CHAPTER 2

Stable Numerical Schemes for Time-Fractional Diffusion Equation with Generalized Memory Kernel

2.1 Introduction

This chapter investigates numerical schemes based on FDM for the generalized time-fractional diffusion equations with smooth and non-smooth solutions. Initially, we formulate the mathematical problem and then we design stable numerical schemes with higher order of accuracy.

2.1.1 Problem formulation

Consider the following initial-boundary value problem for the GTFDEs with variable coefficients [24]:

$$\partial_{0,t}^{\alpha,\omega(t)}\zeta = \mathcal{L}\zeta + \phi(x,t), \qquad x \in (0,1), \quad t \in (0,T], \tag{2.1}$$

$$\zeta(0,t) = 0, \quad \zeta(1,t) = 0, \quad t \in [0,T], \quad \zeta(x,0) = \zeta_0(x), \quad x \in [0,1], \tag{2.2}$$

where,

$$\mathcal{L}\zeta = \frac{\partial}{\partial x}\left(m(x,t)\frac{\partial \zeta}{\partial x}\right) - p(x,t)\zeta,$$

and

$$\partial_{0,t}^{\alpha,\omega(t)}\zeta(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\omega(t-\xi)}{(t-\xi)^{\alpha}}\frac{\partial \zeta}{\partial \xi}(x,\xi)\mathrm{d}\xi,$$

is the fractional derivative of order $\alpha,\ 0 < \alpha < 1$ in generalized Caputo sense with the weight function $\omega(t) \in C^2[0,T]$, $H = \{(x,t) : 0 \le x \le l, 0 \le t \le T\}$, $\omega(t) > 0$ and $\omega'(t) \le 0$ for all $t \in [0,T]$, and $0 < c_1 \le m(x,t) \le c_2$ and $p(x,t) \ge 0$ for all $(x,t) \in H$.

2.1.2 Physical significance of the above model

In this part, we show the physical significance of GTFDEs via the mean square displacement (MSD) factor. MSD is a vital characteristic property of diffusion and relaxation process. In [74], Sandev et al. (2015) proved the variations of the mean square displacement with different weighting functions. The MSD of the traditional Caputo FD (1.6)-(1.7) is found to be

$$\langle x^2(t)\rangle \simeq \frac{2Kt^\alpha}{\Gamma(\alpha+1)}, \tag{2.3}$$

where K is the diffusivity constant, and for the classical diffusion case the MSD is

$$\langle x^2(t)\rangle \propto t. \tag{2.4}$$

For the generalized Caputo FD (1.1) with a tempered memory kernel ($\omega(t) = e^{-bt}$), the MSD in the short time limit is

$$\langle x^2(t)\rangle \simeq 2t^\alpha/\Gamma(1+\alpha), \tag{2.5}$$

and in the long time limit it is

$$\langle x^2(t)\rangle \simeq 2b^{1-\alpha}t,\ b>0\ ,\ 0<\alpha\leq 1. \tag{2.6}$$

From equations (2.3), (2.4), (2.5) and (2.6) it is clear that the behaviour of tempered kernel is such that anomalous diffusion crosses over to normal diffusion, hence resulting in a unified study of various physical problems through a single model. The case of tempered kernel is one of the particular case of the generalized kernel defined in (1.1). Using different weight functions may generate various known mathematical models available in literature. Thus, GTFDEs is an interesting topic of research and deserves much interest.

Hence, we are motivated to design stable and efficient schemes that will be suitable to address mathematical simulation that may naturally arise in numerical methods for GTFDEs with smooth and non-smooth solutions. The main contribution of this chapter can be described by the following points:

- The modified version of the L1 formula for the smooth and non-smooth solutions on a non-uniform grid is developed and shown that for smooth solutions, this scheme presents a better or equivalent result than the uniform mesh. Also, it is shown that for non-smooth solutions, this scheme presents a better result than the uniform mesh discussed in [24].

- Two different non-uniform grids are taken, one for smooth and the other for the non-smooth solution. The motivation behind taking two different meshes is that for the smooth solution, we require a grading of the mesh such that the gridding should be finer near the end point $t = 1$ due to the kernel singularity. Similarly, for the non-smooth solution, we require a grading of the mesh such that the gridding should be finer near $t = 0$ to handle the singularity of the solution near $t = 0$. In our proposed schemes, we use the grid defined in [101] for the smooth solution and the grid defined in [55] for the non-smooth solution, respectively.

- For developed schemes, the analysis is investigated for smooth and non-smooth solutions with respect to L_2-norm and L_∞-norm, respectively. To validate the analysis part, some numerical experiments are given. For both the cases, we have shown a convergence order of $2 - \alpha$.

The rest of the chapter is organized as follows: Section 2.2 presents the derivation of the modified L1 scheme on the non-uniform grid and the fully discrete scheme for GTFDEs is derived and two different grids are introduced, one for the smooth solution, another for the non-smooth solution. Stability and convergence of the scheme for smooth solution in L_2-norm has been established in 2.3. Similarly, in Section 2.4, stability and convergence for non-smooth solution in L_∞-norm are given. Some numerical experiments are performed on the non-uniform grid in Section 2.5. In Section 2.6, some concluding remarks are given.

2.2 The approximation formula and the scheme

2.2.1 Formula derivation

Let $\zeta(x,t)$ be the exact solution of the mathematical model (2.1)-(2.2). For any $N \in \mathbb{Z}^+$, let us partition the interval $[0,T]$ into N sub-intervals with $0 = t_0 < t_1 \ldots < t_N = T$ and $\tau_n = t_n - t_{n-1}$, $1 \leq n \leq N$ where τ_n is the time step size. We approximate the Caputo derivative $\partial_{0,t_n}^{\alpha,\omega(t)} v(t)$. For all $\alpha \in (0,1)$ and $\omega(t) > 0$, $\omega'(t) \leq 0$ and $\omega(t) \in C^2[0,T]$, we have

$$
\begin{aligned}
\partial_{0,t_n}^{\alpha,\omega(t)} v(t) &= \frac{1}{\Gamma(1-\alpha)} \int\limits_0^{t_n} \frac{\omega(t_n-\xi)v'(\xi)\,\mathrm{d}\xi}{(t_n-\xi)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} \int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)v'(\xi)\,\mathrm{d}\xi}{(t_n-\xi)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{s=0}^{n-1} v_{t,s} \int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)\,\mathrm{d}\xi}{(t_n-\xi)^\alpha} + \sum_{s=0}^{n-1} \int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'\,\mathrm{d}\xi}{(t_n-\xi)^\alpha} \right].
\end{aligned}
$$

We use transformation of variables for the first term to convert variable ξ to $t_s + (t_{s+1} - t_s)z$ and then for $\omega(t_n - t_s - \tau_{s+1}z)$, we use the following representation: $\omega(t_n - t_s - \tau_{s+1}z) = \psi + (\omega(t_n - t_s - \tau_{s+1}z) - \psi)$, where

$$
\psi = \omega(t_n - t_s - \frac{\tau_{s+1}}{2}) + [\omega(t_n - t_{s+1}) - \omega(t_n - t_s)](z - 1/2),
$$

and thus in this case, we can write the following equality:

$$
\begin{aligned}
\partial_{0,t_n}^{\alpha,\omega(t)} v(t) &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} v_{t,s}\ \tau_{s+1} \int_0^1 \frac{\omega(t_n - t_s - \frac{\tau_{s+1}}{2}) + [\omega(t_n - t_{s+1}) - \omega(t_n - t_s))](z - \frac{1}{2})}{(t_n - t_s - \tau_{s+1} z)^{\alpha}} \mathrm{d}z \\
&\quad + R_1^n + R_2^n \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{n-1} \left[\omega(t_n - t_s - \frac{\tau_{s+1}}{2}) a_s^n + (\omega(t_n - t_{s+1}) - \omega(t_n - t_s)) b_s^n \right] (v(t_{s+1}) - v(t_s)) \\
&\quad + R_1^n + R_2^n,
\end{aligned}
$$

where,

$$
\begin{aligned}
v_{t,s} &= \frac{v(t_{s+1}) - v(t_s)}{\tau_{s+1}}\ ,\ \Pi_{1,s} v(\xi) = v(t_{s+1}) \frac{\xi - t_s}{\tau_{s+1}} + v(t_s) \frac{t_{s+1} - \xi}{\tau_{s+1}}, \\
a_s^n &= \frac{(t_n - t_s)^{1-\alpha} - (t_n - t_{s+1})^{1-\alpha}}{\tau_{s+1}}, \\
b_s^n &= \frac{1}{\tau_{s+1}^2} \left(\frac{1}{2-\alpha} [(t_n - t_s)^{2-\alpha} - (t_n - t_{s+1})^{2-\alpha}] - \frac{\tau_{s+1}}{2} [(t_n - t_s)^{1-\alpha} + (t_n - t_{s+1})^{1-\alpha}] \right),
\end{aligned}
$$

$$
R_1^n = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} v_{t,s}\ \tau_{s+1} \int_0^1 \frac{\omega(t_n - t_s - \tau_{s+1} z) - \psi}{(t_n - t_s - \tau_{s+1} z)^{\alpha}} \mathrm{d}z,
$$

$$
R_2^n = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} \frac{\omega(t_n - \xi)(v(\xi) - \Pi_{1,s} v(\xi))' \,\mathrm{d}\xi}{(t_n - \xi)^{\alpha}}.
$$

Thus, for the fractional derivative of order α in generalized Caputo sense with the weight function as $\omega(t)$ $(0 < \alpha < 1,\ \omega(t) > 0,\ \omega'(t) \leq 0)$, we have the following numerical approximation formula on non-uniform mesh .

$$\Delta_{0,t_n}^{\alpha,\omega(t)} v = \sum_{s=0}^{n-1} c_s^n (v(t_{s+1}) - v(t_s)), \tag{2.7}$$

where,

$$c_s^n = \frac{1}{\Gamma(2-\alpha)} \left[\omega(t_n - t_s - \frac{\tau_{s+1}}{2}) a_s^n + (\omega(t_n - t_{s+1}) - \omega(t_n - t_s)) b_s^n \right].$$

This can be called as the L1 formula on any non-uniform grid for generalized Caputo fractional derivative for smooth as well as non-smooth solutions.

2.2.2 A difference scheme

Consider the rectangle H. Now, we have the mesh as $\bar{\chi}_{h\tau} = \bar{\chi}_h \times \bar{\chi}_\tau$, where

$\bar{\chi}_h = \{x_i = ih : i = 0, 1, \ldots, M, hM = l\}$ and $\bar{\chi}_\tau = \{t_n : 0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T\}$.

The difference scheme is

$$\Delta_{0,t_n}^{\alpha,\omega(t)} \varpi_i = \wedge \varpi_i^n + \phi_i^n, \quad n = 1, \ldots, N, \ i = 1, 2, \ldots, M-1, \tag{2.8}$$

$$\varpi(0,t) = 0, \ \varpi(l,t) = 0, \ t \in \bar{\chi}_\tau, \ \varpi(x,0) = \zeta_0(x), \ x \in \bar{\chi}_h. \tag{2.9}$$

where,

$$\wedge \varpi_i = ((a\varpi_{\bar{x}})_x - d\varpi)_i = \frac{a_{i+1}\varpi_{i+1} - (a_{i+1} + a_i)\varpi_i + a_i \varpi_{i-1}}{h^2} - d_i \varpi_i, \ \ i = 1, \ldots, M-1,$$

$$\varpi_{\bar{x},i} = \frac{\varpi_i - \varpi_{i-1}}{h}, \ \varpi_{x,i} = \frac{\varpi_{i+1} - \varpi_i}{h}, \ a_i^n = m(x_{i-1/2}, t_n), \ d_i^n = p(x_i, t_n), \ \phi_i^n = \phi(x_i, t_n).$$

2.2.2.1 Non-uniform meshes for smooth and non-smooth solution respectively

This sub-section defines two different types of non-uniform mesh, one for smooth solution and the other for the non-smooth solution.

Mesh for smooth solution

Let N be a positive integer. Consider the partitioning of the interval $[0, T]$ into sub-intervals with $0 = t_0 < t_1 \ldots < t_N = T$ and $\tau_n = t_n - t_{n-1}$, $1 \leq n \leq N$ where τ_n is the time step size as defined in [101].

$$\tau_n = (N + 1 - n)\mu, \ \ 1 \leq n \leq N, \quad \text{where,} \ \ \mu = \frac{2T}{N(N+1)}. \tag{2.10}$$

Mesh for non-smooth solution [55]

Let N $\in \mathbb{Z}^+$. Put

$$t_n = T(n/N)^r \ for\, n = 0, 1, 2 \ldots, N, \tag{2.11}$$

where the constant mesh grading $r \geq 1$ is being choosed by the user. If $r = 1$, the grid is uniform. Also, $\tau_n = t_n - t_{n-1}$ for $1 \leq n \leq N$.

2.3 Stability and convergence of the scheme for smooth solutions

This section deals with the stability and convergence of the proposed scheme on non-uniform mesh for a smooth solution. For the developed scheme, the analysis is investigated for the smooth solution with respect to L_2-norm. The CO is established to be $\mathcal{O}(N^{\alpha-2})$ in the temporal direction.

2.3.1 Stability for smooth solutions

Lemma 2.3.1. [37, 101] For any $n = 1, 2, \ldots, N,\ \{a_s^n | 0 \le s \le n-1\}$ and $\alpha \in (0, 1)$, the following inequality holds:

$$a_{n-1}^n > a_{n-2}^n > \ldots > a_s^n > a_{s-1}^n > \ldots a_0^n > \frac{1-\alpha}{(t_n - t_0)^\alpha},$$

and

$$b_{n-1}^n > b_{n-2}^n > \ldots > b_s^n > b_{s-1}^n > \ldots > b_0^n > 0.$$

Corollary 2.3.1. For any $n = 1, 2, \ldots, N,\ \{c_s^n | 0 \le s \le n-1\},\ \alpha \in (0, 1)$ and $\omega(t) \in C^2[0, T]$, where $\omega(t) > 0,\ \omega'(t) \le 0\ \forall\ t \in [0, T]$, the following inequalities hold:

$$c_{n-1}^n > c_{n-2}^n > \ldots > c_s^n > c_{s-1}^n > \ldots c_0^n > \frac{\omega(t_n - t_0 - \frac{\tau_1}{2})}{\Gamma(1-\alpha)(t_n - t_0)^\alpha}.$$

Proof. From equation (2.7), we get

$$c_s^n = \frac{1}{\Gamma(2-\alpha)}\left[\omega(t_n - t_s - \frac{\tau_{s+1}}{2})a_s^n + (\omega(t_n - t_{s+1}) - \omega(t_n - t_s))b_s^n\right].$$

Since $\omega(t) > 0$ and a decreasing function so the coefficient of b_s^n in the above equation is positive. Also, since a_s^n and b_s^n are increasing functions and the sum of two increasing functions will also be increasing. Hence, c_s^n is monotonically increasing. Now since $b_s^n > 0$,

Thus

$$c_s^n > \frac{\omega(t_n - t_s - \frac{\tau_{s+1}}{2})}{\Gamma(2-\alpha)}a_s^n > \frac{\omega(t_n - t_s - \frac{\tau_{s+1}}{2})}{\Gamma(1-\alpha)(t_n - t_s)^\alpha}.$$

□

The next lemma results from Corollary 2.3.1 and Lemma 3.1 of [24] (or Lemma 1 of [45]).

Lemma 2.3.2. [24] The following inequality is true for any function $v(t)$ defined on the grid $\bar{\chi}_\tau$,

$$v^n \Delta_{0,t_n}^{\alpha,\omega(t)} v \geq \frac{1}{2}\Delta_{0,t_n}^{\alpha,\omega(t)}(v^2).$$

Theorem 2.3.1. The solution of the proposed scheme (2.8)-(2.9) is unconditionally stable and satisfies the following result:

$$||\varpi^n||_0^2 \leq ||\varpi^0||_0^2 + \frac{T^\alpha \Gamma(1-\alpha)}{4c_1\omega(T)} \max_{1\leq n\leq N} ||\phi^{n-1}||_0^2.$$

Proof. For the difference operator $\wedge$, using Green's first difference formula and the embedding theorem [102] for the functions vanishing at $x = 0$ and $x = 1$, we get

$(- \wedge \varpi, \varpi) \geq 4c_1||\varpi||_0^2$, that is, for this operator we take $\varkappa = 4c_1$. Now, as the difference scheme (2.8)-(2.9) has the same form as equation (3.1)-(3.2) in reference [24], where $g_s^n = c_s^n$. Thus, Corollary 2.3.1 implies that following inequalities are valid:

$$g_0^n = c_0^n > \frac{\omega(t_n - t_0 - \frac{\tau_1}{2})}{\Gamma(1-\alpha)(t_n - t_0)^\alpha} > \frac{\omega(T)}{\Gamma(1-\alpha)T^\alpha}, \quad g_{n-1}^n > g_{n-2}^n > \ldots > g_0^n, \quad \sigma = 1.$$

Therefore, Theorem 3.3 in [24] implies the validity of Theorem 2.3.1. Also, Theorem 3.4 in [24] implies that the solution of the proposed scheme (2.8)-(2.9) converges with the rate equal to the order of approximation error $\mathcal{O}(h^2 + N^{\alpha-2})$. □

2.3.2 Convergence of smooth solutions

Lemma 2.3.3. For any $\alpha \in (0,1)$ and $v(t) \in C^2[0, t_n]$, it is true that

$$\partial_{0,t_n}^{\alpha,\omega(t)} v = \Delta_{0,t_n}^{\alpha,\omega(t)} v + \mathcal{O}(N^{\alpha-2}),$$

where $\omega(t) > 0, \omega'(t) \leq 0$ and $\omega(t) \in C^2[0, t_n]$.

Proof.

$$\partial_{0,t_n}^{\alpha,\omega(t)} v - \Delta_{0,t_n}^{\alpha,\omega(t)} v = R_1^n + R_2^n.$$

We estimate the errors R_1^n and R_2^n:

$$|R_1^n| \leq \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} |v_{t,s}| \ \tau_{s+1} \int\limits_0^1 \frac{|\omega(t_n - t_s - \tau_{s+1} z) - \psi|}{(t_n - t_s - \tau_{s+1} z)^\alpha} \mathrm{d}z$$

$$\begin{aligned}
&\leq \frac{M_1^n m_2^n}{4\Gamma(1-\alpha)} \sum_{s=0}^{n-1} \tau_{s+1}^3 \int\limits_0^1 \frac{\mathrm{dz}}{(t_n - t_s - \tau_{s+1} z)^\alpha} = \frac{M_1^n m_2^n}{4\Gamma(1-\alpha)} \sum_{s=0}^{n-1} \tau_{s+1}^2 \int\limits_{t_s}^{t_{s+1}} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} \\
&= \frac{M_1^n m_2^n}{4\Gamma(1-\alpha)} \left[\tau_1^2 \int\limits_{t_0}^{t_1} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} + \tau_2^2 \int\limits_{t_1}^{t_2} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} + \ldots \tau_n^2 \int\limits_{t_{n-1}}^{t_n} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} \right] \\
&\leq \frac{M_1^n m_2^n}{4\Gamma(1-\alpha)} \tau_1^2 \int\limits_{t_0}^{t_n} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} \quad \text{(as } \tau_1^2 > \tau_2^2 > \ldots > \tau_n^2 \text{ for grid defined in [101])} \\
&= \frac{M_1^n m_2^n}{4\Gamma(2-\alpha)} \tau_1^2 t_n^{1-\alpha} = \frac{M_1^n m_2^n}{\Gamma(2-\alpha)} \frac{T^2}{(N+1)^2} t_n^{1-\alpha},
\end{aligned}$$

where,

$$M_1^n = \underset{0 \leq t \leq t_n}{max} |v'(t)| \text{ and } m_2^n = \underset{0 \leq t \leq t_n}{max} |\omega''(t)|;$$

$$
\begin{aligned}
|R_2^n| &= \frac{1}{\Gamma(1-\alpha)}\left|\sum_{s=0}^{n-1}\int\limits_{t_s}^{t_{s+1}}\frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'}{(t_n-\xi)^{\alpha}}d\xi\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\left|\sum_{s=0}^{n-2}\int\limits_{t_s}^{t_{s+1}}\frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'}{(t_n-\xi)^{\alpha}}d\xi\right| \\
&+ \frac{1}{\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n}\frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|}{(t_n-\xi)^{\alpha}}d\xi \\
&\leq \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\int\limits_{t_s}^{t_{s+1}}|v(\xi)-\Pi_{1,s}v(\xi)|\cdot|\omega'(t_n-\xi)|(t_n-\xi)^{-\alpha}\,d\xi \\
&+ \frac{\alpha}{\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\int\limits_{t_s}^{t_{s+1}}|v(\xi)-\Pi_{1,s}v(\xi)|\cdot\omega(t_n-\xi)(t_n-\xi)^{-\alpha-1}\,d\xi \\
&+ \frac{1}{\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n}\frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|d\xi}{(t_n-\xi)^{\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M_2^n m_1^n}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^2\int\limits_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha}\mathrm{d}\xi + \frac{\alpha M_2^n\omega(0)}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^2\int\limits_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha-1}\mathrm{d}\xi \\
&+ \frac{\omega(0)M_2^n}{\Gamma(1-\alpha)}\tau_n\int\limits_{t_{n-1}}^{t_n}\frac{\mathrm{d}\xi}{(t_n-\xi)^{\alpha}},
\end{aligned}
$$

where

$$
M_2^n = \max_{0\leq t\leq t_n}|v''(t)| \text{ and } m_1^n = \max_{0\leq t\leq t_n}|\omega'(t)|.
$$

Now,

$$\frac{M_2^n m_1^n}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^2\int\limits_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha}\mathrm{d}\xi\leq\frac{M_2^n m_1^n}{8\Gamma(1-\alpha)}\tau_1^2\int\limits_{t_0}^{t_n}(t_n-\xi)^{-\alpha}\mathrm{d}\xi=\frac{M_2^n m_1^n}{8\Gamma(1-\alpha)}\tau_1^2(t_n)^{1-\alpha}$$
$$=\frac{M_2^n m_1^n T^2 t_n^{1-\alpha}}{2\Gamma(1-\alpha)(N+1)^2}\ .$$

Also,

$$\frac{\alpha M_2^n\omega(0)}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^2\int\limits_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha-1}\mathrm{d}\xi\leq\frac{\alpha M_2^n\omega(0)}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^3(t_n-t_{s+1})^{-\alpha-1}\ .$$

Since $\mu=\frac{2T}{N(N+1)}$

$$t_n-t_{s+1}=\sum_{l=s+2}^{n}\tau_l=\frac{1}{2}(\tau_n+\tau_{s+2})(n-s-1)=\frac{\mu}{2}[(2N-n-s)(n-s-1)]\ .$$

Therefore,

$$\begin{aligned}\sum_{s=0}^{n-2}\tau_{s+1}^3(t_n-t_{s+1})^{-\alpha-1}&=\sum_{s=0}^{n-2}2^{1+\alpha}\mu^{2-\alpha}(N-s)^3(n-s-1)^{-\alpha-1}(2N-n-s)^{-\alpha-1}\\&\leq 2^{1+\alpha}\mu^{2-\alpha}\sum_{s=0}^{n-2}(N-s)^3(n-s-1)^{-\alpha-1}(N-s)^{-\alpha-1}\\&\leq 2^{1+\alpha}\mu^{2-\alpha}N^{2-\alpha}\sum_{s=0}^{n-2}(n-s-1)^{-\alpha-1}\\&=2^{1+\alpha}\frac{(2T)^{2-\alpha}N^{2-\alpha}}{N^{2-\alpha}(N+1)^{2-\alpha}}\sum_{s=0}^{n-2}(n-s-1)^{-\alpha-1}\\&=\frac{8T^{2-\alpha}}{(N+1)^{2-\alpha}}\sum_{s=0}^{n-2}(n-s-1)^{-\alpha-1}\leq\frac{8T^{2-\alpha}}{(N+1)^{2-\alpha}}\left(\frac{1+\alpha}{\alpha}\right)\ .\end{aligned}$$

Thus,

$$\frac{\alpha M_2^n\omega(0)}{8\Gamma(1-\alpha)}\sum_{s=0}^{n-2}\tau_{s+1}^2\int_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha-1}\mathrm{d}\xi \le \frac{\omega(0)M_2^n}{\Gamma(1-\alpha)(N+1)^{2-\alpha}}T^{2-\alpha}(1+\alpha)$$
$$\le \frac{\omega(0)M_2^n}{\Gamma(1-\alpha)N^{2-\alpha}}T^{2-\alpha}(1+\alpha)\ .$$

Finally,

$$\frac{\omega(0)M_2^n\tau_n}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha}=\frac{\omega(0)M_2^n\tau_n^{2-\alpha}}{\Gamma(2-\alpha)}\le\frac{\omega(0)M_2^n\tau_1^{2-\alpha}}{\Gamma(2-\alpha)}=\frac{\omega(0)M_2^nT^{2-\alpha}2^{2-\alpha}}{\Gamma(2-\alpha)(N+1)^{2-\alpha}}\ .$$

Thus, combining R_1^n and each terms of R_2^n we have

$$|\partial_{0,t_n}^{\alpha,\omega(t)}v-\Delta_{0,t_n}^{\alpha,\omega(t)}v|\le\frac{M_1^nm_2^nT^2(t_n)^{1-\alpha}}{\Gamma(2-\alpha)(N+1)^2}+\frac{M_2^nm_1^nT^2t_n^{1-\alpha}}{2\Gamma(1-\alpha)(N+1)^2}$$
$$+\frac{\omega(0)M_2^nT^{2-\alpha}(1+\alpha)}{\Gamma(1-\alpha)(N+1)^{2-\alpha}}+\frac{\omega(0)M_2^nT^{2-\alpha}2^{2-\alpha}}{\Gamma(2-\alpha)(N+1)^{2-\alpha}}=\mathcal{O}(N^{\alpha-2})\ .$$

□

2.4 Theoretical analysis for non-smooth solutions

2.4.1 Stability analysis

The discrete form of the proposed scheme is :

$$\Delta_{0,t_n}^{\alpha,\omega(t)}\varpi=\sum_{s=0}^{n-1}c_s^n(\varpi_i^{s+1}-\varpi_i^s)$$
$$=c_{n-1}^n\varpi_i^n-c_0^n\varpi_i^0+\sum_{s=1}^{n-1}(c_{n-1-s}^n-c_{n-s}^n)\varpi_i^{n-s}.$$

where $i = 1, 2, \ldots, M-1$ and $n = 1, 2, \ldots, N$.

Let, $d_s^n = c_{n-s}^n$ and $d_{s+1}^n = c_{n-1-s}^n$. Thus,

$$\Delta_{0,t_n}^{\alpha,\omega(t)}\varpi = d_1^n\varpi_i^n - d_n^n\varpi_i^0 + \sum_{s=1}^{n-1}(d_{s+1}^n - d_s^n)\varpi_i^{n-s}$$
$$d_1^n = c_{n-1}^n = \frac{\tau_n^{-\alpha}}{\Gamma(2-\alpha)}\left[\omega\left(\frac{\tau_n}{2}\right) + \{\omega(0) - \omega(\tau_n)\}\frac{\alpha}{2(2-\alpha)}\right].$$

Also, since $\qquad c_{n-s}^n > c_{n-s-1}^n \Rightarrow d_{s+1}^n < d_s^n.$

Hence, the scheme (2.8)-(2.9) becomes

$$\left[d_1^n + \frac{m_{i+1/2}^n + m_{i-1/2}^n}{h^2} + p_i^n\right]\varpi_i^n = \frac{m_{i+1/2}^n\varpi_{i+1}^n}{h^2} + \frac{m_{i-1/2}^n\varpi_{i-1}^n}{h^2} + \phi_i^n$$
$$+ \left[d_n^n\varpi_i^0 + \sum_{s=1}^{n-1}(d_s^n - d_{s+1}^n)\varpi_i^{n-s}\right],$$

for $i = 1, 2, \ldots, M-1$ and $n = 1, 2, \ldots, N$.

We take

$$D_{M,N}\varpi_i^n = \Delta_{0,t_n}^{\alpha,\omega(t)}\varpi_i - \wedge\varpi_i^n = \phi_i^n \qquad \text{for } \ 1 \le i \le M-1, \quad 1 \le n \le N,$$

with

$$\varpi_0^n = \varpi_M^n = 0 \quad \text{for } 1 \le n \le N \ \text{ and } \ \varpi_i^0 = \zeta_0(x) \text{ for } 0 \le i \le M.$$

Lemma 2.4.1. [55] The solution of our discrete problem satisfies

$$||\varpi^n||_\infty \le \frac{1}{d_1^n}\left[||\phi^n||_\infty + d_n^n||\varpi^0||_\infty + \sum_{s=1}^{n-1}(d_s^n - d_{s+1}^n)||\varpi^{n-s}||_\infty\right] \ \text{ for } n = 1, 2, \ldots N.$$

Proof. For $n \in \mathbb{N}$, we choose i_0 such that $|\varpi^n_{i_0}| = ||\varpi^n||_\infty$. The equation of the scheme consisting of the grid point (x_{i_0}, t_n) is

$$\left[d_1^n + \frac{m^n_{i_0+1/2} + m^n_{i_0-1/2}}{h^2} + p^n_{i_0}\right]\varpi^n_{i_0} = \frac{m^n_{i_0+1/2}\varpi^n_{i_0+1}}{h^2} + \frac{m^n_{i_0-1/2}\varpi^n_{i_0-1}}{h^2} + \phi^n_{i_0} + \left[d_n^n\varpi^0_{i_0} + \sum_{s=1}^{n-1}(d_s^n - d_{s+1}^n)\varpi^{n-s}_{i_0}\right],$$

Now, for $p \geq 0$ and the choice of i_0, we get

$$\left[d_1^n + \frac{m^n_{i_0+1/2} + m^n_{i_0-1/2}}{h^2}\right]||\varpi^n||_\infty \leq \left(\frac{m^n_{i_0+1/2} + m^n_{i_0-1/2}}{h^2}\right)||\varpi^n||_\infty + \left|\phi^n_{i_0} + \left[d_n^n\varpi^0_{i_0} + \sum_{s=1}^{n-1}(d_s^n - d_{s+1}^n)\varpi^{n-s}_{i_0}\right]\right|,$$

which implies

$$d_1^n||\varpi^n||_\infty \leq \left|\phi^n_{i_0} + \left[d_n^n\varpi^0_{i_0} + \sum_{s=1}^{n-1}(d_s^n - d_{s+1}^n)\varpi^{n-s}_{i_0}\right]\right|.$$

Using the inequality $d^n_{s+1} < d^n_s$ proved above, one easily gets the desired result. □

Now, we define the real numbers $\theta_{n,k}$, for $n = 1, 2, \dots, N$ and $k = 1, 2, \dots, n-1$ by

$$\theta_{n,n} = 1, \theta_{n,k} = \sum_{s=1}^{n-k}\frac{1}{d_1^{n-s}}(d_s^n - d_{s+1}^n)\theta_{n-s,k}.$$

Lemma 2.4.2. [55] The solution of the discrete problem satisfies

$$||\varpi^n||_\infty \leq ||\varpi^0||_\infty + \frac{1}{d_1^n}\sum_{k=1}^{n}\theta_{n,k}||\phi^k||_\infty$$

$for \;\; n = 1, 2\dots, N.$

Proof. This Lemma can be proved like the Lemma 4.2 in [55]. □

Lemma 2.4.3. Let the parameter γ satisfy $\gamma \leq r\alpha$. Then, for $n \in \mathbb{N}$, one has

$$\frac{1}{d_1^n}\sum_{k=1}^{n} k^{-\gamma}\theta_{n,k} \leq \frac{\Gamma(2-\alpha)}{\omega(T)}\left[\frac{T^\alpha N^{-\gamma}}{1-\alpha}\right]. \tag{2.12}$$

Proof. To prove Lemma 2.4.3, we use the mathematical induction method. For n=1, we have

$$\begin{aligned}\frac{\theta_{1,1}}{d_1^1} &= \frac{\Gamma(2-\alpha)\tau_1^\alpha}{\omega(\frac{\tau_1}{2})+[\omega(0)-\omega(\tau_1)]\frac{\alpha}{2(2-\alpha)}} \leq \frac{\Gamma(2-\alpha)\tau_1^\alpha}{\omega(\frac{\tau_1}{2})} \leq \frac{\Gamma(2-\alpha)\tau_1^\alpha}{\omega(T)} = \frac{\Gamma(2-\alpha)T^\alpha}{\omega(T)N^{r\alpha}}\\ &\leq \frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)} \leq \frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)}.\end{aligned}$$

Let it be true for $n = 1, 2, \ldots l-1$, where $l \in \{2, 3, \ldots, N\}$.

$For \quad n = l$

$$\begin{aligned}\frac{1}{d_1^l}\sum_{k=1}^{l} k^{-\gamma}\theta_{l,k} &= \frac{l^{-\gamma}\theta_{l,l}}{d_1^l} + \frac{1}{d_1^l}\sum_{k=1}^{l-1} k^{-\gamma}\sum_{s=1}^{l-k}\frac{1}{d_1^{l-s}}[d_s^l - d_{s+1}^l]\theta_{l-s,k}\\ &= \frac{l^{-\gamma}}{d_1^l} + \frac{1}{d_1^l}\sum_{s=1}^{l-1}[d_s^l - d_{s+1}^l]\frac{1}{d_1^{l-s}}\sum_{k=1}^{l-s} k^{-\gamma}\theta_{l-s,k}\\ &\leq \frac{l^{-\gamma}}{d_1^l} + \frac{1}{d_1^l}\sum_{s=1}^{l-1}[d_s^l - d_{s+1}^l]\frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)}\\ &\leq \frac{l^{-\gamma}}{d_1^l} + \frac{1}{d_1^l}\left[d_1^l - d_l^l\right]\frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)}\\ &= \frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)} + \frac{1}{d_1^l}\left[l^{-\gamma} - d_l^l\frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)}\right].\end{aligned}$$

Now, we complete our induction proof by showing $\left[l^{-\gamma} - d_l^l\frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)}\right] \leq 0.$ Since,

$$\begin{aligned}d_l^l = c_0^l &= \frac{1}{\Gamma(2-\alpha)}\left[\frac{\omega(t_l-\frac{\tau_1}{2})}{\tau_1}\left[t_l^{1-\alpha}-(t_l-t_1)^{1-\alpha}\right]\right] + [\omega(t_l-t_1)-\omega(t_l)]b_0^l\\ &\geq \frac{1}{\Gamma(2-\alpha)}\left[\frac{\omega(t_l-\frac{\tau_1}{2})}{\tau_1}\left[t_l^{1-\alpha}-(t_l-t_1)^{1-\alpha}\right]\right] \geq \frac{1}{\Gamma(2-\alpha)}\left[\frac{\omega(T)}{\tau_1}\left[t_l^{1-\alpha}-(t_l-t_1)^{1-\alpha}\right]\right].\end{aligned}$$

Hence,

$$d_l^l \frac{\Gamma(2-\alpha)T^\alpha N^{-\gamma}}{\omega(T)(1-\alpha)} \geq \frac{T^\alpha N^{-\gamma}}{(1-\alpha)} \frac{\left[t_l^{1-\alpha} - (t_l - t_1)^{1-\alpha}\right]}{\tau_1} \geq l^{-\gamma} \text{ , as proved in [55].}$$

Thus, (2.12) holds for $n = l$. Hence, by the principle of mathematical induction, Lemma 2.4.3 is true for all n. □

2.4.2 Convergence analysis

Here, we first bound the truncation error and then apply the result of Lemma 2.4.2 for our analysis. It is known that the following results hold for the case of non-smooth solutions when $\omega(t) = \text{const} > 0$.

Theorem 2.4.1. [55] Assume that $\zeta_0(x) \in D(\mathcal{L}^{5/2}), \phi(.,t) \in D(\mathcal{L}^{5/2}), \phi_t(.,t)$ *and* $\phi_{tt}(.,t)$ *are in* $D(\mathcal{L}^{1/2})$ *for each* $t \in (0,T]$ *with*

$$||\phi(.,t)||_{\mathcal{L}^{5/2}} + ||\phi_t(.,t)||_{\mathcal{L}^{1/2}} + t^p||\phi_{tt}(.,t)||_{\mathcal{L}^{1/2}} \leq C_1,$$

$\forall\, t \in (0,T]$ and some constant $\rho < 1$, where C_1 is a constant independent of t. Then (2.1)-(2.2) has a unique solution that satisfies these equations pointwise, and there exist a constant C such that

$$\left|\frac{\partial^k \zeta}{\partial x^k}(x,t)\right| \leq C, \quad for \;\; k = 0,1,2,3,4, \tag{2.13}$$

$$\left|\frac{\partial^l \zeta}{\partial t^l}(x,t)\right| \leq C(1+t^{\alpha-l}), \quad for \;\; l = 0,1,2, \tag{2.14}$$

for all $(x,t) \in [0,l] \times (0,T]$.

For small enough t ($t << 1$) we have $\omega(t) = \omega(0) + \mathcal{O}(t)$. Hence, our problem becomes similar to the classical time-fractional diffusion equation for small enough t. So, we assume that the solution of the problem (2.1)-(2.2) satisfies the conditions (2.13)-(2.14) for such t. From the numerical calculations, it can be observed that these conditions are consistent with the solution of the original problem for test examples.

Also, we use the following bound from [55]

$$\tau_{s+1} = T\left(\frac{s+1}{N}\right)^r - T\left(\frac{s}{N}\right)^r \leq CTN^{-r}s^{r-1}, \tag{2.15}$$

for $s = 0, 1, \ldots, N-1$.

2.4.2.1 Truncation error for diffusion term

Truncation error for diffusion term is estimated by using equation (2.13), which yields

$$\frac{\partial^2 \zeta}{\partial x^2}(x_i, t_n) = \delta_x^2 \zeta(x_i, t_n) + \mathcal{O}(h^2), \tag{2.16}$$

for all grid points $(x_i, t_n) \in H$.

2.4.2.2 Truncation error for Caputo derivative term

Lemma 2.4.4. There exists a constant C such that for any $\alpha \in (0,1)$, function $v(t) \in D(\mathcal{L}^{5/2})$, and all $(x_i, t_n) \in H$, we have

$$|\partial_{0,t_n}^{\alpha,\omega(t)} v - \Delta_{0,t_n}^{\alpha,\omega(t)} v| \leq Cn^{-min\{2-\alpha, r\alpha\}}.$$

Proof. We have

$$\partial_{0,t_n}^{\alpha,\omega(t)} v - \Delta_{0,t_n}^{\alpha,\omega(t)} v = R_1^n + R_2^n,$$

where,

$$\begin{aligned}
|R_1^n| &\le \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n-1} |v_{t,s}|\ \tau_{s+1} \int_0^1 \frac{|\omega(t_n - t_s - \tau_{s+1} z) - \psi)|}{(t_n - t_s - \tau_{s+1} z)^\alpha} \mathrm{d}z \\
&\le \frac{m_2^n}{4\Gamma(1-\alpha)} \sum_{s=0}^{n-1} |v_{t,s}| \tau_{s+1}^3 \int_0^1 \frac{\mathrm{d}z}{(t_n - t_s - \tau_{s+1} z)^\alpha}.
\end{aligned}$$

For $1 \le s < n-1$, we have

$$\begin{aligned}
&\frac{m_2^n}{4\Gamma(1-\alpha)} |v_{t,s}| \tau_{s+1}^3 \int_0^1 \frac{\mathrm{d}z}{(t_n - t_s - \tau_{s+1} z)^\alpha} \le \frac{C m_2^n}{4\Gamma(1-\alpha)} t_s^{\alpha-1} \tau_{s+1}^2 \int_{t_s}^{t_{s+1}} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} \quad \text{by (2.14)} \\
&= \frac{C m_2^n}{4\Gamma(1-\alpha)} \left(T \left[\frac{s}{N} \right]^r \right)^{\alpha-1} \tau_{s+1}^2 \int_{t_s}^{t_{s+1}} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} = C T^{\alpha-1} s^{r(\alpha-1)} N^{-r(\alpha-1)} \tau_{s+1}^2 \int_{t_s}^{t_{s+1}} \frac{\mathrm{d}\xi}{(t_n - \xi)^\alpha} \\
&\le C T^{\alpha-1} s^{r(\alpha-1)} \tau_{s+1}^3 (t_n - t_{s+1})^{-\alpha} N^{-r(\alpha-1)} \\
&\le C T^{\alpha-1} s^{r(\alpha-1)} T^3 N^{-3r} s^{3r-3} (t_n - t_{s+1})^{-\alpha} N^{-r(\alpha-1)} \quad \text{by (2.15)} \\
&= C T^{\alpha+2} s^{r\alpha+2r-3} N^{-r\alpha-2r} \left[T \left(\frac{n}{N} \right)^r - T \left(\frac{s+1}{N} \right)^r \right]^{-\alpha} \\
&= C T^2 s^{r\alpha+2r-3} N^{-2r} \left[n^r - (s+1)^r \right]^{-\alpha}.
\end{aligned}$$

For $1 \le s \le \lceil n/2 \rceil - 1$, we have

$$C T^2 N^{-2r} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r\alpha+2r-3} \left[n^r - (s+1)^r \right]^{-\alpha} \le C T^2 N^{-2r} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r\alpha+2r-3} n^{-r\alpha}$$

$$= C T^2 n^{-r\alpha} N^{-2r} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r\alpha+2r-3} \le C T^2 n^{-r(\alpha+2)} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r(\alpha+2)-3}.$$

By the results of convergence for series, we have

$$CT^2 n^{-r(\alpha+2)} \sum_{s=1}^{\lceil n/2\rceil-1} s^{r(\alpha+2)-3} \le Cn^{-2} \quad \text{as} \quad r(\alpha+2)>2.$$

For $\lceil n/2\rceil \le s < n-1$, we have

$$\begin{aligned}
&\frac{Cm_2^n}{4\Gamma(1-\alpha)}\tau_{s+1}^2 t_s^{\alpha-1}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha} \le C(TN^{-r}s^{r-1})^2 t_s^{\alpha-1}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha}\\
&\le C(TN^{-r}n^{r-1})^2 t_n^{\alpha-1}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha} = CT^{1+\alpha}N^{-2r}n^{2(r-1)}\left(\frac{n}{N}\right)^{r(\alpha-1)}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha}\\
&= CT^{1+\alpha}N^{-r(\alpha+1)}n^{r\alpha+r-2}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha}\ .
\end{aligned}$$

Taking summation, we get

$$\begin{aligned}
&CT^{1+\alpha}N^{-r(\alpha+1)}n^{r\alpha+r-2}\sum_{s=t_{\lceil n/2\rceil}}^{t_{n-2}}\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha} = CT^{1+\alpha}N^{-r(\alpha+1)}n^{r\alpha+r-2}\int_{t_{\lceil n/2\rceil}}^{t_{n-1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^\alpha}\\
&\le \frac{CT^{1+\alpha}n^{r\alpha+r-2}(t_n-t_{\lceil n/2\rceil})^{1-\alpha}}{N^{r(\alpha+1)}} = \frac{CT^{1+\alpha}n^{r\alpha+r-2}\left[T\left(\frac{n}{N}\right)^r - T\left(\frac{\lceil n/2\rceil}{N}\right)^r\right]^{1-\alpha}}{N^{r(\alpha+1)}}\\
&\le CT^2N^{-r(\alpha+1)}n^{r\alpha+r-2}\left(\frac{n}{N}\right)^{r(1-\alpha)} = CT^2\left(\frac{n}{N}\right)^{2r}n^{-2} \le Cn^{-2}\ .
\end{aligned}$$

We established the bound for R_1^n from $s=1$ to $s=n-2$ except for the cases when $s=0$ and $s=n-1$. Now to prove the order of convergence, we also need to find the bound when $s=0$ and $s=n-1$.

Case 1: Bound for $n = 1,\ s = 0$

$$|R_1^1| = \left|\frac{m_2^1\tau_1(v_1 - v_0)}{4\Gamma(1-\alpha)}\int\limits_0^{t_1}(t_1-\xi)^{-\alpha}\mathrm{d}\xi\right| = \left|\frac{m_2^1\tau_1(v_1 - v_0)}{4\Gamma(2-\alpha)}t_1^{1-\alpha}\right| = \left|\frac{m_2^1(v_1 - v_0)}{4\Gamma(2-\alpha)}t_1^{2-\alpha}\right|$$

$$\leq \frac{m_2^1}{4\Gamma(2-\alpha)}t_1^{2-\alpha}\int\limits_{\xi=0}^{t_1}\frac{\partial v}{\partial \xi}\mathrm{d}\xi \leq C\frac{m_2^1}{4\Gamma(2-\alpha)}t_1^{2-\alpha}\int\limits_{\xi=0}^{t_1}\xi^{\alpha-1}\mathrm{d}\xi \leq Ct_1^2 = \frac{CT^2}{N^{2r}} \leq C.$$

Case 2: For $s = 0,\ n > 1$, we have

$$|R_1^n| = \left|\frac{m_2^1\tau_1(v_1 - v_0)}{4\Gamma(1-\alpha)}\int\limits_0^{t_1}(t_n-\xi)^{-\alpha}\mathrm{d}\xi\right| = \left|\frac{m_2^1\tau_1(v_1 - v_0)}{4\Gamma(2-\alpha)}\left[(t_n)^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right]\right|$$

$$\leq \frac{m_2^1\tau_1}{4\Gamma(2-\alpha)}\left[(t_n)^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right]\int\limits_{t_0}^{t_1}\left|\frac{\partial v}{\partial \xi}\right|\mathrm{d}\xi$$

$$\leq C\frac{m_2^1\tau_1}{4\Gamma(2-\alpha)}\left[(t_n)^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right]\int\limits_{t_0}^{t_1}\xi^{\alpha-1}\mathrm{d}\xi$$

$$= C\frac{m_2^1\tau_1}{4\Gamma(2-\alpha)\alpha}\left[(t_n)^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right]t_1^{\alpha} \leq C[(t_n)^{1-\alpha} - (t_n - t_1)^{1-\alpha}]t_1^{1+\alpha}$$

$$\leq C\left(\frac{t_n - t_1}{t_1}\right)^{-\alpha}t_1^2 \qquad \text{(By Mean Value Theorem)}$$

$$= C\left(\frac{t_n - t_1}{t_1}\right)^{-\alpha}\frac{T^2}{N^{2r}} \leq C\left(\frac{t_n - t_1}{t_1}\right)^{-\alpha} \leq Cn^{-r\alpha} \quad \text{for } \ n \geq 1\ .$$

Case 3: For $s = n - 1$

$$|R_1^n| = \left|\frac{m_2^1\tau_n^2 v_{t,n-1}}{4\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n}(t_n-\xi)^{-\alpha}\mathrm{d}\xi\right| \leq \frac{m_2^1\tau_n^2|v'(\eta)|}{4\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n}(t_n-\xi)^{-\alpha}\mathrm{d}\xi$$

for some $\eta \in (t_{n-1}, t_n)$,

$$\leq C\tau_n^2 t_{n-1}^{\alpha-1} \int\limits_{t_{n-1}}^{t_n} (t_n-\xi)^{-\alpha}\mathrm{d}\xi = C\frac{\tau_n^{3-\alpha} t_{n-1}^{\alpha-1}}{1-\alpha} \leq C[TN^{-r}(n-1)^{r-1}]^{3-\alpha} T^{\alpha-1}\left(\frac{n-1}{N}\right)^{r(\alpha-1)}$$
$$= CT^2N^{-2r}(n-1)^{2r+\alpha-3} \leq CT^2(n-1)^{\alpha-3}.$$

Similarly, R_2^n is estimated:

$$\begin{aligned}
|R_2^n| &= \frac{1}{\Gamma(1-\alpha)}\left|\sum_{s=0}^{n-1}\int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'\,\mathrm{d}\xi}{(t_n-\xi)^\alpha}\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\left|\sum_{s=0}^{n-2}\int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'\,\mathrm{d}\xi}{(t_n-\xi)^\alpha}\right| \\
&+ \frac{1}{\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n} \frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|\mathrm{d}\xi}{(t_n-\xi)^\alpha} \\
&\leq \frac{1}{\Gamma(1-\alpha)}\left|\sum_{s=1}^{n-2}\int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)(v(\xi)-\Pi_{1,s}v(\xi))'\,\mathrm{d}\xi}{(t_n-\xi)^\alpha}\right| \\
&+ \frac{1}{\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n} \frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|\mathrm{d}\xi}{(t_n-\xi)^\alpha} + \frac{1}{\Gamma(1-\alpha)}\left|\int\limits_{t_0}^{t_1}\frac{\omega(t_n-\xi)}{(t_n-\xi)^\alpha}\left(\frac{v_1-v_0}{\tau_1}-\frac{\partial v}{\partial \xi}\right)\mathrm{d}\xi\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\sum_{s=1}^{n-2}\int\limits_{t_s}^{t_{s+1}} \frac{|\omega'(t_n-\xi)||v(\xi)-\Pi_{1,s}v(\xi)|\,\mathrm{d}\xi}{(t_n-\xi)^\alpha} \\
&+ \frac{\alpha}{\Gamma(1-\alpha)}\sum_{s=1}^{n-2}\int\limits_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)|v(\xi)-\Pi_{1,s}v(\xi)|\mathrm{d}\xi}{(t_n-\xi)^{\alpha+1}} + \frac{1}{\Gamma(1-\alpha)}\int\limits_{t_{n-1}}^{t_n} \frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|\mathrm{d}\xi}{(t_n-\xi)^\alpha} \\
&+ \frac{1}{\Gamma(1-\alpha)}\left|\int\limits_{t_0}^{t_1}\frac{\omega(t_n-\xi)}{(t_n-\xi)^\alpha}\left(\frac{v_1-v_0}{\tau_1}-\frac{\partial v}{\partial \xi}\right)\mathrm{d}\xi\right|. \qquad (2.17)
\end{aligned}$$

Let the terms in R.H.S of (2.17) be $\gamma_1+\gamma_2+\gamma_3+\gamma_4$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are first, second, third and fourth terms, respectively.

Now,

$$\frac{1}{\Gamma(1-\alpha)}\int_{t_s}^{t_{s+1}}\frac{|\omega'(t_n-\xi)||v(\xi)-\Pi_{1,s}v(\xi)|\,\mathrm{d}\xi}{(t_n-\xi)^{\alpha}}\le\frac{m_1^n}{\Gamma(1-\alpha)}\int_{t_s}^{t_{s+1}}\frac{|(\Phi-\Psi)(\xi)|\,\mathrm{d}\xi}{(t_n-\xi)^{\alpha}},$$

where

$$m_1^n=\max_{0\le t\le t_n}|\omega'(t)|,\quad \Phi(\xi)=\left[\frac{v_{s+1}-v_s}{\tau_{s+1}}\right](\xi-t_s),\quad \Psi(\xi)=v(\xi)-v(t_s).$$

Then, by mean value theorem of integrals,

$$\frac{m_1^n}{\Gamma(1-\alpha)}\int_{t_s}^{t_{s+1}}\frac{(\Phi-\Psi)(\xi)\,\mathrm{d}\xi}{(t_n-\xi)^{\alpha}}=\frac{m_1^n}{\Gamma(1-\alpha)}(\Phi-\Psi)(\beta_1)\int_{t_s}^{t_{s+1}}\frac{\mathrm{d}\xi}{(t_n-\xi)^{\alpha}},$$

for some $\beta_1\in(t_s,t_{s+1})$.

Also,

$$\begin{aligned}(\Phi-\Psi)(\beta_1)&=\left[\frac{v_{s+1}-v_s}{\tau_{s+1}}\right](\beta_1-t_s)-[v(\beta_1)-v(t_s)]\\&=(\beta_1-t_s)[v_t(\beta_2)-v_t(\beta_3)]=(\beta_1-t_s)(\beta_2-\beta_3)v_{tt}(\beta_4),\end{aligned}$$

for some $\beta_2,\beta_3,\beta_4\in(t_s,t_{s+1})$. Therefore, we get for $1\le s<n-1$

$$\begin{aligned}&\left|\frac{m_1^n}{\Gamma(1-\alpha)}\int_{t_s}^{t_{s+1}}\frac{(\Phi-\Psi)(\xi)\,\mathrm{d}\xi}{(t_n-\xi)^{\alpha}}\right|\le C\tau_{s+1}^2\max_{\xi\in[t_s,t_{s+1}]}|v_{tt}(\xi)|\int_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha}\mathrm{d}\xi\\&\le C\tau_{s+1}^2t_s^{\alpha-2}\int_{t_s}^{t_{s+1}}(t_n-\xi)^{-\alpha}\mathrm{d}\xi\le C\tau_{s+1}^2t_s^{\alpha-2}\tau_{s+1}(t_n-t_{s+1})^{-\alpha}\\&=C\tau_{s+1}^3T^{\alpha-2}\left(\frac{s}{N}\right)^{r(\alpha-2)}T^{-\alpha}\left[\left(\frac{n}{N}\right)^r-\left(\frac{s+1}{N}\right)^r\right]^{-\alpha}\le CTN^{-r}s^{r+r\alpha-3}[n^r-(s+1)^r]^{-\alpha}.\end{aligned}$$

For $1 \leq s \leq \lceil n/2 \rceil - 1$, we have

$$
\begin{aligned}
CTN^{-r} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r+r\alpha-3} \left[n^r - (s+1)^r \right]^{-\alpha} &\leq CTN^{-r} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r+r\alpha-3} n^{-r\alpha} \\
&\leq CTn^{-r(\alpha+1)} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r(\alpha+1)-3} \ .
\end{aligned}
$$

Hence, convergence results for series [55] imply

$$
CTn^{-r(\alpha+1)} \sum_{s=1}^{\lceil n/2 \rceil - 1} s^{r(\alpha+1)-3} \leq \begin{cases} Cn^{-r(\alpha+1)} & \text{if } r(\alpha+1) < 2, \\ Cn^{-2} \ln n & \text{if } r(\alpha+1) = 2, \\ Cn^{-2} & \text{if } r(\alpha+1) > 2. \end{cases}
$$

For $\lceil n/2 \rceil \leq s < n - 1$

$$
\begin{aligned}
& C\tau_{s+1}^2 \max_{\xi \in [t_s, t_{s+1}]} |v_{tt}(\xi)| \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi \leq C(TN^{-r}s^{r-1})^2 t_s^{\alpha-2} \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi \\
& \leq CT^2 N^{-2r} n^{2(r-1)} t_n^{\alpha-2} \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi = CT^\alpha N^{-2r} n^{2(r-1)} \left(\frac{n}{N} \right)^{r(\alpha-2)} \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi \\
& = CT^\alpha N^{-r\alpha} n^{r\alpha-2} \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi \ .
\end{aligned}
$$

Taking summation,

$$
\begin{aligned}
& CT^\alpha N^{-r\alpha} n^{r\alpha-2} \sum_{s=t_{\lceil n/2 \rceil}}^{t_{n-2}} \int_{t_s}^{t_{s+1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi = CT^\alpha N^{-r\alpha} n^{r\alpha-2} \int_{s=t_{\lceil n/2 \rceil}}^{t_{n-1}} (t_n - \xi)^{-\alpha} \mathrm{d}\xi \\
& \leq CT^\alpha N^{-r\alpha} n^{r\alpha-2} (t_n - t_{\lceil n/2 \rceil})^{1-\alpha} = CT^\alpha N^{-r\alpha} n^{r\alpha-2} \left[T \left(\frac{n}{N} \right)^r - T \left(\frac{\lceil n/2 \rceil}{N} \right)^r \right]^{1-\alpha} \\
& \leq CT^\alpha N^{-r\alpha} n^{r\alpha-2} \left[T \left(\frac{n}{N} \right)^r \right]^{1-\alpha} = CTN^{-r} n^{r-2} \leq Cn^{-2} \ .
\end{aligned}
$$

Thus, we found the bound for γ_1. Similarly,

$$\begin{aligned}\gamma_2 &= \frac{\alpha}{\Gamma(1-\alpha)} \sum_{s=1}^{n-2} \int_{t_s}^{t_{s+1}} \frac{\omega(t_n-\xi)|v(\xi)-\Pi_{1,s}v(\xi)|\mathrm{d}\xi}{(t_n-\xi)^{\alpha+1}} \\ &\le \frac{\alpha\omega(0)}{\Gamma(1-\alpha)} \sum_{s=1}^{n-2} \int_{t_s}^{t_{s+1}} \frac{|v(\xi)-\Pi_{1,s}v(\xi)|\mathrm{d}\xi}{(t_n-\xi)^{\alpha+1}} \,.\end{aligned}$$

This term can be bounded similar to work [55]. Again the term γ_4 of (2.17) can also be bounded as done in [55]. So, it remains only to bound the term γ_3 in (2.17) for $n > 1$. As for $n = 1$, the case is same as the term γ_4 of (2.17).

$$\begin{aligned}\gamma_3 &= \frac{1}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_n} \frac{\omega(t_n-\xi)|v'(\xi)-v_{t,n-1}|\mathrm{d}\xi}{(t_n-\xi)^{\alpha}} \\ &\le \frac{\omega(0)}{\Gamma(1-\alpha)} \int_{t_{n-1}}^{t_n} |v'(\xi)-v_{t,n-1}|(t_n-\xi)^{-\alpha}\mathrm{d}\xi \\ &= \frac{\omega(0)}{\Gamma(1-\alpha)} \left|\frac{\partial v}{\partial t}(\beta_5) - \frac{\partial v}{\partial t}(\beta_6)\right| \int_{t_{n-1}}^{t_n} (t_n-\xi)^{-\alpha}\mathrm{d}\xi \\ &\le \frac{\omega(0)\tau_n}{\Gamma(2-\alpha)} \left|\frac{\partial^2 v}{\partial t^2}(\beta_7)\right| (t_n-t_{n-1})^{1-\alpha} \le \frac{Ct_{n-1}^{\alpha-2}}{\Gamma(2-\alpha)}\tau_n((t_n-t_{n-1})^{1-\alpha}) \le Ct_{n-1}^{\alpha-2}\tau_n^{2-\alpha} \\ &\le C[TN^{-r}(n-1)^{r-1}]^{2-\alpha}T^{\alpha-2}\left(\frac{n-1}{N}\right)^{\alpha-2} \le Cn^{\alpha-2} \,.\end{aligned}$$

and hence $\gamma_3 \le Cn^{-(2-\alpha)}$ where $\beta_5,\ \beta_6,\ \beta_7 \in (t_{n-1}, t_n)$.

Now, we combine the bounds for $\gamma_1,\ \gamma_2,\ \gamma_3$ and γ_4 and then by using similar steps given in [55], we obtain the desired result. □

Theorem 2.4.2. The solution ϖ_i^n of the scheme (2.8)-(2.9) satisfies

$$\max_{(x_i,t_n)\in H} |\zeta(x_i,t_n) - \varpi_i^n| \le C\left(h^2 + \frac{T^\alpha N^{-min\{2-\alpha, r\alpha\}}}{\omega(T)}\right),$$

for some constant C.

Proof. For a fixed $(x_i, t_n) \in H$ the truncation error of (2.8)-(2.9) at (x_i, t_n) is

$$\left|D_{M,N}[\zeta(x_i,t_n)-\varpi_i^n]\right| = |\Delta_{0,t_n}^{\alpha,\omega(t)}[\zeta(x_i,t_n)-\varpi_i^n]-\wedge[\zeta(x_i,t_n)-\varpi_i^n]| \leq C(h^2+n^{-min\{2-\alpha,r\alpha\}}),$$

by Lemma 2.4.4 and equation (2.16). Now, $D_{M,N}[\zeta(x_i,t_n) - \varpi_i^n] = \chi_i^n$. So, when Lemma 2.4.2 is invoked with $v_i^n = \zeta(x_i,t_n) - \varpi_i^n$ (the initial condition $v_i^0 = 0$ for all i), we get

$$\begin{aligned}
\underset{(x_i,t_n)\in H}{max}|\zeta(x_i,t_n) - \varpi_i^n| &\leq \frac{1}{d_1^n}\sum_{k=1}^{n}\theta_{n,k}||\chi^k||_\infty \\
&\leq C\frac{1}{d_1^n}\sum_{k=1}^{n}\theta_{n,k}(h^2 + k^{-min\{2-\alpha,r\alpha\}}) \\
&\leq \frac{CT^\alpha}{\omega(T)}(h^2 + N^{-min\{2-\alpha,r\alpha\}}),
\end{aligned}$$

by Lemma 2.4.3 with $\gamma = 0$ for the h^2 term and $\gamma = min\{2-\alpha, r\alpha\}$ for the term involving $N^{-min\{2-\alpha,r\alpha\}}$. □

2.5 Numerical examples

In order to test the numerical accuracy of the proposed schemes in this chapter, three test examples are investigated. The following norms are used for the maximum error and CO for the domain and at the final time, respectively: $||.||_{H(\bar{\chi}_{h\tau})}$ and $||.||_{H(T)}$, where $||w||_{H(\bar{\chi}_{h\tau})} = \max_{(x_i,t_j)\in\bar{\chi}_{h\tau}}|w|$, $||w||_{H(T)} = \max_{(1\leq x_i\leq M-1)}|w(x_i,t_N)|$ and $w = \varpi - \zeta$ is the error, where ϖ is the approximate solution. CO $= \log_{\frac{N1}{N2}}\frac{||w_1||}{||w_2||}$ in time and CO $= \log_{\frac{h1}{h2}}\frac{||w_1||}{||w_2||}$ in space. The L_2-norm is defined by $||\cdot||_0$. The CPU time is measured in seconds in all the examples. All the numerical experiments are carried out on a Laptop with (a) 11th Gen Intel(R) Core (TM) i5-1135G7 CPU @ 2.40 GHz (b) RAM 8 GB, and (c) System type 64-bit operating system running on MATLAB 2019a (The MathWorks, Inc., Natick, Massachusetts) programming.

Example 2.5.1. The first example is from [24], where $\zeta(x,t) = \left(\frac{6-e^{-bt}(6bt+3b^2t^2+b^3t^3+6)}{b^4} + 1\right)$ $\sin(\pi x)$ is the exact solution of problem (2.1)-(2.2) with $\omega(t) = e^{-bt}$, $b \geq 0$, and the coefficients $p(x,t) = 1 - sin(xt)$, $m(x,t) = 2 - \cos(xt)$ and T=1. Since the solution of this example is smooth, therefore we use the non-uniform mesh as defined in equation (2.10).

The outcomes of Example 2.5.1 are presented as:

- To verify the CO, we calculate L_2-norm and maximum norm errors by using different step sizes in the temporal and spatial directions in Tables (2.1)-(2.2). From the outcomes of Table 2.1 and Table 2.2, it can be concluded that CO of temporal and spatial errors are $\mathcal{O}(N^{\alpha-2} + h^2)$.
- Table 2.2 shows how the reduction in error takes place as the number of intervals in space and time steps is increased keeping $M^2 = N^{2-\alpha}$.
- Finally, the outcomes of Table 2.1 and Table 2.2 confirm the theoretical results and show the effectiveness of the proposed scheme.

Table 1

			Uniform			Non-Uniform		
b	α	N	$\|\|w\|\|_{H(T)}$	CO	CPU time	$\|\|w\|\|_{H(T)}$	CO	CPU time
3.0	0.2	10	3.1097e-05	-	711.24	2.0562e-05	-	718.09
		20	8.6482e-06	1.8463	767.37	5.4491e-06	1.9159	760.69
		40	2.4174e-06	1.8389	817.03	1.4298e-06	1.9302	860.53
		80	6.8234e-07	1.8249	963.77	3.7700e-07	1.9232	998.00
2.0	0.5	10	1.0377e-04	-	715.34	4.7048e-05	-	725.27
		20	3.4957e-05	1.5698	780.53	1.3994e-05	1.7493	784.63
		40	1.1930e-05	1.5511	847.92	4.2719e-06	1.7119	868.91
		80	4.1179e-05	1.5345	980.87	1.3402e-06	1.6724	1000.76
1.0	0.9	10	02.0e-03	-	720.51	9.5000e-04	-	730.61
		20	9.3665e-04	1.0785	810.36	4.2150e-04	1.1724	762.78
		40	4.4005e-04	1.0898	872.58	1.9116e-04	1.1407	875.47
		80	2.0600e-04	1.0950	1006.31	8.7750e-05	1.1233	985.21

TABLE 2.1: Maximum norm errors and convergence order at final time for Ex. 2.5.1 for $h = 1/10000$.

Table 2

b	α	h	$\|\|w^N\|\|_0$	CO in $\|\|\cdot\|\|_0$	$\|\|w^N\|\|_{H(T)}$	CO in $\|\|.\|\|_{H(T)}$	CPU time
1.0	0.1	1/10	5.5e-03	-	7.8e-03	-	0.76
		1/20	1.4e-03	2.0014	1.9e-03	2.000	0.81
		1/40	3.4648e-04	2.0000	4.8855e-04	1.9963	0.80
		1/80	8.6624e-05	1.9999	1.2214e-04	1.9999	0.85
2.0	0.5	1/10	5.4e-03	-	7.5e-03	-	0.73
		1/20	1.3e-03	2.0029	1.9e-03	2.0024	0.81
		1/40	3.3480e-04	2.0007	4.7207e-04	1.9966	0.85
		1/80	8.3685e-05	2.0002	1.1800e-04	2.0003	1.11
3.0	0.9	1/10	5.3e-03	-	7.4e-03	-	0.82
		1/20	1.3e-03	2.0035	1.9e-03	2.0034	0.91
		1/40	3.2961e-04	2.0009	4.6473e-04	1.9966	2.04
		1/80	8.2391e-05	2.0002	1.1617e-04	2.0002	16.73

TABLE 2.2: L_2-norm and maximum norm errors and convergence order when $N = \lceil M^{2/(2-\alpha)} \rceil$.

Remark 2.5.1. Since the order of convergence in the case of non-smooth solutions is $\mathcal{O}(N^{-min\{2-\alpha, r\alpha\}})$ if we choose r=1 we get a rate of α, while for a choice of $r = (2-\alpha)/(2\alpha)$ we see a rate of $(2-\alpha)/2$ and when $r \geq (2-\alpha)/\alpha$, the optimal rate of $(2-\alpha)$ is attained. Thus, in our examples we used $r = (2-\alpha)/\alpha$.

Example 2.5.2. We take next example with non-smooth solution [24] with data as: $p(x,t) = 0,\ m(x,t) = 1,\ \alpha = 0.5,\ \phi(x,t) = 0,\ \omega(t) = e^{-5t}$ and $\zeta_0(x) = \sin(\pi x)$. In this case, the exact solution of the problem is unknown so we use the double computation method on two different grids $\bar{\chi}_{h\tau_1}$ and $\bar{\chi}_{h\tau_2}$, where $\bar{\chi}_{h\tau_1} \subset \bar{\chi}_{h\tau_2}$ to find CO. Here, we use the grid defined in equation (2.11). The results are given in Table 2.3.

Table 3

		Uniform [24]			**Non-Uniform**		
N_1	N_2	$\|\|\varpi^{N_1} - \varpi^{N_2}\|\|_H$	CO in H	CPU time	$\|\|\varpi^{N_1} - \varpi^{N_2}\|\|_H$	CO in H	CPU time
40	80	6.2671e-02	-	3.04	8.7e-03	-	3.50
80	160	5.7034e-02	0.1360	4.40	3.7e-03	1.2200	4.53
160	320	4.9249e-02	0.2117	7.70	1.5e-03	1.3226	7.99

Table 2.3: Comparison of maximum norm errors for uniform and non-uniform grid when $h = 1/1000$, $\alpha = 0.5$.

Table 4

		Uniform [24]			**Non-Uniform**		
N_1	N_2	$\|\|\varpi_N^{N_1} - \varpi_N^{N_2}\|\|_H$	CO in t_N	CPU time	$\|\|\varpi_N^{N_1} - \varpi_N^{N_2}\|\|_H$	CO in t_N	CPU time
40	80	3.3865e-05	-	3.12	1.2360e-05	-	3.49
80	160	1.5927e-05	1.0883	4.51	4.2148e-06	1.5522	4.55
160	320	7.6690e-06	1.0544	7.76	1.4593e-06	1.5302	7.98

Table 2.4: Comparison of maximum norm errors at final time for uniform and non-uniform grid when $h = 1/1000$, $\alpha = 0.5$.

The numerical results are found and illustrated below:

- Table 2.3 and Table 2.4 present that non-uniform mesh gives CO calculated in the domain and at the final time both, equivalent to $2 - \alpha$, whereas using uniform mesh leads to a decrease in CO due to the presence of singularity in the derivative of the solution. The results obtained from non-uniform mesh are compared with uniform grid results given in [24].

- We can also split the problem into smooth and non-smooth part as done in [24], then the solution can be found in the following form:

$$\zeta(x,t) = y(x,t) - \pi^2 \frac{\sqrt{t}}{\Gamma(1.5)} e^{-5t} \sin(\pi x) + \sin(\pi x),$$

where $y(x,t)$ will be a smooth solution to the problem

$$\partial_{0,t}^{0.5,\omega(t)} y = \frac{\partial^2 y}{\partial x^2} + \pi^4 \sin(\pi x) \frac{\sqrt{t}}{\Gamma(1.5)} e^{-5t} - (1 + te^{-5t} - e^{-5t})\pi^2 \sin(\pi x),$$

$$y(0,t) = 0, \; y(1,t) = 0, \; t \in [0,1], \; y(x,0) = 0, \; x \in [0,1].$$

- From Table 2.5, it can be observed that after using splitting method non-uniform grid gives $\mathcal{O}(N^{\alpha-2})$ CO whereas uniform mesh used in [24] gives CO equivalent to one. At final time, we get similar results from uniform and non-uniform grid.

Table 5

		Uniform [24]			Non-Uniform		
N_1	N_2	$\|\|\varpi^{N_1} - \varpi^{N_2}\|\|_H$	CO in H	CPU time	$\|\|\varpi^{N_1} - \varpi^{N_2}\|\|_H$	CO in H	CPU time
40	80	4.9565e-02	-	2.78	3.3000e-03	-	2.92
80	160	2.6688e-02	0.8931	4.53	1.2000e-03	1.4417	4.69
160	320	1.3354e-02	0.9988	7.85	4.4673e-04	1.4608	7.90

TABLE 2.5: Comparison of maximum norm errors in domain for uniform and non-uniform grid after splitting when $h = 1/1000$, $\alpha = 0.5$.

Example 2.5.3. Here, we construct a new example with non-smooth solution. Let $\zeta(x,t) = t^{\beta} \sin(\pi x) e^{-bt}$ be the exact solution of (2.1)-(2.2) with $\omega(t) = e^{-5t}$, $0 < \beta < 1$, the coefficients $p(x,t) = 1 - sin(xt),\; m(x,t) = 2 - \cos(xt)$ and T=1.

The numerical outcomes of this example are obtained and highlighted below:

- It is observed that the exact solution is non-smooth as the first derivative is singular at $t = 0$. Thus, Table 2.6 shows uniform mesh produces poor results in this case.

- In Table 2.6, maximum norm errors and CO calculated in the domain are given for different values of α and β with fixed $h = 1/1000$ in the temporal direction. The results for uniform and non-uniform mesh are obtained and compared showing non-uniform grid produces CO around $N^{\alpha-2}$ which validates the theoretical results.

- If we fix $N = 5000$ and varying $h = l/M$, $l = 1$, then maximum norm errors and CO in space for Example 2.5.3 are given in Table 2.7.

Table 6

			Uniform			Non-Uniform		
α	β	N	$\|\|w\|\|_{H(\bar{\chi}_{h\tau})}$	CO	CPU time	$\|\|w\|\|_{H(\bar{\chi}_{h\tau})}$	CO	CPU time
0.3	0.3	40	1.640e-02	-	1.68	7.6612e-04	-	2.19
		80	1.460e-02	0.1629	2.27	2.7550e-04	1.4753	2.28
		160	1.330e-02	0.1390	3.47	9.3877e-05	1.5534	3.52
		320	1.210e-02	0.1295	5.82	3.1221e-05	1.5883	5.84
0.5	0.5	40	1.530e-02	-	1.67	1.3000e-03	-	1.65
		80	1.240e-02	0.3002	2.20	5.4104e-04	1.2751	2.17
		160	1.000e-02	0.3106	3.31	2.1220e-04	1.3503	3.33
		320	8.000e-03	0.3313	5.73	8.0172e-05	1.4042	5.66
0.9	0.9	40	5.200e-03	-	1.72	3.4400e-03	-	1.70
		80	2.800e-03	0.8877	2.28	1.7000e-03	0.9880	2.31
		160	1.500e-03	0.9250	3.42	8.4315e-04	1.0150	3.39
		320	7.748e-04	0.9283	5.84	4.1280e-04	1.0303	5.76

TABLE 2.6: Maximum norm error and CO calculated in domain for Ex. 2.5.3 for $h = 1/1000$.

Table 7

α	β	h	$\max\limits_{0\leq n\leq N} \|\|w^n\|\|_0$	CO in $\|\| \cdot \|\|_0$	$\|\|w\|\|_{H(\bar{\chi}_{h\tau})}$	CO in $\|\| \cdot \|\|_{H(\bar{\chi}_{h\tau})}$	CPU time
0.3	0.3	1/10	1.5000e-03	-	2.1000e-03	-	39.75
		1/20	3.7361e-04	2.0029	5.2833e-04	2.0029	39.51
		1/40	9.3565e-05	1.9975	1.3231e-04	1.9975	40.60
0.5	0.5	1/10	9.0033e-04	-	1.3000e-03	-	34.72
		1/20	2.2440e-04	2.0044	3.1729e-04	2.0044	34.56
		1/40	5.5938e-05	2.0042	7.9093e-05	2.0042	35.55
0.9	0.9	1/10	4.0756e-04	-	5.7599e-04	-	38.89
		1/20	9.8349e-05	2.0510	1.3898e-04	2.0511	38.69
		1/40	2.2043e-05	2.1576	3.1142e-05	2.1580	41.05

Table 2.7: L_2-norm and maximum norm errors and spatial CO calculated in the domain when $N = 5000$.

2.6 Conclusion

The presented schemes in this chapter are based on two different non-uniform grids, one for smooth and the other for non-smooth solutions of the mathematical model (2.1)-(2.2). The reason behind using two different meshes is that for the smooth solution, we require a grading of the mesh such that the gridding should be finer near the end point $t = 1$ due to the kernel singularity. Similarly, for the non-smooth solution, we require a grading of the mesh such that the gridding should be finer near $t = 0$ to handle the singularity of the solution near $t = 0$. We see that the non-uniform grid used for smooth solutions in equation (2.10) behaves similar to uniform grid (Tables 1-2) but the non-uniform grid used for non-smooth solutions in eqution (2.11) give much better results than the uniform grid with equivalent CPU time

(Tables 3-6). Stability and convergence order of $\mathcal{O}(N^{\alpha-2})$ of the proposed schemes have been developed for smooth and non-smooth solution with respect to L_2-norm (See Theorem 2.3.1 and Lemma 2.3.3) and L_∞-norm (See Lemma 2.4.4 and Theorem 2.4.2), respectively and also second order of convergence has been established in the spatial direction. The comparative study of our scheme for the non-uniform case with [24] have been shown in Tables (2.3)-(2.5) and it is reflected that our scheme performs with better error efficiency and CO compared to the scheme given in [24].

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