

Chapter 1

Introduction

Let G be a separable locally compact group, and π an irreducible unitary representation of G on a separable Hilbert space \mathcal{H} . Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on \mathcal{H} . An **inductive algebra** is a weakly closed abelian sub-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ such that

$$\pi(g)\mathcal{A}\pi(g^{-1}) = \mathcal{A}, \quad \forall g \in G.$$

\mathcal{A} is said to be maximal if it is a maximal element of the set of inductive algebras, partially ordered by inclusion. If we wish to emphasise the dependence on π , we will use the term π -inductive algebra.

Inductive algebras typically arise from “realizations” of π on a space of sections of a homogeneous vector bundle. Indeed, suppose $\pi : E \rightarrow X$ is a vector bundle, together with a quasi-invariant measure μ on X , and G acts on E and X in such a way that π is equivariant. Then G acts on the space of sections $L^2(X, E, \mu)$ (see [11]) by

$$(g \cdot s)(x) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}} gs(g^{-1}x), \quad g \in G, s \in L^2(X, E, \mu).$$

An equivariant embedding of \mathcal{H} in $L^2(X, E, \mu)$ may be thought of as a realization of π . In this situation the set

$$\mathcal{A} = \{m_f \mid f \in L^\infty(X) \text{ and } fs \in \mathcal{H} \ \forall s \in \mathcal{H}\}$$

is an inductive algebra.

Mackey's results (see [5, §2.5, §2.6, §3.7] and [16, B.10]) give a partial inverse to the previous construction. A self-adjoint inductive algebra arises from a *system of imprimitivity*, which may be thought of as a “measurable vector bundle”. More precisely, there is a standard Borel G -space X , a quasi-invariant measure μ on X , a Hilbert space E and a cocycle A such that π is unitarily equivalent to the action of G on $L^2(X, E, \mu)$ given by

$$(g \cdot f) = A(g, x)f(g^{-1}x),$$

and \mathcal{A} corresponds to the algebra $\{m_f \mid f \in L^\infty(X)\}$ under this equivalence.

Even if \mathcal{A} is not self-adjoint, we hope that inductive algebras will provide a crutch for the construction of hitherto unknown realizations.

Inductive algebras have also found applications in operator theory (see e.g. [4, 6]).

In Chapter 2, we show that each irreducible representation of the affine group of a finite field has a unique maximal inductive algebra, and it is self-adjoint. In the course of the proof, we prove that an inductive algebra does not have non-zero nilpotent elements (Lemma 2.3.2).

In Chapter 3, we show that inductive algebras for a compact group are self-adjoint. This result depends on Lemma 2.3.2.

In Chapter 4, we show that each irreducible representation of the motion group of the plane has a unique maximal inductive algebra, and it is self-adjoint. The

techniques used here are infinitesimal (Lie theoretic).

1.1 Previous work

- Vemuri (1997) classified inductive algebras for the Heisenberg group.
- Steger and Vemuri (2005) classified inductive algebras for $SL(2, \mathbb{R})$.
- Raghavan (2005) showed that every finite dimensional inductive algebra for a connected group is trivial.
- Stegel classified the inductive algebras for the automorphism group of a locally finite homogeneous tree (2004) and the group of even automorphisms of a homogeneous tree (2006).
- Prasad and Vemuri (2010) classified the inductive algebras for the universal cover of $SL(2, \mathbb{R})$.