# Chapter 1

# Preliminaries

In this chapter, we provide an introduction to the essential concepts, notations, and findings related to fractional differential equations (FDEs) that will be utilized in the discussion that follows.

## 1.1 Some Notations, Concepts, and Theorems

Let $(X, \|\cdot\|_X)$ and $(U, \|\cdot\|_U)$ be two Banach spaces, and $J = [0, T], T \in \mathbb{R}^+$; be the closed interval of the set of real numbers $\mathbb{R}$. Consider $C(J, X)$ as a Banach space comprising all continuous functions from $J$ to $X$. The norm in this space is defined as $\|\mathsf{x}\|_c = \sup\{|\mathsf{x}(t)| : t \in J\}$. Moreover, $BL(X, U)$ represents the set of all bounded linear operators from $X$ to $U$, equipped with the operator norm $\|\cdot\|_{BL(X,U)}$. When $X = U$, we denote $BL(X, U)$ simply as $BL(X)$ with the norm $\|\cdot\|_{BL(X)}$ and $\mathbb{R}^+$ denote the set of positive real numbers. The notation $C^n(J, X)$ denotes the set of functions that are $n$-times continuously differentiable on the interval $J$ and take values in $X$ where $n \in \mathbb{N}_0$, and $\mathbb{N}_0$ represents the set of non-negative integers.

Let $\mathcal{G} \subset \mathbb{R}^n,\ n \in \mathbb{N}$. For $1 \le p < \infty$, the space $L^p(\mathcal{G}, X)$ is defined as
$L^p(\mathcal{G}, X) := \left\{ \mathsf{f} : \mathcal{G} \to X : \mathsf{f} \text{ is Bochner measurable and } \left(\int_{\mathcal{G}} \|\mathsf{f}(\tau)\|_X^p d\tau\right)^{\frac{1}{p}} < \infty \right\}.$

This is a Banach space equipped with the norm

$$\|\mathsf{f}\|_{L^p(\mathcal{G},X)} = \left(\int_{\mathcal{G}} \|\mathsf{f}(\tau)\|_X^p d\tau\right)^{\frac{1}{p}}.$$

For $p = \infty$, the space $L^\infty(\mathcal{G}, X)$ is defined as
$L^\infty(\mathcal{G}, X) := \left\{ \mathsf{f} : \mathcal{G} \to X : \mathsf{f} \text{ is Bochner measurable and } \operatorname{ess\,sup}_{\mathsf{x} \in \mathcal{G}} \|\mathsf{f}(\mathsf{x})\|_X < \infty \right\}.$

This is a Banach space equipped with the norm

$$\|\mathsf{f}\|_{L^\infty(\mathcal{G},X)} = \operatorname*{ess\,sup}_{\mathsf{x} \in \mathcal{G}} \|\mathsf{f}(\mathsf{x})\|_X.$$

When $X = \mathbb{R}$, then we represent $L^p(\mathcal{G}, X)$ by $L^p(\mathcal{G})$.
Let $\mathbb{N}^m$ represent the $m$-dimensional set of natural numbers. For example$n \in \mathbb{N}$, the $X$-valued Sobolev space $W^{n,p}(\mathcal{G}, X)$ is defined as

$$W^{n,p}(\mathcal{G}, X) = \{\mathsf{x} \in L^p(\mathcal{G}, X) : \partial^\gamma \mathsf{x} \text{ (weak sense)} \in L^p(\mathcal{G}, X), \forall\ \gamma \in \mathbb{N}^m \text{ with } |\gamma| \le n\},$$

where it is equipped with the norm

$$\|\mathsf{x}\|_{W^{n,p}(\mathcal{G},X)} = \sum_{0 \le |\gamma| \le n} \|\partial^\gamma \mathsf{x}\|_{L^p(\mathcal{G},X)},$$

where for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{N}^m, \quad |\gamma| = \sum_{i=1}^{n} \gamma_i$ and $\partial^\gamma \mathsf{x} = \frac{\partial^{|\gamma|} \mathsf{x}}{\partial_{\tau_1}^{\gamma_1}, \ldots, \partial_{\tau_m}^{\gamma_m}}$.
For $X = \mathbb{R}$, the space $W^{n,p}(\mathcal{G}, X)$ reduces to the standard Sobolev space $W^{n,p}(\mathcal{G})$.

**Definition 1.1.** [1] **(Contraction mapping)** Consider a non-empty complete metric space $(X, d)$. A function $\mathsf{f}$ from $(X, d)$ into itself is said to be a contraction if for all $\mathsf{x}, \bar{\mathsf{x}} \in X$, there exists $q \in [0, 1)$ such that $d\big(\mathsf{f}(\mathsf{x}), \mathsf{f}(\bar{\mathsf{x}})\big) \le q d(\mathsf{x}, \bar{\mathsf{x}})$.

**Lemma 1.2.** *[1]* ***(Banach fixed-point theorem)*** *For a non-empty complete metric space* $(X, d)$*, a contraction mapping* $\mathsf{f} : (X, d) \to (X, d)$ *has a unique fixed-point in* $X$.

**Lemma 1.3.** *[2] Suppose* $X$ *is a Banach space, and* $\mathcal{C} \subset \mathcal{B}$*,* $0 \in \mathcal{C}$ *is an open set, where* $\mathcal{B} \subseteq X$ *is closed and convex. Suppose that* $\mathsf{f}$ *from* $\overline{\mathcal{C}}$ *to* $\mathcal{B}$ *is completely continuous. Then either* $\exists\ \mathsf{x} \in \partial\mathcal{C}$ *such that* $\mathsf{x} = \lambda\mathsf{f}(\mathsf{x}$ *for some* $0 < \lambda < 1$*, or* $\mathsf{f}$ *has a fixed-point in* $\overline{\mathcal{C}}$.

**Definition 1.4. (Completely continuous)** Let $\mathsf{f} : D \subset X \to Y$ be a continuous mapping, with $X$ and $Y$ being Banach spaces. Then, $\mathsf{f}$ is said to be completely continuous if it maps every bounded subset $B$ of $D$ into a relatively compact subset of $Y$.

**Definition 1.5.** [3, 4, 5] Let $\mathcal{A}(t)$ be a closed and linear operator with domain $\mathcal{D}(\mathcal{A})$ defined on a Banach space $X$, and let $\eta > 0$. If $\varrho\big(\mathcal{A}(t)\big)$ represents the resolvent set of $\mathcal{A}(t)$, where $\mathcal{A}(t)$ is the generator of an $\eta$-resolvent family. If there exists a constant $\omega \geq 0$ and function $\mathcal{R}_\eta : \mathbb{R}_+^2 \to BL(X)$ such that $\{\mu^\eta : \mathrm{Re}(\mu) > \omega\} \subset \varrho\big(\mathcal{A}(t)\big)$ and

$$\big(\mu^\eta I - \mathcal{A}(p)\big)^{-1}\mathsf{x} = \int_0^\infty e^{-\mu(t-p)}\mathcal{R}_\eta(t, p)\ \mathsf{x}\ dt, \quad \mathsf{x} \in X.$$

The, the function $\mathcal{R}_\eta(t, p)$ is termed as the two-parameter $\eta$-resolvent family with generator $\mathcal{A}(t)$.

*Remark* 1.6. [5, 6]For $0 \leq s \leq t \leq T < \infty$, the two-parameter $\eta$-resolvent family $\mathcal{R}_\eta(t, p) \in BL(X)$ has the following characteristics.

(*a*) $\mathcal{R}_\eta(p, p) = I$, $\mathcal{R}_\eta(t, \zeta)\mathcal{R}_\alpha(\zeta, p) = \mathcal{R}_\alpha(t, p)$, where $0 \leq p \leq \zeta \leq t \leq T$, and $I$ represents the identity operator.

(*b*) The map $(t, p) \to \mathcal{R}_\eta(t, p)$ is strongly continuous, for $0 \leq p \leq t \leq T$.

(*e*) For all $\hat{\mathsf{x}} \in X$, the map $\mathcal{R}_\eta(t,p)\hat{\mathsf{x}}(t)$ is continuously differentiable in $p \in [0,T]$ and satisfies

$$\frac{\partial \mathcal{R}_\eta(t,p)\hat{\mathsf{x}}}{\partial t} = \mathcal{A}(t)\mathcal{R}_\eta(t,p)\hat{\mathsf{x}}, \quad \frac{\partial \mathcal{R}_\eta(t,p)\hat{\mathsf{x}}}{\partial p} = -\mathcal{R}_\eta(t,p)\mathcal{A}(p)\hat{\mathsf{x}},$$

where $\dfrac{\partial \mathcal{R}_\eta(t,p)\hat{\mathsf{x}}}{\partial t}$ and $\dfrac{\partial \mathcal{R}_\eta(t,p)\hat{\mathsf{x}}}{\partial p}$ are strongly continuous for $0 \le p \le t \le T$.

## 1.2 Basic Fractional Calculus

Fractional calculus (FC) complements classical calculus in addressing specific problems involving fractional-order dynamics and nonlocal behavior. The primary motivations for using these derivatives in various applications are their memory effect and nonlocality [7, 8, 9]. Over the past few decades, FC's growing applications and widespread use have attracted many mathematicians, scientists, and engineers. Significant progress has been made in the theoretical and practical aspects of FDEs (see [10, 11, 12] and references therein).

FC is a powerful mathematical tool that extends traditional calculus to fractional orders, enabling the modeling and analysis of phenomena with long-range memory, nonlocal interactions, and anomalous behaviors. As researchers explore its potential, FC's applications continue to grow across various disciplines. It is used in scientific and engineering fields such as physics, signal processing, control systems, viscoelasticity, fractional finance, and fractional-order modeling of complex systems.

Before delving into fractional calculus, let's introduce some commonly used functions that serve as its building blocks. These functions are essential in the definition of integrals and fractional derivatives.

## 1.3 Gamma Function

The gamma function, denoted by $\Gamma(z)$, is a fundamental mathematical function particularly important in fractional calculus. It extends the concept of factorials to non-integer and complex values of $z$. For complex numbers $z$ with positive real parts, it is defined [9] as:

$$\Gamma(z) = \int_0^\infty p^{z-1} e^{-p} dp. \tag{1.1}$$

For positive integers $n$, the gamma function simplifies to the factorial function: $\Gamma(n) = (n-1)!$. Additionally, the Gamma function satisfies key properties, such as the recurrence relation: $\Gamma(z+1) = z\Gamma(z)$. The gamma function is essential in various mathematical and physical applications, including fractional calculus, where it plays a key role in defining fractional derivatives and integrals [9]. By means of the gamma function, we can define a function's fractional integrals and derivatives, allowing us to solve FDEs that arise in many fields of science and engineering.

## 1.4 Beta Function

In certain situations, using the beta function is more convenient and efficient than working directly with the values of the gamma function. This is because the beta function simplifies the expression and computation of integrals involving products of powers, making it particularly useful in probability theory, statistics, and various areas of mathematical analysis. The beta function, denoted as $\mathbf{B}(z_1, z_2)$, is a special function defined [9] for $z_1 > 0$ and $z_2 > 0$ by the integral:

$$\mathbf{B}(z_1, z_2) = \int_0^1 p^{z_1-1}(1-p)^{z_2-1}dp, \;\; Re(z_1) > 0, \;\; Re(z_2) > 0. \tag{1.2}$$

The beta function is closely related to the gamma function through the following relationship:

$$\mathbf{B}(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

## 1.5 Mittag-Leffler Function

The Mittag-Leffler function is a special function that generalizes the exponential function and plays a significant role in the study of fractional calculus. It is denoted by $E_\eta(z)$ and is defined [7, 9] by the following series representation:

$$E_\eta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta z + 1)}, \tag{1.3}$$

where $\eta$ is a positive parameter. For $\eta = 1$, the Mittag-Leffler function reduces to the exponential function.

There is also a two-parameter generalization of the Mittag-Leffler function, defined [7, 9] as:

$$E_{\eta,\zeta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta z + \zeta)}, \tag{1.4}$$

where $\eta$ and $\zeta$ are positive parameters.

The Mittag-Leffler function is important in solving FDEs and has applications in various fields such as physics, engineering, and control theory. It is helpful in modeling systems with memory and hereditary properties.

After providing some essential definitions, we can now introduce a few definitions for fractional integral and differential operators. For this, we begin with the integral operator.

## 1.6 Riemann-Liouville Operators

The Riemann-Liouville integral is a frequently used integral transform for defining an integral of fractional order. This definition is motivated by Cauchy's formula for repeated integration. For this, first, we recall Cauchy's formula for repeated integration to a function $\mathsf{f}(t) \in C([a,b],X)$, is defined as follows:

$$I^n\mathsf{f}(t) = \frac{1}{(n-1)!}\int_a^t (t-p)^{n-1}\mathsf{f}(p)dp, \quad t \in [a,b]. \tag{1.5}$$

If we replace $n \in \mathbb{R}$ by $\eta$, where $\eta$ is a positive real number. Then, the above-defined integral can be generalized as the following Lemma:

**Lemma 1.7.** *Let $\mathsf{f} \in L^1([a,b],X)$ for $a,b \in \mathbb{R}$ with $a < b$, and $\eta \in \mathbb{R}^+$. Then, the integral (Bochner's sense) defined by*

$$I_a^\eta\mathsf{f}(t) = \frac{1}{\Gamma(\eta)}\int_a^t (t-p)^{\eta-1}\mathsf{f}(p)dp, \quad t > a, \tag{1.6}$$

*exists for almost every $t \in [a,b]$ and $I_a^\eta\mathsf{f} \in L^1([a,b],X)$.*

*Proof.* First, we define $\mathsf{h}_\eta : \mathbb{R} \to \mathbb{R}$ by

$$\mathsf{h}_\eta(t) = \begin{cases} \frac{t^{\eta-1}}{\Gamma(\eta)} & \text{for} \quad t \in (0, b-a], \\ 0 & \text{elsewhere.} \end{cases} \tag{1.7}$$

Now, we define the extension of $\mathsf{f}$ as follows

$$\tilde{\mathsf{f}_\eta}(t) = \begin{cases} 0 & \text{for} \quad t \neq [a,b], \\ \mathsf{f}(t) & \text{for} \quad t \in [a,b]. \end{cases} \tag{1.8}$$

Then, we can write $I_a^\eta \mathsf{f}(t) = (\mathsf{h}_\eta * \tilde{\mathsf{f}_\eta})(t)$, where $*$ is the convolution of $\mathsf{h}_\eta$ and $\tilde{\mathsf{f}_\eta}$. Since $\mathsf{h}_\eta \in L^1([a,b], \mathbb{R})$ and $\mathsf{f} \in L^1([a,b], X)$, by Young's convolution inequality, $(\mathsf{h}_\eta * \|\mathsf{f}(\cdot)\|_X)(t)$ exists for almost every $t \in \mathbb{R}$. So, by the Bochner's theorem, $(\mathsf{h}_\eta * \mathsf{f})(t)$ exists for almost every $t \in \mathbb{R}$, and $I_a^\eta \mathsf{f} \in L^1([a,b], X)$ satisfying $\|I_a^\eta \mathsf{f}\|_{L^1([a,b],X)} \leq \frac{b^\eta}{\Gamma(\eta+1)} \|\mathsf{f}\|_{L^1([a,b],X)}$. □

Later on, Riemann and Liouville generalized Cauchy's formula for repeated integration (1.5) to the fractional integral. They replace the integer $n$ in Cauchy's formula for repeated integration with the real number $\eta$ and replace the discrete factorial $(n-1)!$ with the continuous gamma function $\Gamma(n)$.

**Definition 1.8.** [9, 13] Let $\mathsf{f} \in L^1([a,b], X)$, $\eta \in \mathbb{R}^+$. Then, the following fractional integral is defined as follows,

$${}_aI_t^\eta \mathsf{f}(t) = \frac{1}{\Gamma(\eta)} \int_a^t (t-p)^{\eta-1} \mathsf{f}(p) dp,$$

$${}_tI_b^\eta \mathsf{f}(t) = \frac{1}{\Gamma(\eta)} \int_t^b (p-t)^{\eta-1} \mathsf{f}(p) dp,$$

provided the right-side integrals are well-defined, are referred to as the left and the right Riemann-Liouville fractional integrals of order $\eta$ respectively, where $\Gamma(\cdot)$ denotes the gamma function.

Now, we define the most used version of the Riemann-Liouville fractional integral.

*Remark* 1.9. [14] For $\mathsf{f}$ and $\eta$ mentioned in Definition 1.8, the Riemann-Liouville fractional integral of $\mathsf{f}$ is defined as follows:

$$ {}_aI_b^{\eta}\mathsf{f}(t) = \frac{1}{\Gamma(\eta)} \int_a^b |t-p|^{\eta-1}\mathsf{f}(p)dp. $$

## 1.7 Caputo Fractional Derivatives

We introduce an important concept in FC known as the Caputo fractional derivative. This definition is particularly favored in applied sciences and engineering because it seamlessly integrates with initial conditions expressed in terms of integer-order derivatives, making it practical for modeling real-world phenomena.

**Definition 1.10.** [9, 15] For $\eta \in \mathbb{R}$, $\lceil \eta \rceil = n$, and $\mathsf{f} \in C^n([a,b],X)$. The left and right Caputo fractional derivative of $\mathsf{f}$ of order $\eta$ is given by respectively:

$$ {}_a^CD_t^{\eta}\mathsf{f}(t) = I^{n-\eta}\mathsf{f}^{(n)}(t) = \frac{1}{\Gamma(n-\eta)} \int_a^t \frac{\mathsf{f}^{(n)}(p)}{(t-\tau)^{n-\eta-1}}dp, $$

$$ {}_t^CD_b^{\eta}\mathsf{f}(t) = \frac{(-1)^n}{\Gamma(n-\eta)} \int_t^b \frac{\mathsf{f}^{n}(p)}{(p-t)^{\eta+1-n}}dp, $$

provided the right-side integrals are pointwise defined.

## 1.8 Riesz-Caputo Fractional Operator

The Riesz-Caputo derivative significantly advances modeling systems with non-local dependencies and complex temporal behaviors. It is particularly effective in capturing phenomena influenced by both past and future dynamics, making it a preferred choice over traditional right or left derivatives. Its applications span fields like geophysics, finance, control theory, and mechanics, as evidenced by studies such as [16, 17].

Using the Riesz derivative in the Caputo sense rather than in the Riemann-Liouville sense addresses issues like non-zero derivatives of constants, mass balance errors, and hyper-singular integrals, offering more accurate and physically consistent solutions [18, 19].

**Definition 1.11.** [15] For the function $\mathsf{f}$ as defined in Definition 1.10, and for $a \leq t \leq b$, the integral expressed below is referred to as the classical Riesz-Caputo derivative of order $\eta$:

$$\begin{aligned} {}^{RC}_{a}D^{\eta}_{b}\mathsf{f}(t) &= \frac{1}{\Gamma(n-\eta)} \int_a^b \frac{\mathsf{f}^n(p)}{|t-p|^{\eta+1-n}} dp \\ &= \frac{1}{2}\left( {}^{C}_{a}D^{\eta}_{t} + (-1)^n \, {}^{C}_{t}D^{\eta}_{b} \right)\mathsf{f}(t), \end{aligned}$$

where ${}^{C}_{t}D^{\eta}_{b}$ and ${}^{C}_{a}D^{\eta}_{t}$ represent the right and left Caputo fractional derivatives in the corresponding order.

## 1.9 Semigroup Theory

The semigroup theory of operators provides a robust mathematical framework for analyzing the evolution of systems governed by linear or nonlinear partial differential equations (PDEs). The concept of continuous semigroups is fundamental, as it facilitates the study of the existence, uniqueness, and stability of solutions to time-dependent PDEs. This theory establishes a connection between the abstract dynamics of semigroups and differential operators by linking a semigroup to its infinitesimal generator, a closed linear operator, enabling a rigorous approach to solving complex problems in both physical and engineering contexts.

Applications of the semigroup theory of operators span diverse fields. In mathematical physics, it is used to model heat conduction, wave propagation, and quantum mechanics by studying heat equations, wave equations, and Schrödinger equations, respectively. In control theory, semigroups enable the analysis of the stability and controllability of dynamical systems, which is crucial for designing systems with desired behaviors. Moreover, they are widely applied in stochastic processes, where semigroups describe the evolution of probability distributions over time. This theoretical foundation also underpins numerical methods for approximating PDEs solutions, providing insights into long-term system behavior and facilitating simulations in science and engineering.

**Definition 1.12.** [12, 20] A one-parameter family $\{\mathcal{T}(t)\}_{t\geq 0} \subset BL(X)$ is called a semigroup of operators on $X$, if it satisfies the following properties:

(i) $\mathcal{T}(0) = I$, where $I$ is the identity operator on $X$.

(ii) $\mathcal{T}(t+p) = \mathcal{T}(t)\mathcal{T}(p),\ \forall\ t, p \geq 0,$

**Definition 1.13.** [12, 20] Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be a semigroup on $X$. Then, $\{\mathcal{T}(t)\}_{t\geq 0}$ is said to be uniformly continuous if

$$\lim_{t\to 0^+} \|\mathcal{T}(t) - \mathcal{T}(0)\|_{BL(X)} = 0,$$

*i.e.*,

$$\lim_{|t-p|\to 0} \|\mathcal{T}(t) - \mathcal{T}(p)\|_{BL(X)} = 0.$$

**Definition 1.14.** [12, 20] A semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $X$ is said to be a $C_0$-semigroup if the map $t \to \mathcal{T}(t)\mathsf{x}$ is strongly continuous for all $\mathsf{x} \in X$, *i.e.*,

$$\lim_{t\to 0^+} \mathcal{T}(t)\mathsf{x} = \mathsf{x}, \ \forall \ \mathsf{x} \in X.$$

**Definition 1.15.** [12, 20] Let $\mathcal{T}(t)$ be a $C_0$-semigroup. Then, the infinitesimal generator $\mathcal{A}$ of $\mathcal{T}(t)$ is a linear operator defined as follows:

$$\mathcal{A}(\mathsf{x}) = \lim_{t\to 0^+} \frac{\mathcal{T}(t)\mathsf{x} - \mathsf{x}}{t}, \quad \text{for } \mathsf{x} \in D(\mathcal{A}), \tag{1.9}$$

where $D(\mathcal{A}) = \left\{\mathsf{x} \in X : \lim_{t\to 0^+} \frac{\mathcal{T}(t)\mathsf{x} - \mathsf{x}}{t} \text{ exists}\right\}$ is the domain of $\mathcal{A}$.

**Example 1.1.** *Assume that $\mathcal{A} : X \to X$ is a bounded linear operator. For $t \geq 0$, we define $\mathcal{T}(t) = e^{t\mathcal{A}}$. Then, $\mathcal{T}(t)$ is a uniformly continuous semigroup of bounded linear operators on $X$, and its infinitesimal generator is $\mathcal{A}$. Here,*

$$e^{t\mathcal{A}} := I + \frac{t\mathcal{A}}{1!} + \frac{t^2\mathcal{A}^2}{2!} + \dots,$$

*is convergent as $\mathcal{A} \in BL(X)$.*

Thus far, in this section, we have provided a brief overview of semigroup along with the key definitions related to semigroup theory. Building on this foundation, we now proceed to recall some fundamental properties of semigroup theory to advance our study in this direction.

**Theorem 1.16.** *[20] Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be a $C_0$-semigroup. Then, for every $t > 0$*

$$\|\mathcal{T}(t)\| \leq \mathcal{M}e^{\beta t}, \tag{1.10}$$

*where $\beta \geq 0$ and $\mathcal{M} \geq 1$ are constants.*

**Theorem 1.17.** *[20] Consider a uniformly continuous semigroup defined by $e^{t\mathcal{A}}$, then the linear operator $\mathcal{A}$ is the infinitesimal generator of $e^{t\mathcal{A}}$ iff $\mathcal{A} \subset BL(X)$, where $BL(X)$ denotes the set of bounded linear operators on $X$. Further, if $\{\mathcal{T}(t)\}_{t\geq 0}$ and $\{\mathcal{S}(t)\}_{t\geq 0}$ be two uniformly continuous semigroup, and $\mathcal{A}$ is the infinitesimal generator of both, then $\mathcal{T}(t) = \mathcal{S}(t)$ for all $t \geq 0$.*

One of the foundational results in semigroup theory, the Hille-Yosida Theorem, was introduced in 1948 by Hille [21] and Yosida [22, 23]. It characterizes the infinitesimal generator of a $C_0$-semigroup of contractions by providing necessary and sufficient conditions on the resolvent set $\rho(\mathcal{A})$ of an operator $\mathcal{A}$. This theorem is pivotal in understanding the conditions under which a linear operator $\mathcal{A}$ generates a $C_0$-semigroup and is fundamental to the study of evolution equations. Before presenting this result, we recall the definition of the resolvent set $\rho(\mathcal{A}) := \{\mu \in \mathbb{C} : (\mu I - \mathcal{A})^{-1}$ exists, and it is a bounded linear operator in $X\}$. For $\mu \in \rho(\mathcal{A})$, $\mathcal{R}(\mu; \mathcal{A}) = (\mu I - \mathcal{A})^{-1}$ is called the resolvent operator (not necessarily bounded) of a linear operator $\mathcal{A} : D(\mathcal{A}) \to X$, where $D(\mathcal{A}) \subset X$.

**Theorem 1.18.** *(Hille-Yosida Theorem) [20] Consider a linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, which is not necessarily bounded. Then, $\mathcal{A}$ is the infinitesimal generator of $C_0$-semigroup of contraction $\{\mathcal{T}(t)\}_{t\geq 0}$ iff*

*(i) $\mathcal{A}$ is closed and $D(\mathcal{A})$ is dense in $X$.*

*(ii) $\mathbb{R}^+ \subset \rho(\mathcal{A})$ and for all $\mu > 0$, $\|\mathcal{R}(\mu;\mathcal{A})\|_{BL(X)} \leq \dfrac{1}{\mu}$.*

*Remark* 1.19. Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be a $C_0$-semigroup with infinitesimal generator $\mathcal{A}$ such that for some $\beta \leq 0$, $\|\mathcal{T}(t)\|_{BL(X)} \leq e^{\beta t}$, then $\mathcal{S}(t) := e^{-\beta t}\mathcal{T}(t)$ is a $C_0$-semigroup contraction generated by $\mathcal{A} - \beta I$. Further, for a $C_0$-semigroup contraction $\{\mathcal{S}(t)\}_{t\geq 0}$ generated by $\mathcal{A}$, $\mathcal{T}(t) = e^{\beta t}\mathcal{S}(t)$ is a $C_0$-semigroup generated by $\mathcal{A} + \beta I$ such that $\|\mathcal{T}(t)\|_{BL(X)} \leq e^{\beta t}$.

**Theorem 1.20.** *[20] Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be a $C_0$-semigroup generated by $\mathcal{A}$. Then*

*(i) For $\mathsf{x} \in X$,*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathcal{T}(p)\ \mathsf{x}\ d\mathsf{p} = \mathcal{T}(t)\mathsf{x}.$$

*(ii) For $\mathsf{x} \in X$,*

$$\int_0^t \mathcal{T}(p)\ \mathsf{x}\ dp \in D(\mathcal{A}), \ \ \textit{and}\ \mathcal{A}\left(\int_0^t \mathcal{T}(p)\ \mathsf{x}\ dp\right) = \mathcal{T}(t)\mathsf{x} - \mathsf{x}.$$

*(iii) For $\mathsf{x} \in D(\mathcal{A})$ and $t > 0$,*

$$\mathcal{T}(t)\mathsf{x} \in D(\mathbf{A}), \ \ \textit{and}\ \frac{d}{dt}\mathcal{T}(t)\mathsf{x} = \mathcal{A}\mathcal{T}(t)\mathsf{x} = \mathcal{T}(t)\mathcal{A}\mathsf{x}.$$

*(iv) For* $\mathsf{x} \in D(\mathcal{A})$,

$$\mathcal{T}(t)\mathsf{x} - \mathcal{T}(p)\mathsf{x} = \int_p^t \mathcal{T}(\zeta)\mathcal{A}\mathsf{x}d\zeta = \int_p^t \mathcal{A}\mathcal{T}(\zeta)\mathsf{x}d\zeta.$$

*(v)* $\overline{D(\mathcal{A})} = X$, *and* $\mathcal{A}$ *is a closed operator, i.e.,* $\mathcal{A}$ *is a densely defined closed linear operator on* $X$.

*Remark* 1.21. As established in Theorem 1.17, the semigroup $\mathcal{T}(t)$ associated with a bounded linear operator $\mathcal{A}$ can be expressed as $e^{t\mathcal{A}} = \sum_{m=0}^{\infty} \frac{t^m \mathcal{A}^m}{m!}$. However, when $\mathcal{A}$ is an unbounded linear operator, identifying the representation of the corresponding semigroup, if it exists, becomes more challenging. This issue is addressed through the complex operator calculus, which provides a representation of the semigroup, if it exists, in terms of a complex integral. The following theorem not only establishes this representation but also outlines the criteria for an operator $\mathcal{A}$ to be the infinitesimal generator of a $C_0$-semigroup.

**Theorem 1.22.** *[20] Let* $\mathcal{A}$ *be a densely defined operator in* $X$ *which satisfies the following properties:*

*(i)* $\Sigma_\delta = \{\mu : |arg(\mu)| < (\pi/2) + \delta\} \cup \{0\} \subset \rho(\mathcal{A})$, *for some* $0 < \delta < \frac{\pi}{2}$.

*(ii) For* $\mu \in \Sigma_\delta$ *with* $\mu \neq 0$ $\exists$ *a constant* $\mathcal{M}$ *such that,* $||(\mu I - \mathcal{A})^{-1}|| \leq \frac{\mathcal{M}}{|\mu|}$.

*Then,* $\mathcal{A}$ *is the infinitesimal generator of a* $C_0$*-semigroup* $\mathcal{T}(t)$. *Moreover, for some constant* $\mathcal{K}$, $||\mathcal{T}(t)|| \leq \mathcal{K}$ *and*

$$\mathcal{T}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\mu t}(\mu I - \mathcal{A})^{-1} d\mu,$$

*where* $\Gamma$ *is a smooth curve in* $\Sigma_\delta$ *running from* $\infty e^{-i\theta}$ *to* $\infty e^{i\theta}$ *for* $\pi/2 < \theta < (\pi/2)+\delta$. *The above integral converges for* $t > 0$ *in the uniform operator topology.*

Previously, we focused on semigroups defined on the nonnegative real axis. Now, we extend the domain of the parameter to a region of the complex plane that includes the negative real axis. This extension is important from an application perspective, as it broadens the scope of problems that can be addressed. Such an extension is made possible by a specific type of semigroup, known as an analytic semigroup, which is defined below.

**Definition 1.23.** [20] Assume $\Delta = \{z \in \mathbb{C} : \phi_1 < arg\ z < \phi_2, \phi_1 < 0 < \phi_2\}$, and let $\mathcal{T}(z)$ be a bounded linear operator for $z \in \Delta$. The family $\mathcal{T}(z),\ z \in \Delta$, is called an analytic semigroup in $\Delta$ if the following conditions are satisfied:

(i) $z \to \mathcal{T}(t)$ is analytic in $\Delta$,

(ii) $\mathcal{T}(0) = I$ and $\lim_{z \to 0} \mathcal{T}(t)\mathsf{x} = \mathsf{x},\ \forall\ \mathsf{x} \in X$,

(iii) $\mathcal{T}(z_1 + z_2) = \mathcal{T}(z_1)\mathcal{T}(z_2),\ \forall\ z_1, z_2 \in \Delta$.

A semigroup $\mathcal{T}(t)$ is considered as analytic if it is analytic in a sector $\Delta$ containing the non-negative real axis.

The possibility or impossibility of extending a $C_0$-semigroup $\mathcal{T}(t)$ to an analytic semigroup in a sector $\Delta$ remains unaffected by multiplying $\mathcal{T}(t)$ by a function $e^{\omega t}$. Thus, in most cases, we can focus on a uniformly bounded $C_0$-semigroup for simplicity. Further, suppose a uniformly bounded $C_0$-semigroup $\mathcal{T}(t)$ is multiplied by $e^{\epsilon t}$ $(\epsilon > 0)$. In that case, it is possible to include 0 in the resolvent set $\rho(\mathcal{A})$ of the infinitesimal generator $\mathcal{A}$ of $\mathcal{T}(t)$. Therefore, for convenience, we will assume that $0 \in \rho(\mathcal{A})$.

Recall Theorem 1.22, which provides sufficient conditions for an operator $\mathcal{A}$ to serve as the infinitesimal generator of a $C_0$-semigroup. Even more is true:

the $C_0$-semigroup $\mathcal{T}(t)$, generated by a densely defined linear operator $\mathcal{A}$ satisfying conditions $(i)$ and $(ii)$ of the Theorem 1.22, can be extended to an analytic semigroup in the sector $\Delta_\delta = \{z : |arg\ z| < \delta\}$ and in every closed sector, $\overline{\Delta}_{\delta'} = \{z : |arg\ z| \leq \delta' < \delta\}$, $\|S(t)\|$, $\|\mathcal{T}(t)\|$ is uniformly bounded. These results and additional insights follow from the theorem presented below.

**Theorem 1.24.** *[20] Let* $\{\mathcal{T}(t)\}_{t\geq 0}$ *be a uniformly bounded* $C_0$*-semigroup of operators generated by the infinitesimal generator* $\mathcal{A}$*, and* $0 \in \rho(\mathcal{A})$*. Then the following are equivalent*

*(i) The semigroup* $\mathcal{T}(t)$ *can be extended to an analytic semigroup in a sector* $\Delta_\delta = \{z : |arg\ z| < \delta\}$ *and for each closed subsector* $\overline{\Delta}_{\delta'}$, $\delta' < \delta$ *of* $\Delta_\delta$, $\|\mathcal{T}(z)\|$. *is uniformly bounded.*

*(ii) For each* $p > 0$*, there exists a constant* $\mathcal{K} > 0$ *such that*

$$\|R(p + i\zeta; \mathcal{A})\|_{BL(X)} \leq \frac{\mathcal{K}}{|\zeta|},\ \zeta \neq 0.$$

*(iii) There exist* $0 < \delta < \pi/2$ *such that*

$$\rho(\mathcal{A}) \supset \Sigma_\delta = \{\mu : |arg(\mu)| < (\pi/2) + \delta\} \cup \{0\},$$

*and*

$$\|R(\mu; \mathcal{A})\|_{BL(X)} \leq \frac{\mathcal{M}}{|\mu|},\ \mathcal{M} > 0,\ \mu \neq 0,$$

*for* $\mu \in \Sigma_\delta$.

*(iv)* $\mathcal{T}(t)$ *is differentiable for* $t > 0$ *and there is a constant* $\mathcal{K}$ *such that*

$$||\mathcal{A}\mathcal{T}(t)|| \leq \frac{\mathcal{K}}{t}, \text{ for } t > 0.$$

## 1.10 Fractional Powers of Closed Linear Operators

This subsection introduces fractional powers of certain unbounded linear operators, focusing on densely defined closed operators $\mathcal{A}$, where $-\mathcal{A}$ is the generator of an analytic semigroup. Fractional powers generalize integer powers of operators to non-integer values, offering flexibility in analyzing differential operators and their regularity. These concepts, essential in fields like PDEs and mathematical physics, will be applied in subsequent chapters to study abstract FDEs. Now, we consider the following assumptions, which are required to define the fractional powers of $\mathcal{A}$.

**Assumption 1:**

(i) Let $\mathcal{N}_0$ be a neighborhood of zero. Then, for some $\beta > 0$, we define

$$\rho(\mathcal{A}) \supset \Sigma^+ := \{\mu \in \mathbb{C} : \beta < |arg(\mu)| \leq \pi\} \cup \mathcal{N}_0.$$

(ii) For all $\mu \in \Sigma^+$, there exists $\mathcal{M} > 0$ such that

$$\|\mathcal{R}(\mu; \mathcal{A})\| \leq \frac{\mathcal{M}}{1 + |\mu|}.$$

As stated in Theorem 1.22, $-\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup when $\beta = \frac{\pi}{2}$ and $\mathcal{M} = 1$. While, Theorem 1.24 says that $-\mathcal{A}$ generates an analytic semigroup for $\beta < \pi/2$. Although most results concerning the fractional powers of $\mathcal{A}$ remain valid even if $0 \notin \rho(\mathcal{A})$, for convenience, we assume that a neighborhood $\mathcal{N}_0$ of zero is contained in $\rho(\mathcal{A})$.

For $\eta > 0$, we define the fractional power of an operator $\mathcal{A}$ satisfying the **Assumption 1** as follows:

$$\mathcal{A}^{-\eta} = \frac{1}{2\pi i} \int_{\Gamma} z^{-\eta} (\mathcal{A} - zI)^{-1} dz, \tag{1.11}$$

where the path $\Gamma \subset \rho(\mathcal{A})$ is from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, where $\theta \in (\beta, \pi)$, and avoids both the negative real axis and the origin. We note that, $z^{-\eta}$ is positive for real positive values of $z$. Further, for all $\eta > 0$, the above integral (1.11) converges in the uniform operator topology. Thus, $\mathcal{A}^{-\eta}$ is a bounded linear operator. Moreover, the operator $\mathcal{A}^{-\eta}$ satisfies the semigroup property,

$$i.e., \quad \mathcal{A}^{-(\eta_1+\eta_2)} = \mathcal{A}^{-\eta_1} \mathcal{A}^{-\eta_2}, \text{ for } \eta_1, \eta_2 \geq 0. \tag{1.12}$$

Furthermore, there is a constant $\mathcal{K}$ such that $\|\mathcal{A}^{-\eta}\| \leq \mathcal{K}$. Additionally, for $\mathcal{A}$ which satisfies **Assumption 1** with $\beta < \dfrac{\pi}{2}$, we have

$$\mathcal{A}^{-\eta} = \frac{1}{\Gamma(\eta)} \int_0^{\infty} p^{\eta-1} \mathcal{T}(p) dp, \tag{1.13}$$

where $-\mathcal{A}$ is the generator of the analytic semigroup $\mathcal{T}(t)$. The above integral converges in the uniform operator topology for all $\eta > 0$, and it can be used as the definition of the operator $\mathcal{A}^{-\eta}$ whenever $-\mathcal{A}$ is the generator of an analytic semigroup $\mathcal{T}(t)$.

Since $\mathcal{A}$ satisfies the **Assumption 1** for $\beta < \frac{\pi}{2}$, we define

$$\begin{cases} \mathcal{A}^{\eta} = (\mathcal{A}^{-\eta})^{-1}, & \text{for } \eta > 0 \\ \mathcal{A}^{\eta} = I, & \text{for } \eta = 0. \end{cases} \tag{1.14}$$

Now, for $\mathcal{A}^{\eta}$, we have the following result:

**Theorem 1.25.** *[20] For $\mathcal{A}^{\eta}$ given by (1.14), we have the following:*

*(i) $\mathcal{A}^{\eta}$ is densely defined closed operator, for $\eta > 0$.*

*(ii) $D(\mathcal{A}^{\eta_1}) \subset D(\mathcal{A}^{\eta_2})$, for $\eta_1 \geq \eta_2$.*

*(iii) For every $\eta \geq 0$, $D(\mathcal{A}^{\eta})$ is dense in $X$.*

*(iv) For $\eta_1, \eta_2 > 0$*

$$\mathcal{A}^{\eta_1+\eta_2}\mathsf{x} = \mathcal{A}^{\eta_1}\mathcal{A}^{\eta_2}\mathsf{x}, \forall\ x \in D(\mathcal{A}^{\gamma}),$$

*where $\gamma = \max\{\eta_1, \eta_2, \eta_1 + \eta_2\}$.*

By the Theorem 1.25, it follows that for all $\eta > 0$, the space $D(\mathcal{A}^{\eta})$ is a Banach space with respect to its graph norm. Moreover, for $\eta_1 \geq \eta_2 > 0$, we have the following embedding:

$$D(\mathcal{A}^{\eta_1}) \hookrightarrow D(\mathcal{A}^{\eta_2}).$$

This subsection concludes with several results related to $\mathcal{A}^{\eta}$ and the analytic semigroup $\mathcal{T}(t)_{t\geq 0}$ with generator $-\mathcal{A}$.

**Theorem 1.26.** *[20] Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be an analytic semigroup generated by the infinitesimal generator $-\mathcal{A}$ and $0 \in \rho(\mathcal{A})$. Then*

(i) *For all* $t > 0$, $\mathcal{A}(t) : X \to D(\mathcal{A}^\eta)$, $\eta \geq 0$.

(ii) $\mathcal{T}(t)\mathcal{A}^\eta \mathsf{x} = \mathcal{A}^\eta \mathcal{T}(t)\mathsf{x}, \forall\ \mathsf{x} \in D(\mathcal{A}^\eta)$.

(iii) *The operator* $\mathcal{A}^\eta \mathcal{T}(t)$ *is bounded for all* $\eta > 0$ *and* $t > 0$. *Further, there exist* $\mathcal{C}_\eta > 0, \delta > 0$ *such that*

$$\|\mathcal{A}^\eta \mathcal{T}(t)\| \leq \mathcal{C}_\eta t^{-\eta} e^{-\delta t}.$$

(iv) *For* $0 < \eta \leq 1$ *and* $\mathsf{x} \in D(\mathcal{A}^\eta)$, *there exists a constant* $\mathcal{M}_\eta$ *such that*

$$\|\mathcal{T}(t)\mathsf{x} - \mathsf{x}\| \leq \mathcal{M}_\eta t^\eta \|\mathcal{A}^\eta \mathsf{x}\|.$$

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