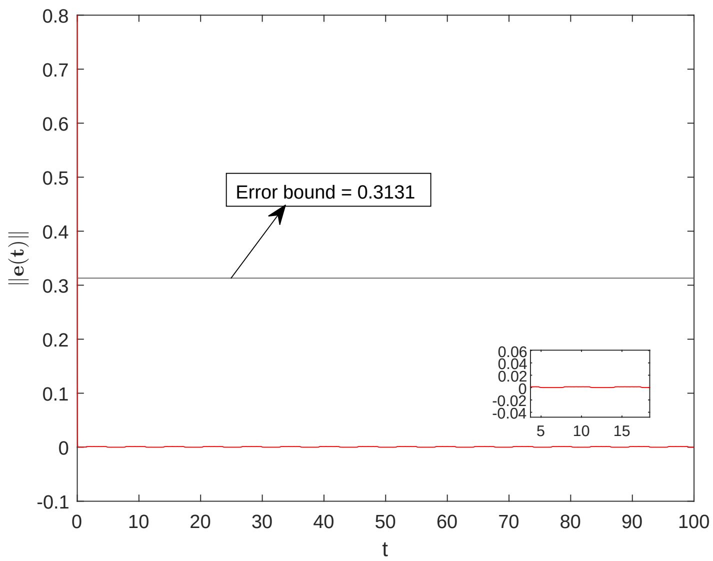

Figure 3.7: Plots of the global quasi-synchronization error $e(t)$ of the systems (3.3.1) and (3.3.2) with time varying delay term as $\tau(t) = 0.1cos(t)$ having error bound $= 0.3131$.

3.4 Conclusion

In this chapter, a drive has been taken to obtain the global quasi-synchronization criteria for CVRNNs in the presence of time-varying delays and interaction terms. The synchronization condition is obtained by using matrix measure method and Lyapunov stability theory. The numerical example is taken to show the feasibility and effectiveness of the obtained global quasi-synchronization results. The main contribution is the showcasing of proper selection of the controller gain matrix to obtain the global quasi-synchronization criteria by using the proposed method. Another important contribution is the accomplishment of the estimation of upper bound of synchronization error.

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