Abstract

Optimization is a key tool in fields like science, engineering, and economics, providing a clear method for decision-making. It helps to find the best solutions within certain limits. Conventional optimization methods and theories often struggle with real-world uncertainty. This leads to the development of specialized areas such as interval and set optimization. These areas in optimization address imprecise data by using intervals or sets of values, offering robust solutions adaptable to dynamic environments.

This thesis aims to develop both theoretical frameworks and numerical methods in the fields of interval and set optimization. Firstly, the focus of this thesis is on interval optimization. In this regard, the concepts of a support function for interval vectors and subdifferentiability for interval-valued functions (gH-subdifferentiability) using the generalized Hukuhara difference are defined. Next, the notion of weak sharp minima for interval-valued functions is introduced. Further, primal and dual characterizations of weak sharp minima are provided. The primal characterization is given in terms of gH-directional derivatives. On the other hand, to derive dual characterizations, the notions of the support function of a subset of $I(\mathbb{R})^n$ and gH-subdifferentiability are used. To extend this theoretical framework in interval optimization, a more general form of gH-subdifferentiability, known as ϵ-subdifferentiability or gH_ϵ-subdifferentiability, is proposed. Several key properties of gH_ϵ-subdifferential set, such as nonemptiness, closedness, convexity, and boundedness, are established subsequently. To prove the convexity of gH_ϵ-subdifferential set, the concept of gH_ϵ-directional derivative is defined. The boundedness of gH_ϵ-subdifferential set at an interior point of the effective domain is demonstrated using a weighted mapping for intervals. However, it is also observed that gH_ϵ-subdifferential set may be unbounded at the boundary of the effective domain. Additionally, we introduce a new solution concept known as the approximate solution, or ϵ-solution, for interval optimization problems. Subsequently, necessary and sufficient optimality conditions to identify an ϵ-solution for unconstrained interval optimization problems are provided. Furthermore, a result is also demonstrated to show how the concept of gH_ϵ-subdifferentiability can be applied to find approximate solutions for interval minimax optimization problems.

The second part of the thesis deals with set optimization problems. The study of numerical methods has recently gained attention in set optimization. We extend this numerical development in set optimization by proposing conjugate gradient and projected gradient methods along with their variants for unconstrained and constrained set optimization problems, respectively, where the objective functions are collections of finitely many vector-valued functions. These methods are based on the set approach to identify the weakly minimal solutions with respect to the lower set less order relation. First, a general algorithm for the conjugate gradient method using Wolfe line search but without imposing an explicit restriction on the conjugate parameter is developed. Thereafter, two variants of the algorithm, namely Fletcher-Reeves and conjugate descent, using two different choices of the conjugate parameter but with the same line search rule, are given. Subsequently, two particular types of projected gradient methods with backtracking Armijio line search rule are proposed and analyzed: one with constant step size and the other with variable step size along the negative gradient before taking the projection on the constraint set. In the proposed conjugate and projected gradient methods, the direction of movement at each iterate is identified by solving the families of unconstrained vector optimization problems (for conjugate gradient methods) and constrained vector optimization problems (for projected gradient methods). These families of vector optimization problems (both unconstrained and constrained) are identified with the help of the concept of partition set at the current iterate. As these vector optimization problems are commonly different at different iterations, the conventional conjugate and projected gradient methods for vector optimization cannot be straightly extended to solve the set optimization problems under consideration. It is also established that the proposed methods are well-defined. Subsequently, the global convergence of both conjugate and projected methods is proved under some reasonable assumptions. It is also noticeable that no convexity assumption is required to prove the global convergence of the proposed methods. Numerical demonstrations of these methods, along with a comparative analysis with the existing steepest descent method for the considered unconstrained set optimization problems, are also provided.