Chapter 5

Numerical study of the dynamics of travelling waves through fourth-order nonlinear time-space fractional sub-diffusion equations of variable-order

5.1 Introduction

Fractional sub-diffusion equations is an important class of anomalous diffusion system which have gained wide attention of researchers in different disciplines such as turbulence, wave propoagation, signal processing, porous media and anomalous diffusion. Time-fractional sub-diffusion equations can be obtained from the classical diffusion equation by replacing the time derivative by a fractional derivative of order τ $(0 < \tau < 1)$. In the time-fractional sub-diffusion equation, anomalous diffusion is described by the nonlinear relation between the mean square displacement of the particles and time i.e.

$$\langle r^2(t) \rangle \sim \frac{2K_\tau}{\Gamma(1+\tau)} t^\tau,$$

where τ is the anomolous diffusion exponent, t is the elapsed time and K_τ is the so-called generalized diffusion coefficient. There are several theoretical and numerical work including second-order time-fractional sub-diffusion equations ([164], [165],

[166], [167]). Time-fractional sub-diffusion models with second-order space derivative cannot describe many physical phenomena such as ice formation [168], wave propagation in beams, fluids on lungs [169], modeling formation of grooves on a flat surface, designing special curves on surfaces, brain warping [170], the description of unique phenomena such the propagation of powerful laser beams in a bulk medium with Kerr nonlinearity [171] etc. So a fourth-order space derivative term is needed to describe such phenomena precisely.

In the few decades it has been observed that fractional differential equations gained much attention of a huge number of mathematicians since it can model many complex physical problem in the fields of sciences, thermodynamics, physics, mechatronics, image processing, aerodynamic, engineering, hydrology finance, geophysics, fluid dynamics and control theory [172–174]. In terms of order dependency, fractional differential operators is influenced into two types. One is characterized by constant-order while the other is characterized by variable-order which is based on a variety of factors such as space, time coordinates. Numerous physical phenomena such as anomalous diffusion, signal processing, viscoelastic materials, noise reduction, turbulence, signature verification and wave propagation exhibit memory and non-local behaviour, are described by fractional-order differential equations with the change of spatial and temporal variables ([54, 175–177]). Fractional differential equations can be used to capture the long-term effects of past events on a system's current state, which makes them very useful for modeling the real-life problem. For example, to analyze the dynamics of coronavirus diffusion, Zika virus, HIV and turberculosis diseases [178–180]. In conclusion, fractional-operators has become an indispensable property are their greater degree of freedom than integer order derivatives and their heritability [94]. Additionaly, there are much considerable fractional partial differential equations which do not have analytical solutions. For this reason many numerical methods have been developed as a suitable alternative to simulate such problems so existing numerical approach which have been applied in recent years such as Petrov-Galerkin algorithm [181, 182], Jacobi Tau method [183], Sinc-Legendre collocation techiques [184], local discontinuous Galerkin techniques [185, 186], meshless methods, finite element techniques [187, 188], spectral techniques [189], finite difference techniques [190]. There are lots of prominent work among them. For example, Zeng et al. [191] discussed the time-fractional sub-diffusion equations with space discretized by the finite element method and time discretized by the fractional linear multistep method. Cui ji et al. [192] presented a compact difference scheme for the fractional sub-diffusion equations and also discussed the unconditional stability and convergency by discrete energy method. M. H. Heydari [193] proposed Galerkin method based on the Legendre wavelets for solving time fractional sub-diffusion equations in the Riemann-Liouville sense.

It has been seen that the fractional-order partial differential equation is failed to model a variety of significant dynamical problems which may vary with time, space, or other conditions. To model an effective mathematical framework for accurately describing such phenomenon, variable-order fractional calculus is introduced. The concept of variable-order differental operator have been investigated by Hartly and Lorenzo [56] in which the order can change with time or space. Even more Coimbra [194] also have been studied the new introduced variable-order operator. They are also discussed the variable-order differentiation and integration as well. In 1993 and 1995, the basic properties of Variable-order Marchaud and Reimann-Liouville fractional calculus are given by Ross and Samko [195], [196]. By using the variable-order (VO) subordination, the author [197] derived the Filler semigroups. Pseudo-differential operators have been studied in [198] utilizing the VO-generating Filler semigroups and by applying this form of VO pseudo-differential operators, the

Markov processes have been characterized by Kikuchi and Negoro [199]. Lorenzo and Hartley have explored the distributed order and VO fractional operators in [200].

In the last decades, various work based on time-fractional sub-diffusion with fourth-order space derivative terms have been done. Zhang et al. [201] presented a general solution for fourth-order time-fractional sub-diffusion equations with the help of compact difference scheme. Ran et al. [202] solved the fourth-order time-fractional sub-diffusion equations of the distributed order defined in bounded domain. Cui ji et al. [203] investigated the general solution of fourth-order time-fractional sub-diffusion equations with first Dirichlet boundary conditions depicted the wave propagation in intense laser beams. Nandal et al. [204] discussed the nonlinear fourth-order distributed fractional sub-diffusion equations with time delay. Zhang et al. [205] developed a compact finite difference scheme for solving a fourth-order time-fractional sub-diffusion equations of variable-order. From the literature survey, the authors have found that there are few reports available on fractional sub-diffusion model of fourth-order. So in this chapter, the authors present the generalized fourth-order non-linear time-space fractional sub-diffusion equation of variable-order with numann boundary conditions for one and two dimensional case

(1+1)-dimensional equation:

$$ {}_0^C D_t^{\alpha(x,t)} u(x,t) + {}_0^C D_x^{\theta(x,t)} \left(\kappa(x,t) {}_0^C D_x^{\theta(x,t)} u(x,t) \right) = h(u,x,t), \tag{5.1} $$

where, $x \in [0, L]$, $t \in (0, T]$, $0 < \alpha(x,t) < 1$, $1 < \theta(x,t) \leq 2$, with the conditions

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq L, \tag{5.2}$$

$$u(0,t) = \omega_1(t), \quad u(0,t) = \omega_2(t), \quad 0 < t \leq T, \tag{5.3}$$

$$\frac{\partial^2 u(0,t)}{\partial \vartheta^2} = \omega_3(t), \quad \frac{\partial^2 u(L,t)}{\partial \vartheta^2} = \omega_4(t), \quad 0 < t \leq T, \tag{5.4}$$

(1+2)-dimensional equation:

$${}_0^C D_t^{\delta(x,y,t)} u(x,y,t) + \Delta_{\beta(x,y,t),\gamma(x,y,t)}(w(x,y,t)\Delta_{\beta(x,y,t),\gamma(x,y,t)} u(x,y,t)) = f(u,x,y,t), \tag{5.5}$$

where $(x,y) \in \Omega = [0,L] \times [0,L], \quad t \in (0,T]$,

combined with the following initial and boundary conditions

$$u(x,y,0) = \sigma(x,y), \qquad (x,y) \in \Omega, \tag{5.6}$$

$$u(0,y,t) = \phi_1(y,t), \quad u(L,y,t) = \phi_2(y,t), \quad y \in [0,L], \quad 0 < t \leq T, \tag{5.7}$$

$$u(x,0,t) = \phi_3(x,t), \quad u(x,L,t) = \phi_4(x,t), \quad x \in [0,L], \quad 0 < t \leq T, \tag{5.8}$$

$$\Delta u(0,y,t) = \phi_5(y,t), \quad \Delta u(L,y,t) = \phi_6(y,t), \quad y \in [0,L], \quad 0 < t \leq T, \tag{5.9}$$

$$\Delta u(x,0,t) = \phi_7(x,t), \quad \Delta u(x,L,t) = \phi_8(x,t), \quad x \in [0,L], \quad 0 < t \leq T, \tag{5.10}$$

where Δ is the Laplacian operator, $\Delta_{\beta(x,y,t),\gamma(x,y,t)} = {}_0^C D_x^{\beta(x,y,t)} + {}_0^C D_y^{\gamma(x,y,t)}$ and the fractional order are $1 < \beta(x,y,t),\ \gamma(x,y,t) \leq 2$, and for $m-1 < \beta(x,y,t), \gamma(x,y,t) < m$, $m \in \mathbb{N}$, ${}_0^C D_x^{\beta(x,y,t)}$, and ${}_0^C D_y^{\beta(x,y,t)}$ are Caputo fractional derivatives of variable order with respect to the space variables x and y, respectively. Here $\kappa(x,t)$, $h(u,x,t)$, $\varphi(x)$, $w_1(t)$, $w_2(t)$, $w_3(t)$, $w_4(t)$, $\sigma(x,t)$, $\phi_1(y,t)$, $\phi_2(y,t)$, $\phi_3(x,t)$, $\phi_4(x,t)$, $\phi_5(y,t)$, $\phi_6(y,t)$, $\phi_7(x,t)$, $\phi_8(x,t)$, $w(x,y,t)$, $g(u)$ and $f(x,y,t)$ are given smooth functions.

Also $0 < \delta(x, y, t) < 1$.

The following is a breakdown for chapter 5. In section 5.2, the features of the shifted airfoil polynomials of second kind are described. The approximation of an arbitrary function in terms of shifted airfoil polynomials of second kind and formation of operational matrices for differentiation of the Shifted airfoil polynomials of second kind are also discussed in section 5.3 and 5.4. A brief description of the scheme for general case is described in section 5.5. section 5.6 will wrap up the conversation on convergence and error analysis. The numerical computation of this study are described in the section 5.7, and finally conclusion is presented in section 5.8.

5.2 Shifted Airfoil Polynomials and their Properties

The shifted airfoil polynomials of second kind $\mathrm{A}_m(x)$ for degree n on the interval [0,1] satisfies the following recurrence relation:

$$\mathrm{A}^*_{m+1}(x) = 2(2x-1)\mathrm{A}^*_m(x) - \mathrm{A}^*_{m-1}(x), \quad m = 2, 3, ... \tag{5.11}$$

with the values

$$\mathrm{A}^*_0(x) = 1, \qquad \mathrm{A}^*_1(x) = 4x - 1. \tag{5.12}$$

The explicit expansion form of the $\mathrm{A}_m(x)$ can be expressed as

$$\mathrm{A}^*_m(x) = \sum_{r=0}^{m} (-1)^{m-r} \frac{2^r \Gamma(m+r+1)}{\Gamma(m-r+1)\Gamma(2r+1)} x^r, \quad m = 2, 3, ..., \tag{5.13}$$

The family of shifted airfoil polynomials of second kind constitute an orthogonal system with regard to weight function $\sqrt{\frac{1-x}{x}}$ as given below:

$$\langle \mathrm{A}^*_{m_1}(x), \mathrm{A}^*_{m_2}(x)\rangle = \int_0^1 \sqrt{\frac{1-x}{x}}\mathrm{A}^*_{m_1}(x)\mathrm{A}^*_{m_2}(x)dx = \begin{cases} 0, & m_1 \neq m_2, \\ \frac{\pi}{2}, & m_1 = m_2. \end{cases} \tag{5.14}$$

where m_1, $m_2 \in \mathbb{N} \cup \{0\}$.

5.3 Function Approximation

Now, suppose the approximate function $u(x,t)$ which belongs to $L^2[0,1] \times L^2[0,1]$ of two variables can be written in the following ways:

$$u(x,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij}\mathrm{A}^*_i(x)\mathrm{A}^*_j(t), \tag{5.15}$$

where the coefficients c_{ij} can be calculated as

$$c_{ij} = \frac{4}{\pi^2}\int_0^1\int_0^1 \sqrt{\frac{1-x}{x}}\sqrt{\frac{1-t}{t}}u(x,t)\mathrm{A}^*_i(x)\mathrm{A}^*_j(t)dxdt, \qquad i,j = 0,1,2,... \tag{5.16}$$

To approximate $u(x,t)$, an endeavour will use the above infinte series as follows

$$u(x,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij}\mathrm{A}^*_i(x)\mathrm{A}^*_j(t) = \varphi(x)^T\mathbb{U}\varphi(t), \tag{5.17}$$

where

$$\varphi(x) = [\mathrm{A}^*_0(x), \mathrm{A}^*_1(x), ..., \mathrm{A}^*_m(x)]^T, \tag{5.18}$$

and $\mathbb{U}$=$[c_{ij}]_{i,j=0}^{m,m}$ is the $(m+1) \times (m+1)$ matrix of entries.

Similiarly, a desired function $u(x,y,t) \in L^2[0,1] \times L^2[0,1] \times L^2[0,1]$ can be written in terms of shifted airfoil polynomials of second kind as:

$$u(x,y,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t) = \varphi^T(t)\mathbb{V}(\varphi(x)\otimes\varphi(y)), \tag{5.19}$$

where matrix $\mathbb{V}$=$[c_{ijk}]_{i,j,k=0}^{m,m,m}$ is of order $(m+1)\times(m+1)^2$ whose entries are

$$c_{ijk} = \frac{8}{\pi^3}\int_0^1\int_0^1\int_0^1 \sqrt{\frac{1-x}{x}}\sqrt{\frac{1-y}{y}}\sqrt{\frac{1-t}{t}}u(x,y,t)\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t)dxdydt, \tag{5.20}$$

where $i,j,k = 0,1,2,...$

5.4 Construction of Operational Matrix for Variable Derivatives

The first-order derivative of the shifted airfoil vector $\varphi(x)$ is defined as follows:

$$\frac{d\varphi(x)}{dx} = \mathbf{D}_{ij}\varphi(x), \tag{5.21}$$

where $\mathbf{D}_{ij}$ is the $(m+1)\times(m+1)$ operational matrix of the shifted airfoil vector $\varphi(x)$ for first-order derivative with the following entries:

$$\mathbf{D}_{ij} = \begin{cases} 0, & \text{for } i \geq j, \\ 2(-i+j), & \text{for } i < j,\ (i+j) \text{ is even}, \\ 2(i+j+1), & \text{for } i < j,\ (i+j) \text{ is odd}. \end{cases} \tag{5.22}$$

Theorem 5.1. *The fractional Caputo derivative of order $\mu(x,y,t)$ for the shifted airfoil vector $\varphi(x)$ is presented as*

$$D_x^{\mu(x,y,t)}\varphi(x) = \Psi_x^{\mu(x,y,t)}\varphi(x), \tag{5.23}$$

where $\Psi_x^{\mu(x,y,t)}$ is the $(m+1)\times(m+1)$ operational matrix of Caputo fractional derivative of variable-order, $\mu(x,y,t)$ that is defined as follows:

$$\Psi_x^{\mu(x,y,t)} = x^{-\mu(x,y,t)}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{i=q}^{q}\Theta_{q,l,i,0}^{\mu(x,y,t)} & \sum_{i=q}^{q}\Theta_{q,l,i,1}^{\mu(x,y,t)} & \cdots & \sum_{i=q}^{q}\Theta_{q,l,i,m}^{\mu(x,y,t)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=q}^{n}\Theta_{n,l,i,0}^{\mu(x,y,t)} & \sum_{i=q}^{n}\Theta_{n,l,i,1}^{\mu(x,y,t)} & \cdots & \sum_{i=q}^{n}\Theta_{n,l,i,m}^{\mu(x,y,t)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=q}^{m}\Theta_{m,l,i,0}^{\mu(x,y,t)} & \sum_{i=q}^{m}\Theta_{m,l,i,1}^{\mu(x,y,t)} & \cdots & \sum_{i=q}^{m}\Theta_{m,l,i,m}^{\mu(x,y,t)} \end{pmatrix}, \tag{5.24}$$

where

$$\Theta_{n,l,i.j}^{\mu(x,y,t)} = \frac{1}{\sqrt{\pi}}\sum_{l=0}^{j}\frac{(-1)^{(n+j-l-i)}(2)^{2(l+i)}\Gamma(n+j+l+i+2)\Gamma(i+1)}{\Gamma(n-i+1)\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2i+1)\Gamma(i+1-\mu(x,y,t))} \times\left(\frac{\Gamma(i+l+1/2)}{\Gamma(i+l+2)}\right). \tag{5.25}$$

Proof. By using the operator $D_x^{\mu(x,y,t)}$ on the function $\mathrm{A}_n^*(x)$, we get

$$\begin{aligned} D_x^{\mu(x,y,t)}\mathrm{A}_n^*(x) &= \sum_{i=0}^{n} \frac{(-1)^{n-i}(2)^{2i}\Gamma(n+i+1)}{\Gamma(n-i+1)\Gamma(2i+1)} D_x^{\mu(x,y,t)}(x^i), \quad n=q,...,m, \\ &= \sum_{i=q}^{n} \frac{(-1)^{n-i}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i+1)\Gamma(2i+1)\Gamma(i+1-\mu(x,y,t))} x^{i-\mu(x,y,t)}, \\ &= x^{-\mu(x,y,t)} \sum_{i=q}^{n} \frac{(-1)^{n-i}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i+1)\Gamma(2i+1)\Gamma(i+1-\mu(x,y,t))} x^{i}. \end{aligned} \tag{5.26}$$

Now, we approximate x^i as

$$x^i = \sum_{j=0}^{m} b_{ij}\mathrm{A}_j^*(x), \tag{5.27}$$

where

$$\begin{aligned} b_{ij} &= \frac{2}{\pi}\int_0^1 x^i \sqrt{\frac{1-x}{x}}\mathrm{A}_j^*(x)dx, \\ &= \frac{2}{\pi}\sum_{l=0}^{j} \frac{(-1)^{j-l}(2)^{2l}\Gamma(j+l+1)}{\Gamma(j-l+1)\Gamma(2l+1)} \int_0^1 x^{i+l}\sqrt{\frac{1-x}{x}}dx, \\ &= \frac{2}{\pi}\sum_{l=0}^{j} \frac{(-1)^{j-l}(2)^{2l}\Gamma(j+l+1)}{\Gamma(j-l+1)\Gamma(2l+1)} \left(\frac{\sqrt{\pi}}{2}\frac{\Gamma(i+l+1/2)}{\Gamma(i+l+2)}\right). \end{aligned} \tag{5.28}$$

Substituting Eq. (5.27) into Eq. (5.26), we get

$$D_x^{\mu(x,y,t)}\mathrm{A}_n^*(x) = x^{-\mu(x,y,t)} \sum_{i=q}^{n}\sum_{j=0}^{m} \frac{(-1)^{n-i}(2)^{2i}\Gamma(n+i+1)\Gamma(i+1)}{\Gamma(n-i+1)\Gamma(2i+1)\Gamma(i+1-\mu(x,y,t))} b_{ij}\mathrm{A}_j^*(x), \tag{5.29}$$

where $n = q, ..., m$.

Now, let us write

$$D_x^{\mu(x,y,t)} \mathrm{A}_n^*(x) = \sum_{j=0}^{m} \left(\sum_{i=q}^{n} \Theta_{n,l,i,j}^{\mu(x,y,t)} \right) \mathrm{A}_j^*(x), \quad n = q, ..., m, \tag{5.30}$$

where

$$\Theta_{n,l,i.j}^{\mu(x,y,t)} = \frac{1}{\sqrt{\pi}} \sum_{l=0}^{j} \frac{(-1)^{(n+j-l-i)}(2)^{2(l+i)}\Gamma(n+j+l+i+2)\Gamma(i+1)}{\Gamma(n-i+1)\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2i+1)\Gamma(i+1-\mu(x,y,t))} \times \left(\frac{\Gamma(i+l+1/2)}{\Gamma(i+l+2)} \right). \tag{5.31}$$

Let us write Eq. (5.30) in a vector form as

$$D_x^{\mu(x,y,t)} \mathrm{A}_n^*(x) \simeq \left[\sum_{i=q}^{n} \Theta_{n,l,i,0}^{\mu(x,y,t)}, \sum_{i=q}^{n} \Theta_{n,l,i,1}^{\mu(x,y,t)}, ..., \sum_{i=q}^{n} \Theta_{n,l,i,m}^{\mu(x,y,t)} \right] \varphi(x), \tag{5.32}$$

where $n = q, ..., m$.

Also according to Lemma, we have

$$D_x^{\mu(x,y,t)} \mathrm{A}_n^*(x) = [0, 0, ..., 0] \, \varphi(x), \quad n = 1, 2, ..., q-1. \tag{5.33}$$

The Eqs. (5.32) and (5.33) gives the desired result. □

5.5 Solution Methodology

In this section, we are trying to find the approximate solution of a nonlinear fractional reaction sub-diffusion equation of fourth-order for one-dimensional (5.1)-(5.4) and two-dimensional (5.5)-(5.10) based on the operational matrix scheme associated with the shifted airfoil polynomials of second kind.

Case 1: Firstly, we shall use the approximation given in Eq. (5.15) to approximate each terms of (5.1)-(5.4). Now, substituting the approximation in the Eqs. (5.1)-(5.4), we get

$$\begin{aligned}\varphi(x)^T\mathbb{U}({}_0^{MABC}D_t^{\alpha(x,t)}\varphi(t)) + D_x^{\theta(x,t)}\left(\kappa(x,t)(D_x^{\theta(x,t)}\varphi(x))^T\mathbb{U}\varphi(t)\right)\\ = u(x,t) + v(\varphi(x)^T\mathbb{U}\varphi(t))\end{aligned} \tag{5.34}$$

$$\begin{aligned}&\varphi(x)^T\mathbb{U}\varphi(0) = \phi(x),\\ &\varphi(0)^T\mathbb{U}\varphi(t) = \omega_1(t),\\ &\varphi(L)^T\mathbb{U}\varphi(t) = \omega_2(t),\\ &(D_x^2\varphi(0))^T\mathbb{U}\varphi(t) = \omega_3(t),\\ &(D_x^2\varphi(L))^T\mathbb{U}\varphi(t) = \omega_4(t),\end{aligned} \tag{5.35}$$

To obtain the numerical solution of eqs. (5.1)-(5.4) firstly, we collocate Eq. (5.34) at points $x_i = t_i = \frac{2i+1}{2m+2}$ for i=0,1,2,..,m to get $m(m-3)$ algebraic equations and Eq. (5.35) gives $m(m-3)$ algebraic equations at the same points.After solving this we can get a desirable solution of our proposed fractional reaction sub-diffusion equation of fourth-order.

Case 2: Now, we use the approximation of $u(x, y, t)$ by shifted Airfoil polynomials of second kind which is given in (5.19). Now, substituting the following approximations

$$\begin{aligned}
&\Delta_{\beta(x,y,t),\gamma(x,y,t)}(w(x, y, t)\Delta_{\beta(x,y,t),\gamma(x,y,t)}u(x, y, t))\\
&= 2w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(x)) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ D_x^{\beta(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(y)) \otimes \varphi(y))\\
&+ D_x^{\beta(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ D_y^{\gamma(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(x)) \otimes \varphi(y))\\
&+ D_y^{\gamma(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{2\beta(x,y,t)}\varphi(x)) \otimes \varphi(x))\\
&+ w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{2\gamma(x,y,t)}\varphi(y))),
\end{aligned} \tag{5.36}$$

in the Eq. (5.5), we get

$$\begin{aligned}
&(D_t^{\delta(x,y,t)}\varphi(t))^T\mathbb{V}(\varphi(x) \otimes \varphi(y)) + w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{2\gamma(x,y,t)}\varphi(y)))\\
&+ 2w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(y)) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ D_x^{\beta(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(x)) \otimes \varphi(y))\\
&+ D_x^{\beta(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ D_y^{\gamma(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{\beta(x,y,t)}\varphi(x)) \otimes \varphi(y))\\
&+ D_y^{\gamma(x,y,t)}w(x, y, t)\varphi^T(t)\mathbb{V}(\varphi(x) \otimes (D_y^{\gamma(x,y,t)}\varphi(y)))\\
&+ w(x, y, t)\varphi^T(t)\mathbb{V}((D_x^{2\beta(x,y,t)}\varphi(x)) \otimes \varphi(y))\\
&= f(x, y, t) + g(\varphi^T(t)\mathbb{A}(\varphi(x) \otimes \varphi(y))),
\end{aligned} \tag{5.37}$$

from the conditions (5.6)-(5.10) and the approximation given in Eq. (5.19), we get

$$\begin{aligned}
&\varphi^T(0)\mathbb{V}(\varphi(x)\otimes\varphi(y)) = \sigma(x,y),\\
&\varphi^T(t)\mathbb{V}(\varphi(0)\otimes\varphi(y)) = \phi_1(y,t),\\
&\varphi^T(t)\mathbb{V}(\varphi(L)\otimes\varphi(y)) = \phi_2(y,t),\\
&\varphi^T(t)\mathbb{V}(\varphi(x)\otimes\varphi(0)) = \phi_3(x,t),\\
&\varphi^T(t)\mathbb{V}(\varphi(x)\otimes\varphi(L)) = \phi_4(x,t),\\
&\varphi^T(t)\mathbb{V}((D^2\varphi(0))\otimes\varphi(y)) + \varphi^T(t)\mathbb{V}(\varphi(0)\otimes(D^2\varphi(y))) = \phi_5(y,t),\\
&\varphi^T(t)\mathbb{V}((D^2\varphi(L))\otimes\varphi(y)) + \varphi^T(t)\mathbb{V}(\varphi(L)\otimes(D^2\varphi(y))) = \phi_6(y,t),\\
&\varphi^T(t)\mathbb{V}((D^2\varphi(x))\otimes\varphi(0)) + \varphi^T(t)\mathbb{V}(\varphi(x)\otimes(D^2\varphi(0))) = \phi_7(x,t),\\
&\varphi^T(t)\mathbb{V}((D^2\varphi(x))\otimes\varphi(L)) + \varphi^T(t)\mathbb{V}(\varphi(x)\otimes(D^2\varphi(L))) = \phi_8(x,t).
\end{aligned} \tag{5.38}$$

In the same ways, we collocate Eq. (5.37) and Eq. (5.38) at points $x_i = y_i = t_i = \frac{2i-1}{2m+1}$ for $i = 0, 1, ..., m$ to get a set of $(m+1)^3$ nonlinear algebraic equations.This non-linear system produces the matrix $\mathbb{V}$ which yields the numerical solution of our proposed fractional fourth-order sub-diffusion equation (5.5)-(5.10).

5.6 Error and Convergence Analysis

Theorem 5.2. *Suppose that $\varphi(t)^T\mathbb{V}(\varphi(x)\otimes\varphi(y))$ is the approximation of $u(x,y,t)$ by shifted airfoil polynomials of second kind. If the function $u(x,y,t)$ has sixth-order* *continuous derivatives, then*

$$
\begin{aligned}
&\|c_{000}\| \leq 8L_{0,0,0},\ \|c_{100}\| \leq 8L_{0,0,0},\ \|c_{010}\| \leq 8L_{0,0,0},\ \|c_{001}\| \leq 8L_{0,0,0},\\
&\|c_{110}\| \leq 8L_{0,0,0},\ \|c_{011}\| \leq 8L_{0,0,0},\ \|c_{101}\| \leq 8L_{0,0,0},\ \|c_{111}\| \leq 8L_{0,0,0},\\
&\|c_{i00}\| \leq \frac{L_{2,0,0}}{(i-1)^2},\ \|c_{i01}\| \leq \frac{L_{2,0,0}}{(i-1)^2},\ \|c_{i10}\| \leq \frac{L_{2,0,0}}{(i-1)^2},\ \|c_{i11}\| \leq \frac{L_{2,0,0}}{(i-1)^2},\\
&\|c_{0j0}\| \leq \frac{L_{0,2,0}}{(j-1)^2},\ \|c_{1j0}\| \leq \frac{L_{0,2,0}}{(j-1)^2},\ \|c_{0j1}\| \leq \frac{L_{0,2,0}}{(j-1)^2},\ \|c_{1j1}\| \leq \frac{L_{0,2,0}}{(j-1)^2},\\
&\|c_{00k}\| \leq \frac{L_{0,0,2}}{(k-1)^2},\ \|c_{01k}\| \leq \frac{L_{0,0,2}}{(k-1)^2},\ \|c_{10k}\| \leq \frac{L_{0,0,2}}{(k-1)^2},\ \|c_{11k}\| \leq \frac{L_{0,0,2}}{(k-1)^2},\\
&\|c_{ij0}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2},\ \|c_{ij1}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2},\ \|c_{0jk}\| \leq \frac{L_{0,2,2}}{(i-1)^2(k-1)^2},\\
&\|c_{1jk}\| \leq \frac{L_{0,2,2}}{(i-1)^2(k-1)^2}, \|c_{i0k}\| \leq \frac{L_{2,0,2}}{(i-1)^2(k-1)^2},\ \|c_{i1k}\| \leq \frac{L_{2,0,2}}{(i-1)^2(k-1)^2},\\
&\|c_{ijk}\| \leq \frac{L_{2,2,2}}{(i-1)^2(j-1)^2(k-1)^2},\quad \textit{for}\ \ i,j,k > 2,
\end{aligned}
$$

where

$$
\begin{aligned}
L_{0,0,0} &= max\left\{\|u(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,0} &= max\left\{\|u_{xx}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,0} &= max\left\{\|u_{yy}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,0,2} &= max\left\{\|u_{tt}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,0} &= max\left\{\|u_{xxyy}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{0,2,2} &= max\left\{\|u_{yytt}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,0,2} &= max\left\{\|u_{xxtt}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\},\\
L_{2,2,2} &= max\left\{\|u_{xxyytt}(x,y,t)\| : (x,y,t) \in [0,1]\times[0,1]\times[0,1]\right\}.
\end{aligned}
$$

Proof. Applying Eq. (5.20), the coefficient c_{111} is computed as

$$c_{000} = \frac{8}{\pi^3}\int_0^1\int_0^1\int_0^1 u(x,y,t)\chi(x)\chi(y)\chi(t)dxdydt, \tag{5.39}$$

and by change of variables $2x-1=\cos(\theta_1)$, $2y-1=\cos(\theta_2)$, and $2t-1=\cos(\theta_3)$, we obtain

$$c_{000} = \frac{8}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right) \sin^2\left(\frac{\theta_1}{2}\right)\sin^2\left(\frac{\theta_2}{2}\right)\sin^2\left(\frac{\theta_3}{2}\right)d\theta_1 d\theta_2 d\theta_3. \tag{5.40}$$

From Eq. (5.40), we have $\|c_{000}\| \le 8L_{0,0,0}$.

Next, from Eq. (5.20) we have

$$c_{100} = \frac{8}{\pi^3}\int_0^1\int_0^1\int_0^1 u(x,y,t)\mathrm{A}_1^*(x)\chi(x)\chi(y)\chi(t)dxdydt. \tag{5.41}$$

The same change of variables and the following property

$$\mathrm{A}_r^*(x) = \frac{\sin\left((r+\frac{1}{2})x\right)}{\sin(\frac{1}{2}x)}, \quad s=\frac{\cos(x)+1}{2}, \tag{5.42}$$

leads to

$$c_{100} = \frac{8}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right) \times\sin\left(\frac{3}{2}\theta_1\right)\sin\left(\frac{\theta_1}{2}\right)\sin^2\left(\frac{\theta_2}{2}\right)\sin^2\left(\frac{\theta_3}{2}\right)d\theta_1 d\theta_2 d\theta_3, \tag{5.43}$$

which implies $\|c_{100}\|, \le 8L_{0,0,0}$. Similarly one can conclude that $\|c_{010}\| \le 8L_{0,0,0}$, $\|c_{001}\| \le 8L_{0,0,0}$, $\|c_{110}\| \le 8L_{0,0,0}$, $\|c_{011}\| \le 8L_{0,0,0}$, $\|c_{101}\| \le 8L_{0,0,0}$ and $\|c_{111}\| \le 8L_{0,0,0}$.

The coefficients c_{i00} for $i \leq 2$ are computed as:

$$c_{i00} = \frac{8}{\pi^3} \int_0^1 \int_0^1 \int_0^1 u(x,y,t) \mathrm{A}_i^*(x) \chi(x)\chi(y)\chi(t) dx dy dt. \tag{5.44}$$

With the help of previous change of variables and property (5.42), Eq. (5.44) becomes

$$c_{i00} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2}, \frac{\cos(\theta_2)+1}{2}, \frac{\cos(\theta_3)+1}{2}\right) \times \sin\left((i+\frac{1}{2})\theta_1\right) \sin\left(\frac{\theta_1}{2}\right) \sin^2\left(\frac{\theta_2}{2}\right) \sin^2\left(\frac{\theta_3}{2}\right) d\theta_1 d\theta_2 d\theta_3, \tag{5.45}$$

or equivalently

$$c_{i00} = \frac{4}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2}, \frac{\cos(\theta_2)+1}{2}, \frac{\cos(\theta_3)+1}{2}\right) \times [\cos(i)\theta_1 - \cos(i+1)\theta_1] \sin^2\left(\frac{\theta_2}{2}\right) \sin^2\left(\frac{\theta_3}{2}\right) d\theta_1 d\theta_2 d\theta_3, \tag{5.46}$$

Now, integrating Eq. (5.46) twice with respect to θ_1, we get

$$c_{i00} = \frac{1}{2\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\theta_1\theta_1}\left(\frac{\cos(\theta_1)+1}{2}, \frac{\cos(\theta_2)+1}{2}, \frac{\cos(\theta_3)+1}{2}\right) \times \Lambda_i(\theta_1) \sin^2\left(\frac{\theta_2}{2}\right) \sin^2\left(\frac{\theta_3}{2}\right) d\theta_1 d\theta_2 d\theta_3, \tag{5.47}$$

where

$$\Lambda_i(\theta_1) = \left(\frac{\sin(i-1)\theta_1}{i(i-1)} - \frac{\sin(i+1)\theta_1}{i(i+1)} - \frac{\sin(i\theta_1)}{i(i+1)} + \frac{\sin(i+2)\theta_1}{(i+1)(i+2)}\right) \sin(\theta_1). \tag{5.48}$$

By an easy calculation, we obtain for $i \geq 2$

$$\int_0^\pi \Lambda_i(\theta_1)d\theta_1 = \begin{cases} -\frac{8}{i^2(i+2)^2}, & \text{if } i \text{ is odd,} \\ \frac{8}{(i-1)^2(i+2)^2}, & \text{if } i \text{ is even.} \end{cases} \tag{5.49}$$

After some manipulation, we get

$$\left| \int_0^\pi \Lambda_i(\theta_1)d\theta_1 \right| \leq \frac{8}{(i-1)^2} \tag{5.50}$$

Now from Eqs. (5.47)-(5.50), we have $\|c_{i00}\| < \frac{L_{2,0,0}}{(i-1)^2}$. Similiarly, it can be computed that $\|c_{i01}\| < \frac{L_{2,0,0}}{(i-1)^2}$, $\|c_{i10}\| < \frac{L_{2,0,0}}{(i-1)^2}$, $\|c_{i11}\| < \frac{L_{2,0,0}}{(i-1)^2}$, $\|c_{0j0}\| < \frac{L_{0,2,0}}{(j-1)^2}$, $\|c_{1j0}\| < \frac{L_{0,2,0}}{(j-1)^2}$, $\|c_{0j1}\| < \frac{L_{0,2,0}}{(j-1)^2}$, $\|c_{1j1}\| < \frac{L_{0,2,0}}{(j-1)^2}$, $\|c_{00k}\| < \frac{L_{0,0,2}}{(k-1)^2}$, $\|c_{01k}\| < \frac{L_{0,0,2}}{(k-1)^2}$, $\|c_{10k}\| < \frac{L_{0,0,2}}{(k-1)^2}$ and $\|c_{11k}\| < \frac{L_{0,0,2}}{(k-1)^2}$.

Moreover, for $i, j \geq 2$, we have

$$c_{ij0} = \frac{8}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2}, \frac{\cos(\theta_2)+1}{2}, \frac{\cos(\theta_3)+1}{2}\right) \times \sin\left((i+\frac{1}{2})\theta_1\right) \sin\left(\frac{\theta_1}{2}\right) \sin\left((j+\frac{1}{2})\theta_2\right) \sin\left(\frac{\theta_2}{2}\right) \sin^2\left(\frac{\theta_3}{2}\right) d\theta_1 d\theta_2 d\theta_3, \tag{5.51}$$

Now, integrating Eq. (5.51) twice with respect to θ_1, we get

$$c_{ij0} = \frac{1}{2\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi u_{\theta_1\theta_1}\left(\frac{\cos(\theta_1)+1}{2}, \frac{\cos(\theta_2)+1}{2}, \frac{\cos(\theta_3)+1}{2}\right) \times \Lambda_i(\theta_1) \sin\left((j+\frac{1}{2})\theta_2\right) \sin\left(\frac{\theta_2}{2}\right) \sin^2\left(\frac{\theta_3}{2}\right) d\theta_1 d\theta_2 d\theta_3, \tag{5.52}$$

Again, integrating Eq. (5.44) twice with respect to θ_2, we obtain

$$c_{ij0} = \frac{1}{32\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u_{\theta_1\theta_1\theta_2\theta_2}\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right)$$
$$\times\Lambda_i(\theta_1)\Lambda_j(\theta_2)\sin^2\left(\frac{\theta_3}{2}\right)d\theta_1 d\theta_2 d\theta_3, \qquad (5.53)$$

Taking the absolute value of Eq. (5.53), we obtain

$$\|c_{ij0}\| = ||\frac{1}{32\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u_{\theta_1\theta_1\theta_2\theta_2}\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right)$$
$$\times\Lambda_i(\theta_1)\Lambda_j(\theta_2)\sin^2\left(\frac{\theta_3}{2}\right)d\theta_1 d\theta_2 d\theta_3||. \qquad (5.54)$$

After some mathematical manipulation, we get

$$\|c_{ij0}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2}. \qquad (5.55)$$

Similiarly $\|c_{ij1}\| \leq \frac{L_{2,2,0}}{(i-1)^2(j-1)^2}$, $\|c_{0jk}\| \leq \frac{L_{0,2,2}}{(j-1)^2(k-1)^2}$, $\|c_{1jk}\| \leq \frac{L_{0,2,2}}{(j-1)^2(k-1)^2}$, $\|c_{i0k}\| \leq \frac{L_{0,2,2}}{(i-1)^2(k-1)^2}$, and $\|c_{i1k}\| \leq \frac{L_{2,0,2}}{(i-1)^2(k-1)^2}$.

Moreover, for $i, j, k \geq 2$, we have

$$c_{ijk} = \frac{8}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right)$$
$$\times\sin\left(i+\frac{1}{2}\right)\theta_1\sin\frac{\theta_1}{2}\sin\left(j+\frac{1}{2}\right)\theta_2\sin\frac{\theta_2}{2}\sin\left(k+\frac{1}{2}\right)\theta_3\sin\frac{\theta_3}{2}d\theta_1 d\theta_2 d\theta_3, \qquad (5.56)$$

and by using the above process, we obtain

$$c_{ijk} = \frac{1}{512\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi u_{\theta_1\theta_1\theta_2\theta_2\theta_3\theta_3}\left(\frac{\cos(\theta_1)+1}{2},\frac{\cos(\theta_2)+1}{2},\frac{\cos(\theta_3)+1}{2}\right)$$
$$\times\Lambda_i(\theta_1)\Lambda_j(\theta_2)\Lambda_k(\theta_3)d\theta_1 d\theta_2 d\theta_3, \qquad (5.57)$$

and consequently

$$\|c_{ijk}\| \leq \frac{L_{2,2,2}}{(i-1)^2(j-1)^2(k-1)^2}, \tag{5.58}$$

which completes the proof. □

Theorem 5.3. *Let $\varphi(t)^T\mathbb{V}(\varphi(x)\otimes\varphi(y))$ be the approximation of $u(x,y,t)$ in the terms of shifted airfoil polynomials of second kind. If the function $u(x,y,t)$ has continuous derivatives of the sixth order, then*

$$\left\|u(x,y,t)-\varphi(t)^T\mathbb{V}(\varphi(x)\otimes\varphi(y))\right\|_{L^2} \leq \sqrt{G(i,j,k)}, \tag{5.59}$$

where

$$\begin{aligned}
G(i,j,k) =& \frac{\pi^3}{2}\sum_{k=m+1}^{\infty}\frac{L_{0,0,2}^2}{(k-2)^4} + \frac{\pi^3}{2}\sum_{j=m+1}^{\infty}\frac{L_{0,2,0}^2}{(j-2)^4} + \frac{\pi^3}{2}\sum_{i=m+1}^{\infty}\frac{L_{2,0,0}^2}{(i-2)^4} \\
&+ \frac{\pi^3}{4}\sum_{j=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4} + \frac{\pi^3}{4}\sum_{j=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4} \\
&+ \frac{\pi^3}{4}\sum_{i=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(j-2)^4} + \frac{\pi^3}{4}\sum_{i=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(k-2)^4} \\
&+ \frac{\pi^3}{4}\sum_{i=2}^{m}\sum_{j=m+1}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4} + \frac{\pi^3}{4}\sum_{i=m+1}^{\infty}\sum_{j=2}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4} \\
&+ \frac{\pi^3}{8}\sum_{i=2}^{m}\sum_{j=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&+ \frac{\pi^3}{8}\sum_{i=2}^{m}\sum_{j=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&+ \frac{\pi^3}{8}\sum_{i=m+1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
&||u(x,y,t)-\varphi(t)^T\mathbb{V}(\varphi(x)\otimes\varphi(y))||_{L^2}^2\\
&=\int_0^1\int_0^1\int_0^1(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t)\\
&-\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m}c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t))^2\chi(x)\chi(y)\chi(t)dxdydt,
\end{aligned}
$$

$$
\begin{aligned}
=\int_0^1\int_0^1\int_0^1(&\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty}c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t)\\
&+\sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=m+1}^{\infty}c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t)\\
&+\sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{ijk}\mathrm{A}_i^*(x)\mathrm{A}_j^*(y)\mathrm{A}_k^*(t))^2\chi(x)\chi(y)\chi(t)dxdydt,
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty}c_{ijk}^2\int_0^1\mathrm{A}_i^{*^2}(x)\chi(x)dx\int_0^1\mathrm{A}_j^{*^2}(y)\chi(y)dy\int_0^1\mathrm{A}_k^{*^2}(t)\chi(t)dt\\
&+\sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=m+1}^{\infty}c_{ijk}^2\int_0^1\mathrm{A}_i^{*^2}(x)\chi(x)dx\int_0^1\mathrm{A}_j^{*^2}(y)\chi(y)dy\int_0^1\mathrm{A}_k^{*^2}(t)\chi(t)dt\\
&+\sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{ijk}^2\int_0^1\mathrm{A}_i^{*^2}(x)\chi(x)dx\int_0^1\mathrm{A}_j^{*^2}(y)\chi(y)dy\int_0^1\mathrm{A}_k^{*^2}(t)\chi(t)dt. \quad (5.60)
\end{aligned}
$$

Eq. (5.60) and orthogonality of shifted airfoil polynomials of second kind yield

$$
=\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=m+1}^{\infty}c_{ijk}^2\frac{\pi^3}{8}+\sum_{i=0}^{m}\sum_{j=m+1}^{\infty}\sum_{k=m+1}^{\infty}c_{ijk}^2\frac{\pi^3}{8}+\sum_{i=m+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{ijk}^2\frac{\pi^3}{8}. \quad (5.61)
$$

From Theorem 5.2 and Eq. (5.61), we have

$$
\begin{aligned}
||u(x,y,t)-&\varphi(t)^T\mathbb{V}(\varphi(x)\otimes\varphi(y))||_{L^2}^2 \\
&< \frac{\pi^3}{2}\sum_{k=m+1}^{\infty}\frac{L_{0,0,2}^2}{(k-2)^4}+\frac{\pi^3}{2}\sum_{j=m+1}^{\infty}\frac{L_{0,2,0}^2}{(j-2)^4}+\frac{\pi^3}{2}\sum_{i=m+1}^{\infty}\frac{L_{2,0,0}^2}{(i-2)^4} \\
&+\frac{\pi^3}{4}\sum_{j=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4}+\frac{\pi^3}{4}\sum_{j=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{0,2,2}^2}{(j-2)^4(k-2)^4} \\
&+\frac{\pi^3}{4}\sum_{i=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{4}\sum_{i=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,0,2}^2}{(i-2)^4(k-2)^4} \\
&+\frac{\pi^3}{4}\sum_{i=2}^{m}\sum_{j=m+1}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4}+\frac{\pi^3}{4}\sum_{i=m+1}^{\infty}\sum_{j=2}^{\infty}\frac{L_{2,2,0}^2}{(i-2)^4(j-2)^4} \\
&+\frac{\pi^3}{8}\sum_{i=2}^{m}\sum_{j=2}^{m}\sum_{k=m+1}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&+\frac{\pi^3}{8}\sum_{i=2}^{m}\sum_{j=m+1}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4} \\
&+\frac{\pi^3}{8}\sum_{i=m+1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{L_{2,2,2}^2}{(i-2)^4(j-2)^4(k-2)^4}. \qquad (5.62)
\end{aligned}
$$

Finally, by taking the square roots of both sides of Eq. (5.62), the proof is completed.

□

Theorem 5.4. *Under the assumptions of Theorem 2, we have*

$$
\|u_{N+1}-u_N\|_w^2 = O\left(\frac{1}{N}\right). \qquad (5.63)
$$

Proof.

$$
\begin{aligned}
&\|u_{N+1}-u_N\|^2_{L^2}\\
&=\left\|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{ijk}\mathrm{A}^*_i(x)\mathrm{A}^*_j(y)\mathrm{A}^*_k(t)-\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}c_{ijk}\mathrm{A}^*_i(x)\mathrm{A}^*_j(y)\mathrm{A}^*_k(t)\right\|^2_{L^2},\\
&=\|\sum_{j=0}^{N}\sum_{k=0}^{N}c_{N+1jk}\mathrm{A}^*_{N+1}(x)\mathrm{A}^*_j(y)\mathrm{A}^*_k(t)+\sum_{i=0}^{N}\sum_{k=0}^{N}c_{iN+1k}\mathrm{A}^*_i(x)\mathrm{A}^*_{N+1}(y)\mathrm{A}^*_k(t)\\
&+\sum_{i=0}^{N}\sum_{j=0}^{N}c_{ijN+1}\mathrm{A}^*_i(x)\mathrm{A}^*_j(y)\mathrm{A}^*_{N+1}(t)+\sum_{i=0}^{N}c_{iN+1N+1}\mathrm{A}^*_i(x)\mathrm{A}^*_{N+1}(y)\mathrm{A}^*_{N+1}(t)\\
&+\sum_{j=0}^{N}c_{N+1jN+1}\mathrm{A}^*_{N+1}(x)\mathrm{A}^*_j(y)\mathrm{A}^*_{N+1}(t)+\sum_{k=0}^{N}c_{N+1N+1k}\mathrm{A}^*_{N+1}(x)\mathrm{A}^*_{N+1}(y)\mathrm{A}^*_k(t)\\
&+c_{N+1N+1N+1}\mathrm{A}^*_{N+1}(x)\mathrm{A}^*_{N+1}(y)\mathrm{A}^*_{N+1}(t)\|^2_{L^2},
\end{aligned}
$$

$$
\begin{aligned}
\|u_{N+1}-u_N\|^2_{L^2}&=\sum_{j=0}^{N}\sum_{k=0}^{N}c^2_{N+1jk}\frac{\pi^3}{8}+\sum_{i=0}^{N}\sum_{k=0}^{N}c^2_{iN+1k}\frac{\pi^3}{8}+\sum_{i=0}^{N}\sum_{j=0}^{N}c^2_{ijN+1}\frac{\pi^3}{8}\\
&+\sum_{i=0}^{N}c^2_{iN+1N+1}\frac{\pi^3}{8}+\sum_{j=0}^{N}c^2_{N+1jN+1}\frac{\pi^3}{8}+\sum_{k=0}^{N}c^2_{N+1N+1k}\frac{\pi^3}{8}+c^2_{N+1N+1N+1}\frac{\pi^3}{8},
\end{aligned}
$$

$$= c_{N+1,0,0}^2\frac{\pi^3}{8} + c_{N+1,0,1}^2\frac{\pi^3}{8} + \sum_{k=2}^{N} c_{N+1,0,k}^2\frac{\pi^3}{8} + c_{N+1,1,0}^2\frac{\pi^3}{8} + c_{N+1,1,1}^2\frac{\pi^3}{8} + \sum_{k=2}^{N} c_{N+1,1,k}^2\frac{\pi^3}{8}$$
$$+ \sum_{j=2}^{N} c_{N+1,j,0}^2\frac{\pi^3}{8} + \sum_{j=2}^{N} c_{N+1,j,1}^2\frac{\pi^3}{8} + \sum_{j=2}^{N}\sum_{k=2}^{N} c_{N+1,j,k}^2\frac{\pi^3}{8} + c_{0,N+1,0}^2\frac{\pi^3}{8} + c_{0,N+1,1}^2\frac{\pi^3}{8}$$
$$+ \sum_{k=2}^{N} c_{0,N+1,k}^2\frac{\pi^3}{8} + c_{1,N+1,0}^2\frac{\pi^3}{8} + c_{1,N+1,1}^2\frac{\pi^3}{8} + \sum_{k=2}^{N} c_{1,N+1,k}^2\frac{\pi^3}{8} + \sum_{i=2}^{N} c_{i,N+1,0}^2\frac{\pi^3}{8}$$
$$+ c_{1,0,N+1}^2\frac{\pi^3}{8} + \sum_{i=2}^{N} c_{i,N+1,1}^2\frac{\pi^3}{8} + \sum_{i=2}^{N}\sum_{k=2}^{N} c_{i,N+1,k}^2\frac{\pi^3}{8} + c_{0,0,N+1}^2\frac{\pi^3}{8} + c_{0,1,N+1}^2\frac{\pi^3}{8}$$
$$+ \sum_{j=2}^{N} c_{0,j,N+1}^2\frac{\pi^3}{8} + c_{1,1,N+1}^2\frac{\pi^3}{8} + \sum_{j=2}^{N} c_{1,j,N+1}^2\frac{\pi^3}{8} + \sum_{i=2}^{N} c_{i,0,N+1}^2\frac{\pi^3}{8} + \sum_{i=2}^{N} c_{i,1,N+1}^2\frac{\pi^3}{8}$$
$$+ \sum_{i=2}^{N}\sum_{j=2}^{N} c_{i,j,N+1}^2\frac{\pi^3}{8} + c_{0,N+1,N+1}^2\frac{\pi^3}{8} + c_{1,N+1,N+1}^2\frac{\pi^3}{8} + \sum_{i=2}^{N} c_{i,N+1,N+1}^2\frac{\pi^3}{8}$$
$$+ c_{N+1,0,N+1}^2\frac{\pi^3}{8} + c_{N+1,1,N+1}^2\frac{\pi^3}{8} + \sum_{j=2}^{N} c_{N+1,j,N+1}^2\frac{\pi^3}{8} + c_{N+1,N+1,0}^2\frac{\pi^3}{8}$$
$$+ c_{N+1,N+1,1}^2\frac{\pi^3}{8} + \sum_{k=2}^{N} c_{N+1,N+1,k}^2\frac{\pi^3}{8} + c_{N+1,N+1,N+1}^2\frac{\pi^3}{8}$$

Following the procedure of Theorem 2, we can show that

$$c_{N+1,0,0}^2 < \frac{K_1}{N^4}, \quad c_{N+1,0,1}^2 < \frac{K_2}{N^4}, \quad c_{N+1,1,0}^2 < \frac{K_3}{N^4}, \quad c_{N+1,1,1}^2 < \frac{K_4}{N^4}, \dots \tag{5.64}$$

$$c_{0,N+1,N+1}^2 < \frac{M_1}{N^8}, \quad c_{1,N+1,N+1}^2 < \frac{M_2}{N^8}, \quad c_{N+1,0,N+1}^2 < \frac{M_3}{N^8}, \quad c_{N+1,1,N+1}^2 < \frac{M_4}{N^8}, \dots \tag{5.65}$$

$$c_{N+1,N+1,N+1}^2 < \frac{M_1}{N^{12}}. \tag{5.66}$$

Also, from the result of Theorem 2, we have

$$
\begin{aligned}
c^2_{N+1,0,k} &< \frac{B_1}{N^4(k-1)^4},\\
\sum_{k=2}^{N} c^2_{N+1,0,k} &< \frac{B_1^1}{N^3}.
\end{aligned}
\tag{5.67}
$$

Similiarly

$$
\sum_{k=2}^{N} c^2_{N+1,1,k} < \frac{B_2^1}{N^3}, \quad \sum_{j=2}^{N} c^2_{N+1,j,0} < \frac{B_3^1}{N^3}, \quad \sum_{k=2}^{N} c^2_{0,N+1,k} < \frac{B_4^1}{N^3}, \dots \tag{5.68}
$$

Again, from the conclusion of Theorem 2, we obtain

$$
\begin{aligned}
c^2_{N+1,j,k} &< \frac{H_1}{N^4(j-1)^4(k-1)^4},\\
\sum_{j=2}^{N} c^2_{N+1,j,k} &< \frac{H_1^1}{N^3(k-1)^4}\\
\sum_{j=2}^{N}\sum_{k=2}^{N} c^2_{N+1,j,k} &< \frac{H_1^{11}}{N^2},
\end{aligned}
\tag{5.69}
$$

Similiarly

$$
\sum_{i=2}^{N}\sum_{j=2}^{N} c^2_{i,j,N+1} < \frac{H_2^{11}}{N^2}, \quad \sum_{i=2}^{N}\sum_{k=2}^{N} c^2_{i,N+1,k} < \frac{H_3^{11}}{N^2}. \tag{5.70}
$$

Thus we get

$$
\|u_{N+1} - u_N\|^2_{L^2} \leq K\frac{\pi^3}{8}\frac{1}{N^2}. \tag{5.71}
$$

Consequently, $|u_{N+1}| \to 0$ with convergence order $O\left(\frac{1}{N}\right)$ as $N \to \infty$. $\square$

5.7 Numerical Simulation

In this section, we implement the proposed approach for solving three test problems to demonstrate the correctness of the proposed method. To evaluate the accuracy of the method, the L_2 and L_∞ errors for $u(x, y, t)$ at time t are measured based on the following formulae.

$$L_2 = \sqrt{\int_0^1 \int_0^1 \left|u(x,y,t) - u^{'}(x,y,t)\right|^2 dxdy}$$
$$L_\infty = \max_{\{0<x<1\}} \max_{\{0<y<1\}} \left|u(x,y,t) - u^{'}(x,y,t)\right|$$

where $u(x, y, t)$ and $u^{'}(x, y, t)$ are the approximate and exact solutions.

To testify the convergence orders (CO) of our developed scheme, the following formula is used:

$$CO = \frac{\log\left(\frac{\chi_1}{\chi_2}\right)}{\log\left(\frac{N_2}{N_1}\right)}$$

where χ_1 and χ_2 are the values of the first and second maximum absolute error respectively. Moreover, N_1 and N_2 are the number of the shifted airfoil polynomials of second kind which are used in the first and second simulations, respectively.

Example 5.7.1 Let us consider the following time fractional fourth-order sub-diffusion equation [201]:

$$\frac{\partial^{\alpha(x,t)} u(x,t)}{\partial t^{\alpha(x,t)}} + \frac{\partial^4 u(x,t)}{\partial x^4} = \cos(\pi x)\left(\frac{\Gamma(\alpha(x,t)) + 3}{2} + (t^{\alpha(x,t)} + 1)(\pi^4 + 1)\right) - u(x,t), \tag{5.72}$$

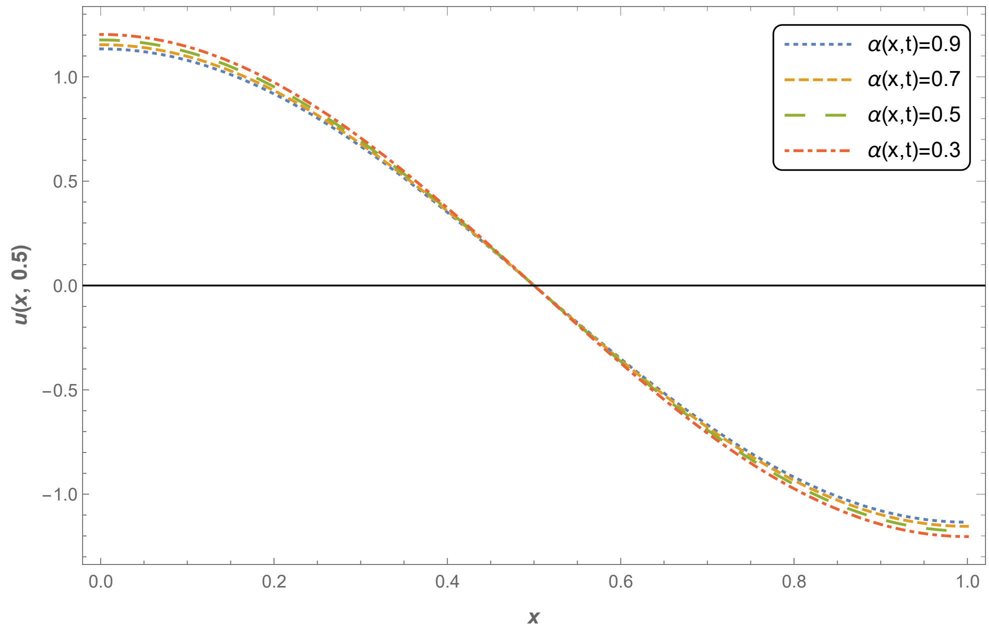

FIGURE 5.1: Numerical solutions for example 5.7.1 at different value of $\alpha(x,t)$.

with initial and boundary conditions

$$u(x,0) = \cos(\pi x), \quad x \in (0,1), \tag{5.73}$$

$$u(0,t) = t^{\alpha(x,t)+2} + 1, \quad u(1,t) = -t^{\alpha(x,t)+2} - 1, \quad t \in [0,T], \tag{5.74}$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = -\pi^2(t^{\alpha(x,t)+2} + 1), \ \frac{\partial^2 u(1,t)}{\partial x^2} = \pi^2(t^{\alpha(x,t)+2} + 1), \quad t \in [0,T], \tag{5.75}$$

The exact solution of the problem (5.72)-(5.75) is $u(x,t) = \cos(\pi x)(t^{\alpha(x,t)+2} + 1)$.

For example 5.7.1, table (5.1) provided the results getting from our suggested technique which analyzing the L_2 and L_∞ errors with increasing order of approximate polynomials for different time fractional which confirm the high accuracy of the method and also support the applicability and accuracy of the presented method. Also figures (5.1) and (5.2) show the nature of numerical solution and absolute error for example 5.7.1 in finite domain.

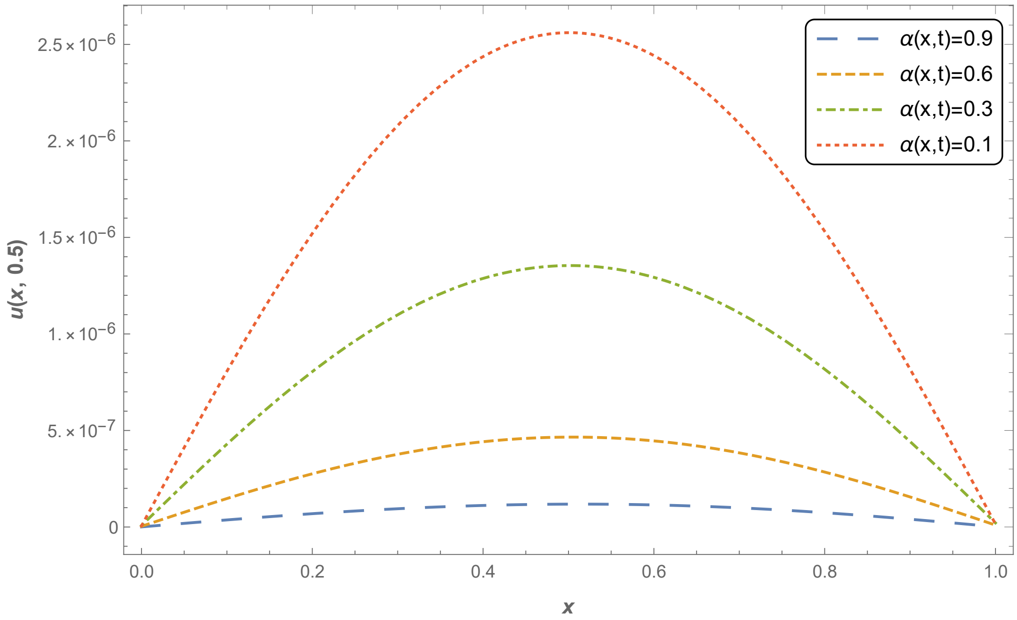

FIGURE 5.2: Absolute errors for example 5.7.1 at different value of $\alpha(x,t)$.

TABLE 5.1: The computational error and CO for example 5.7.1

$\alpha(x,t)$	N	L_∞ error	CO	L_2 error	CO
	10	2.01584×10^{-6}	-	1.16895×10^{-6}	-
	11	1.73846×10^{-6}	0.85066	1.00201×10^{-6}	0.88548
$\frac{\exp^{-t}}{300}$	12	1.50473×10^{-6}	0.90236	8.68363×10^{-7}	0.89423
	13	1.30758×10^{-6}	0.94751	7.56482×10^{-7}	0.93061
	14	1.13962×10^{-6}	0.99636	6.61793×10^{-7}	0.96913
$\alpha(x,t)$	q	L_∞ error	CO	L_2 error	CO
	10	1.93652×10^{-6}	-	1.27648×10^{-6}	-
	11	1.63588×10^{-6}	0.96948	1.09678×10^{-6}	0.87188
$\frac{2+\sin(t)}{400}$	12	1.39739×10^{-6}	0.98432	9.48743×10^{-7}	0.90576
	13	1.20649×10^{-6}	0.99106	8.21493×10^{-7}	0.97165
	14	1.05152×10^{-6}	0.99632	7.15792×10^{-7}	0.99818
$\alpha(x,t)$	q	L_∞ error	CO	L_2 error	CO
	10	1.69467×10^{-6}	-	1.08142×10^{-6}	-
	11	1.42684×10^{-6}	0.98853	9.18593×10^{-7}	0.93774
$\frac{2t+1}{300}$	12	1.21673×10^{-6}	0.99506	7.88225×10^{-7}	0.95611
	13	1.04969×10^{-6}	0.99633	6.81184×10^{-7}	0.98467
	14	9.14081×10^{-7}	1.00250	5.93757×10^{-7}	0.99548

Table 5.2: The computational error and CO for example 5.7.3

$\delta(x,y,t)$	N	L_∞ error	CO	L_2 error	CO
	12	1.20759×10^{-7}	-	1.17483×10^{-7}	-
	13	1.04948×10^{-7}	0.94681	9.96947×10^{-8}	0.94344
0.8+0.1sin(xyt)	14	9.15583×10^{-8}	0.98915	8.53769×10^{-8}	0.96846
	15	8.04926×10^{-8}	0.99794	7.39369×10^{-8}	0.70635
	16	7.12834×10^{-8}	1.00208	6.44195×10^{-8}	0.98618
$\delta(x,y,t)$	N	L_∞ error	CO	L_2 error	CO
	12	1.21594×10^{-7}	-	1.17485×10^{-7}	-
	13	1.05486×10^{-7}	0.95880	1.02548×10^{-7}	0.91744
0.007-0.003x^2+0.003y^2	14	9.21935×10^{-8}	0.97611	8.98574×10^{-8}	0.95739
	15	8.12726×10^{-8}	0.97678	7.91953×10^{-8}	0.97854
	16	7.19692×10^{-8}	1.00265	7.01636×10^{-8}	0.99866
$\delta(x,y,t)$	N	L_∞ error	CO	L_2 error	CO
	12	2.01582×10^{-7}	-	1.18794×10^{-7}	-
	13	1.74948×10^{-7}	0.95611	1.03483×10^{-7}	0.93096
0.35+0.12exp($-(xyt)$)	14	1.52983×10^{-7}	0.97228	9.05735×10^{-8}	0.96564
	15	1.34738×10^{-7}	0.98386	7.96523×10^{-8}	0.99545
	16	1.19486×10^{-7}	0.99079	6.99879×10^{-8}	1.06680

Example 5.7.3 Considering the problem of two-dimension sub-diffusion equation on a bounded domain Ω_1=[0,1]×[0,1]:

$$\frac{\partial^{\delta(x,y,t)}u(x,y,t)}{\partial t^{\delta(x,y,t)}} + \Delta^2 u(x,y,t) = f(x,y,t) + u(x,y,t), \tag{5.76}$$

with the given conditions

$$u(x,y,0) = 0, \qquad (x,y) \in \Omega_1, \tag{5.77}$$

$$u(0,y,t) = e^y t^4, \quad u(1,y,t) = e^{1+y} t^4, \quad y \in [0,1], \quad 0 \le t \le T, \tag{5.78}$$

$$u(y,0,t) = e^y t^4, \quad u(x,1,t) = e^{x+1} t^4, \quad x \in [0,1], \quad 0 \le t \le T, \tag{5.79}$$

$$\Delta u(0,y,t) = 2e^y t^4, \quad \Delta u(1,y,t) = 2e^{1+y} t^4, \quad y \in [0,1], \quad 0 \le t \le T, \tag{5.80}$$

$$\Delta u(x,0,t) = 2e^x t^4, \quad \Delta u(x,1,t) = 2e^{x+1} t^4, \quad x \in [0,1], \quad 0 \le t \le T, \tag{5.81}$$

where

$$f(x,y,t) = \frac{24}{\Gamma(5-\delta(x,y,t))} e^{x+y} t^{4-\delta(x,y,t)} + 3e^{x+y} t^4.$$

The exact solution to this problem is given by,

$$u(x,y,t) = e^{x+y} t^4.$$

We have applied the operational matrix method for this equation at different values of $\delta(x,y,t)$. Table (5.2) reports L_2 and L_∞ errors of results for increasing value of degree of orthonormal shifted Airfoil polynomials of second kind. The reported results confirm that the proposed method is very accurate for solving this problem.

5.8 Conclusion

In this chapter, the author successfully applied the second kind shifted Airfoil operational matrix method and collocation technique to present the numerical solutions of non-linear fractional fourth-order sub-diffusion equation of variable-order for one and two-dimensional. From this study, it is observed that the proposed numerical technique is efficient and simple to discuss the wide class of a nonlinear sub-diffusion model with neumann boundary conditions. It is analytically found that our scheme generates convergence order of nearly $O\left(\frac{1}{N}\right)$ this fact is verified by some examples. Therefore, this study shows that the proposed scheme is sufficiently accurate and effective to solve time-space fractional non-linear fourth-order sub-diffusion equation of variable-order.

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