Chapter 2

The flavour problem of the Standard model

In this chapter[1], we will revisit the various approaches and solutions that were introduced to address the flavour problem, tracing back to the early development of the SM as a robust theoretical framework. We will then shift our attention to the some of the novel and unconventional approaches we have proposed into this context, and discuss their theoretical structure in detail.

The problem of flavour of the standard model (SM) is one of the most challenging observational puzzles and is defined by the lack of any mechanism to explain the observed fermionic mass spectrum and mixing, including that of neutrino as well. This problem is so fascinating that in his remarkable review “The Problem of Mass" in 1977, when the SM was finally starting to come together, Weinberg wrote, “*Over the last decade, we have seen a satisfying synthesis of the theories of weak, electromagnetic, and strong interactions, which has provided explanations for many of the ad-hoc hypotheses that had been previously introduced into particle physics on chiefly empirical grounds. However, one essential element of this systematic theory*

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has remained obscure: we must take the masses of the leptons and quarks as input parameters, without any real idea of where they came from"[2]. Reviews of the flavour problem can be found in references [3–7].

The earliest attempt to predict the quark-mixing were made by Cabibbo and Franceschi by suggesting that the requirement of vanishing divergent contribution in the lowest order in weak interactions could produce the correct value of the Cabibbo angle[8]. This observation was used by Gatto, Sartori, and Tonin to relate the Cabibbo angle to parameters of strong interactions by requiring that the quadratic divergence of the weak correction does not contribute to the $SU(3)$ and isospin breaking [9]. In the same issue of Phys. Lett. B, Cabibbo and Maiani produced the Cabibbo angle by proposing that the weak and electromagnetic corrections determine the structure of symmetry breaking in the strong Hamiltonian [10]. In 1969, Oakes predicted the Cabibbo angle by assuming that chiral $SU(2) \times SU(2)$ breaking in strong interactions and the non-conservation of strangeness in weak interactions can produce the Cabibbo angle[11]. Following this, in 1970, Cabibbo and Maiani again came up with a new model to predict the Cabibbo angle by relating it to the observed breaking of the $SU(3) \times SU(3)$ symmetry [12].

In this chapter, after reviewing the main standard solutions to the flavour problem, we shall discuss some new typical and atypical ideas to address the flavour problem and dark matter together. For instance, a typical unified solution to the flavour problem and dark matter is the models based on the $\mathcal{Z}_N \times \mathcal{Z}_M$ flavour symmetry [13–16]. We notice that the Froggatt-Nielsen (FN) mechanism within minimal framework of the SM (non-supersymmetric) based on the $\mathcal{Z}_N \times \mathcal{Z}_M$ flavour symmetry or any discrete symmetry is the first-ever such construction in literature. This idea replaces the requirement of a gauged symmetry $U(1)$ symmetry, thus, providing an entirely unique framework with unique signatures at colliders

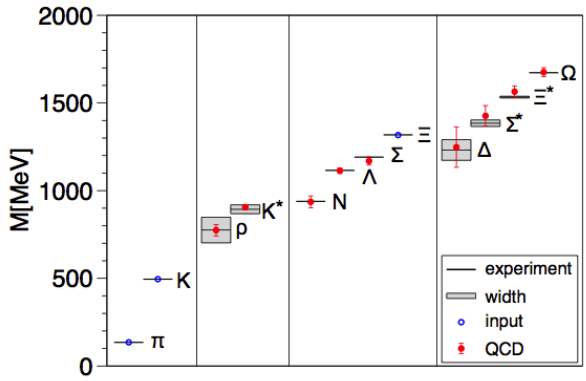


Fig. 2.1 The light hadron spectrum of QCD. Figure is taken from reference [21].

[16]. The reference [16] is a phenomenological guide to flavour models based on the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetries, and can be used to probe flavour models based on $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetries at the high-luminosity Large-hadron Collider (HL-LHC), high-energy LHC (HE-LHC) [17], and 100 TeV collider such as FCC-hh [18].

Further motivation for this work comes from recent hints of new physics, referred to as anomalies, emerging from different experiments [19]. The energy scales of these anomalies range from approximately 1 GeV to the TeV scale probed by the Large Hadron Collider (LHC) [19]. In particular, a recent review by the ATLAS experiment at the LHC provides hints of several anomalies spanning from 10 GeV to the TeV scale [20]. For instance, a part of the summary table given in reference [20] is shown in table 2.1.

We observe from table 2.1 that the mass spectrum is quite random, and apparently is very difficult to fit in any model based on the extended scalar sector of the SM. We notice that if these anomalies continue to show up in future runs of the LHC, the situation would be quite similar to the 1950s, when bubble chambers and spark chambers experiments discovered so many hadrons. For instance, see figure 2.1, taken from reference [21].

Decay Channel	Production Mode	Mass (GeV)
$H \to \tau\tau$	b-associated	400
$H \to \tau\tau$	ggF	400
$H \to \mu\mu$	b-associated	480
$H \to t\bar{t}$	ggF	800
$H \to t\bar{t}/t\bar{q}$	qq and qg	900
$H \to ZZ \to 4\ell/2\ell 2\nu$	ggF	240
$H \to ZZ \to 4\ell/2\ell 2\nu$	VBF	620
$H \to \gamma\gamma$	ggF	684
$H \to \gamma\gamma$	ggF	95.4
$H \to Z(\ell\ell)\gamma$	ggF	420
$H \to Z(q\bar{q})\gamma$	ggF	3640
$A \to Zh_{125}(b\bar{b})$	ggF	500
$A \to Zh_{125}(b\bar{b})$	b-associated	500
$A \to ZH \to \ell\ell b\bar{b}$	ggF	610 (A), 290 (H)
$A \to ZH \to \ell\ell b\bar{b}$	b-associated	440 (A), 220 (H)
$A \to ZH \to \ell\ell WW$	ggF	440 (A), 310 (H)
$A \to ZH \to \ell\ell t\bar{t}$	ggF	650 (A), 450 (H)
$A \to ZH \to Zh_{125}(b\bar{b})h_{125}(b\bar{b})$	VH	420 (A), 320 (H)
$H^+ \to cb$	$t\bar{t}$ decay	130
$H^+ \to Wa(\mu\mu)$	$t\bar{t}$ decay	120–160 (H^+), 27 (a)
$H^{++} \to WW$	VBF	450
$H \to h_{125}h_{125} \to 4b$	ggF	1100
$H \to h_{125}h_{125} \to 4b$	VBF	550
$H \to h_{125}h_{125} \to b\bar{b}\tau\tau$	ggF	1000
$H \to h_{125}h_{125}$ (combination)	ggF	1100
$X \to Sh_{125} \to b\bar{b}\gamma\gamma$	ggF	575 (X), 200 (S)
$h_{125} \to Z_d Z_d \to 4\ell$	ggF	28
$h_{125} \to ZZ_d \to 4\ell$	ggF	39
$h_{125} \to aa \to b\bar{b}\mu\mu$	ggF, VBF, VH	52
$h_{125} \to aa \to 4\gamma$	ggF	10–25
$h_{125} \to e\tau$ and $h_{125} \to \mu\tau$	ggF, VBF, VH	125

Table 2.1 The anomalies reported by the ATLAS experiment. This table is adopted from reference [20].

The first guess in that scenario would be to assume that these anomalies are not fundamental particles, and rather bound states of a new strong force. Along this line of argument, we shall discuss new models of flavour based VEVs hierarchy [22, 23]. In particular, the Standard Hierarchical VEVs Model (SHVM) [23], may have potential to address the hierarchical anomalies given in table 2.1.

Remarkably, an ultra-violet (UV) completion of these models lies in a strong dark technicolour paradigm based on $\mathscr{G} \equiv SU(\mathrm{N_{TC}}) \times \mathrm{SU}(\mathrm{N_{DTC}}) \times \mathrm{SU}(\mathrm{N_F})$ symmetry, where TC stands for technicolour, which may be a conformal strong dynamics, DTC for dark-technicolour, and F represents strong dynamics of dark-QCD of vector-like fermions [24, 25].

In this framework, the hierarchical VEVs are chiral multi-fermion bound states of the $SU(\mathrm{N_{DTC}})$ dynamics. Moreover, $SU(\mathrm{N_F})$ is a QCD-like dynamics. Therefore, its spectrum can be predicted by rescaling the spectrum of QCD. Thus, it may explain several anomalies given in table 2.1, and early investigation shows that the 95.4 GeV excess can easily accounted in the SHVM framework [26]. Furthermore, the spectrum of $SU(\mathrm{N_{DTC}})$ dynamics, a part of which exhibits itself as the gauge singlet scalar fields in the SHVM, may account for several anomalous data given in table 2.1. An additional purpose of this review is to introduce the community to the unconventional approach of the SHVM, and its ultra-violet (UV) completion based on the dark-technicolour paradigm.

Apart from above discussion, we have presented unified solutions of the flavour and dark matter in this chapter. The SHVM may also provide a new kind of “possible" dark matter particle. Thus, apart from collider researchers, this idea may also be interesting for the people working on dark matter searches, and for people working on theoretical models of scalar dark matter.

The chapter is organized as follows: In section 2.1, we review the flavour models based on continuous and discrete symmetries, GUT theories, compositeness hypothesis, and extra dimensions. Section 2.2 and 2.3 has the discussion of the novel framework based on the $\mathscr{Z}_{\mathrm{N}} \times \mathscr{Z}_{\mathrm{M}}$ flavour symmetry that we proposed and its different prototype forms and their theoretical structure. In section 2.4, we discuss another unconventional approach based on the VEVs hierarchy to address the flavour problem.

2.1 Solutions of the flavour problem

Weinberg was the first to pay attention to the problem of the fermionic mass spectrum in theories with spontaneously broken gauge symmetries[27]. He incorporated the weak and electromagnetic interactions into a parity-conserving $SU(3) \times SU(3)$ parity invariant gauge theory, which is spontaneously broken to the $SU(2)_L \times U(1)_Y$ model. In this model, the leptons μ^+, e, and ν form a Konopinski-Mahmoud triplet, and a scalar field ϕ is added, transforming as $(3, \bar{3})$ representation and coupled to the leptonic triplet in a $SU(3) \times SU(3)$ invariant manner. The scalar field ϕ acquires the VEVs only for the diagonal elements due to the charge conservation. It turns out that due to the smallness of leptonic masses, only non-zero VEV is $\langle \phi_{11} \rangle$. The scalar field components (ϕ_{11}, ϕ_{21}) form a scalar Higgs doublet which further breaks the SM $SU(2)_L \times U(1)_Y$ symmetry spontaneously. The mass of the electron is predicted to be αm_μ from the interactions of leptonic triplet and the scalar field ϕ.

In fact, Weinberg was so obsessed with the flavour problem that it became the only mystery that he wanted to see being solved in his lifetime[22]. He recalled this in his interview with CERN Courier, *"It was the worst summer of my life! I mean, obviously, there are broader questions, such as: Why is there something rather than nothing? But if you ask for a very specific question, that's the one. And I'm no closer now to answering it than I was in the summer of 1972"* [28].

However, even the worst summer of his life gave birth to the first pioneer and standard paradigm for the solution of the flavour problem by exploiting the fact that the representation content of the fields and symmetry can be used to eliminate the masses of light fermions at tree-level, and as a consequence, the masses of light fermions can be calculated as finite higher-order effects [29]. The main obstacle in front of Weinberg was that, a symmetry, which forbids some mass or mass differences at zeroth order, also does not allow the mass or mass difference nonzero in all higher orders. However, the renormalizability of spontaneously broken gauge symmetries, such as the SM, provides a way out of this impasse. In a renormalizable spontaneously broken gauge theory, if the zeroth-order contribution to a mass or mass difference vanishes for a given set of fields in some representation of the gauge symmetry, then the higher order contribution to the mass or mass difference must be finite. Assuming an extreme scenario, Weinberg argued that if there are no scalar fields belonging to the irreducible representation of the symmetry at hand, the Yukawa couplings are completely forbidden. However, the VEVs of the scalar fields (for example, the Higgs field in the SM) contribute to masses of the vector bosons of the corresponding gauge symmetry. The higher order corrections introduced by the vector gauge bosons can produce the finite fermion masses. This powerful idea was immediately used by Georgi and Glashow to explore the fermionic mass patterns using the different representations of scalar multiplets in Abelian and non-Abelian models[30, 31].

One of the most important ideas for solving the flavour problem was put forward by the Froggatt and Nielsen (FN) in 1979 [32]. FN were inspired by the idea of Weinberg of generating fermionic masses through symmetry and the representation of scalar fields. In the FN mechanism, an Abelian flavour symmetry $U(1)$ and a gauge singlet scalar are used to produce the masses of fermions through the effective operators having the following structure,

$$\mathcal{O} = y(\frac{\chi}{\Lambda})^{(\theta_i+\theta_j)}\bar{\psi}_i\varphi\psi_j, \tag{2.1}$$

where y is the coupling constant, χ is the gauge singlet scalar field known as flavon, φ represents the SM Higgs field, θ_i and θ_j are the charges of the fermions ψ_i^c and ψ_j under $U(1)$ symmetry, and Λ is the scale at which these interactions become part of a renormalized underlying theory.

The solutions to the flavour problem can be divided into two classes, bottom-up and top-down. Bottom-up solutions are designed for relatively higher and accessible energies, and they may be a subset of a larger underlying theory. On the other side, the top-down scenarios are motivated by GUT and other problems, such as the hierarchy problem of the SM.

Bottom-up solutions are mainly based on continuous and discrete symmetries [13, 14, 16, 32–45]. Soon after Weinberg's 1972 paper, serious efforts were being made to address the flavour problem in parts. For instance, Georgi and Pais discussed calculability conditions for local gauge theories with spontaneous symmetry breaking and applied them to create a mechanism to compute the Cabibbo angle in models based on $SU(2)\times U(1)\times U(1)$ and $O(4)\times U(1)$ symmetries [46].

Georgi and Pais used the fact that the counter-terms required for renormalizability of the SM possess the symmetries of the Lagrangian before spontaneous breaking of the symmetry. This produces non-trivial relations among the counter-terms, resulting in zeroth-order relations among masses and couplings. This important result can be used to create non-trivial relations between the Cabibbo angle and the masses of quarks. For example, $\cos^2\theta_W = M_W^2/M_Z^2$ is an example of the zeroth-order relation in the SM. For instance, in the case of $SU(2)\times U(1)\times U(1)$ model with two quark doublets, the Higgs multiplets required to generate leptonic masses are different from those which create quark masses. For quark masses, the model contains three Higgs doublets ϕ, χ, and η of the type $(\frac{1}{2}^{(0,-1/2)})$. Now, by imposing

$\mathscr{Z}_2$-type discrete symmetries, one can derive the Cabibbo angle in terms of the masses of the quarks of two doublets [46].

The first serious attempt to explain the flavour mixing in a gauge theory with spontaneous symmetry breaking was made by Rujula, Georgi, and Glashow (RGG) in an $SU(2)_L \times SU(2)_R \times U(1)$ model where quark mixing is produced through radiative corrections in the spirit of Weinberg's 1972 model[47]. RGG used a discrete symmetry to forbid the off-diagonal terms in fermionic mass matrices such that the charged weak current is flavour diagonal. The flavour mixing is obtained by the soft breakdown of the discrete symmetry in the mass terms of the Higgs field resulting in a calculable Cabibbo angle in terms of a small soft breaking parameter. Shortly after this, Fritzsch calculated the Cabibbo angle in terms of the quark mass ratios in an $SU(2)_L \times SU(2)_R \times U(1)$ model and a discrete symmetry[48].

In 1979, Wilczek and Zee introduced the idea of gauged horizontal continuous symmetries, which can determine the weak mixing angles in terms of quark masses[49]. The idea was to introduce a gauge symmetry along the horizontal direction, which acts among the left-handed doublets and among the right-handed singlets. This is implemented by extending the SM by the continuous symmetry $SU(2)_H$, under which the vertical fermionic doublets and singlet fermions of the SM transform as triplets of the $SU(2)_H$. By adding two scalar fields, which are behaving as tensors and vectors under the $SU(2)_H$, one can obtain the Cabibbo angle in terms of masses of d, s, u, and c quarks.

The FN mechanism was published in the same year. The models based on the FN mechanism with continuous symmetries are discussed in the references [50–64]. In the framework of the SM, considering only one Higgs doublet with a $U(1)$ flavour symmetry, the flavour problem has been addressed in [63]. Flavour symmetry is spontaneously broken at energy scale M around the TeV scale. The effective theory below energy scale M corresponds to the SM with one Higgs doublet but with non–renormalizable terms in the Higgs Yukawa couplings. Only the top quark has renormalizable Yukawa couplings. The couplings of other fermions are suppressed by different powers of $(H^{\dagger}H/M^{2})$, where H is the Standard Model Higgs doublet. This suppression factor provides the small expansion parameter ε, and is subjected to the flavour symmetry, resulting in different powers of ε in the fermionic mass matrices. However, for neutrinos whose masses are expected to be in the eV scale, seesaw mechanism has been considered in the work of E. Ma [65]. In this work, in addition to SM particles, heavy right-handed neutrinos with TeV scale mass has been considered as the new physics scale. In the scalar sector, an additional scalar doublet is also considered. A $U(1)$ flavour symmetry with lepton number has been considered in such a way that the heavy fermionic singlets interact with the SM leptons only through the additional scalar doublet but not through the SM higgs doublet. As the VEV of the additional doublet is in the MeV scale, the Dirac mass term in the seesaw mass matrix is much lower, and appropriate light neutrino masses are naturally obtained with TeV scale mass of heavy right-handed neutrinos.

In another work [66] in obtaining small neutrino masses, certain conditions of Dirac and Majorana masses have been found under which Type-I seesaw mechanism can lead to different number of massless neutrinos. Such conditions are found to be related with some underlying $U(1)$ symmetry. In such cases, higher-order corrections at one or two loop level provide the appropriate light neutrino mass with seesaw scale in the TeV range.

Flavour deconstruction is another interesting approach to address the flavour problem [67]-[85]. In the flavour deconstruction framework, the hierarchy of the fermionic masses is obtained through a successive hierarchical spontaneous symmetry breaking (SSB) of the gauge symmetries. For instance, we could have a gauge group $G_1 \times G_2 \times G_3$, where each G_i represents a gauge symmetry for each fermionic family of the SM. Now, a chain of SSBs along the track $G_1 \times G_2 \times G_3 \rightarrow G_{12} \times G_3 \rightarrow G$ facilitated by the VEVs of some scalar fields, provides the masses to each generation of fermions.

An alternative way to address the flavour problem is through discrete symmetries. The first attempt in this direction was made by Wilczek and Zee (WZ) in 1977 by predicting the relation $\tan^2\theta_c = m_d/m_s$[86]. WZ employed the gauge group $SU(2)_L \times SU(2)_R \times U(1)$ with four quark doublets with two bi-doublets followed by the imposition of a discrete symmetry. The discrete symmetry allows a particular mass matrix, which produces $\tan^2\theta_c = m_d/m_s$. The idea was further explored in references [87–89].

The masses of fermions are generated through the radiative corrections using a horizontal non-Abelian discrete symmetry S_3 in reference [90]. In this model, the non-Abelian discrete symmetry S_3 allows only top and bottom quark masses at tree-level. The masses of charm, strange quarks and τ-lepton originate at the one-loop level. The masses of u, d quarks, and μ-lepton arise at the two-loop level, and the mass of electron at the three-loop level.

A minimal symmetry $\mathcal{Z}_2 \times \mathcal{Z}_5$ can solve the flavour problem by implementing the FN mechanism[13]. This symmetry is a prototype of the general $\mathcal{Z}_N \times \mathcal{Z}_M$ flavour symmetry[13–15]. One can also use a large Z_N symmetry to construct the FN mechanism in a SUSY framework[44]. Discrete symmetries are also extensively used in neutrino model building[91–94]. Phenomenological investigations of the FN mechanism using continuous and discrete symmetries are performed in references [14, 95–103].

Recently, modular symmetries are also used to address the flavour problem [104]. A modular symmetry is created by a geometry of a two-dimensional torus

T^2. In models based on modular symmetries, the modular invariance acts like a flavour symmetry. The chiral multiplets reside in unitary representations of the modular group, the Yukawa couplings are the modular forms of level N. In this setup, the flavon field is not an essential requirement, and the modular invariance can be broken by the VEV of the modulus. For a review of the modular symmetries, see reference [105].

Top-down solutions to the flavour problem, in particular, involve GUT theories based on Pati-Salam and $SU(5)$ unification [106, 107]. One of the earliest attempts to provide an explanation to the fermion masses was using $SU(N)$ GUT with $N > 5$, which are broken to $SU(5)$ GUT [108]. In this approach, fermions are accommodated in a so-called primitive representation, which is left-handed and totally anti-symmetric $SU(N)$ tensor. Moreover, this representation is anomaly-free and produces three fermionic families of the SM with the existence of a hierarchy of masses among light fermions. To achieve this, replication of the primitive representation is also required. This is followed by solutions based on $SO(10)$ GUT[37, 109]. A different solution is constructed using $O(10) \times U(1)_F$ GUT in reference [110].

The renormalization-group equations can also provide the evolution of Yukawa coupling from a large scale, such as the GUT scale, through fixed-points leading to predictions of fermions masses [111]. In this approach, the low-energy structure of the renormalization-group equations determines the SM fermion masses through the evolution from a scale M_X to m_f. This is obtained by a solution of the renormalization-group equations, referred to as intermediate-fixed-point, that is relevant at $\mu^2 = m_f^2$. More recent studies are in references [112–123].

Another class of top-down solutions to the flavour problem is based on extra-dimensions models[124, 125]. For instance, a five-dimensional model where the bulk is a slice of AdS, and warped extra dimensions can provide a solution to the flavour problem [126–128]. For instance, reference [126] allows the SM fermion fields to propagate in the AdS_5 bulk, and localized Higgs field originates from a Kaluza-Klein excitation on the TeV-brane. This results in a warp factor similar to that is used to provide a solution of the gauge hierarchy problem by creating the TeV scale from the Planck scale. This setup produces the four-dimensional Yukawa couplings [126],

$$Y_{ij} = \frac{Y_{ij}^{(5)} k}{N_{iL} N_{jR}} e^{(1-c_{iL}-c_{jR})\pi kR} , \tag{2.2}$$

where

$$\frac{1}{N_{iL}^2} \equiv \frac{1/2 - c_{iL}}{e^{(1-2c_{iL})\pi kR} - 1} , \tag{2.3}$$

We can generate exponentially small Yukawa couplings for values of c_{iL} and c_{jR} slightly larger than $1/2$ [126]. This results in an explanation for the fermion mass hierarchy through the metric warp factor. Moreover, the issue of neutrino masses can also be addressed in an extra-dimensional framework without invoking a see-saw mechanism[129]. Furthermore, there are Randall-Sundrum-type models, which are also capable of addressing the flavour problem[130, 131].

The models based on dynamical symmetry breaking[132, 133] are also very attractive scenarios for solving the flavour problem. For instance, the idea of Pati and Salam [134] that the SM fermions may be the bound states of more fundamental PRE-entities can provide a solution to the flavour problem[135]. In this model, the spontaneous breaking of the $SU(2)_L \times U(1)_Y$ of the SM is performed by a technicolour dynamics. Unlike conventional technicolour models, this is achieved by not coupling the SM fermions directly to the order parameter of the $SU(2)_L \times U(1)_Y$ breaking. Instead, the SM fermions mix with technibaryons. The diagonalization of the fermion/technibaryon mass matrix provides the masses of the SM fermions. Thus, in this model, the SM fermions are semi-composite states, having an admixture of technicoloured states.

The assumption of composite fermions and an approximate $U(2)^3$ symmetry can also provide a solution, for instance, see reference [136] and references therein. A dark technicolour symmetry can be used to solve the flavour problem either through the VEVs hierarchy [22–25]. This will be discussed in the section 2.6.

2.2 The $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry

The $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry[13] provides a novel framework for the celebrated Froggatt-Nielsen (FN) mechanism that eventually furnishes an elegant solution to the flavour problem of the standard model (SM)[32]. Unlike the conventional continuous $U(1)$ flavour symmetry employed in the FN mechanism, the $\mathscr{Z}_N \times \mathscr{Z}_M$ symmetry consists of a product of two discrete symmetries. This discrete framework implements the FN mechanism uniquely, parametrizing the flavour structure of the SM— including neutrino masses and mixing parameters— in terms of a small parameter, ϵ, defined as the ratio of the vacuum expectation value (VEV) of the flavon field to the flavour scale, Λ [13].

The theoretical origin of the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry may come from spontaneous breaking of a $U(1) \times U(1)$ symmetry, which could be a global or a local symmetry. It is also possible that this symmetry contains only one $U(1)$ factor as a local symmetry. The crucial difference between the FN mechanism based on the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry and the FN mechanism based on a $U(1)$ symmetry is that the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry may also originate from the breaking of a global $U(1)_A \times U(1)_A$ symmetry. This keeps apart the FN mechanism based on the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry from that based on the conventional $U(1)$ symmetry. Such a theoretical scenario is discussed in the section 2.6 of this chapter.

The FN mechanism is produced by imposing the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry on the SM. This provides the following mass Lagrangian for fermions,

$$\begin{aligned} -\mathscr{L}_{\rm Yukawa} &= \left[\frac{\chi(\text{or } \chi^\dagger)}{\Lambda}\right]^{n^u_{ij}} y^u_{ij}\bar{\psi}^q_{L_i}\tilde{\varphi}\psi^u_{R_j} + \left[\frac{\chi(\text{or } \chi^\dagger)}{\Lambda}\right]^{n^d_{ij}} y^d_{ij}\bar{\psi}^q_{L_i}\varphi\psi^d_{R_j} \\ &+ \left[\frac{\chi(\text{or } \chi^\dagger)}{\Lambda}\right]^{n^\ell_{ij}} y^\ell_{ij}\bar{\psi}^\ell_{L_i}\varphi\psi^\ell_{R_j} + \text{H.c.}, \end{aligned} \tag{2.4}$$

The complex singlet scalar field χ acquires VEV $\langle\chi\rangle = \frac{f}{\sqrt{2}}$, and the Lagrangian corresponding to the SM fermions (charged leptons and quarks) mass is given by

$$-\mathscr{L}_{\rm Yukawa} = Y^u_{ij}\bar{\psi}^q_{L_i}\tilde{\varphi}\psi^u_{R_j} + Y^d_{ij}\bar{\psi}^q_{L_i}\varphi\psi^d_{R_j} + Y^\ell_{ij}\bar{\psi}^\ell_{L_i}\varphi\psi^\ell_{R_j} + \text{H.c.}, \tag{2.5}$$

where i and j show the generation indices, ψ^q_L, ψ^ℓ_L are left-handed doublets of the quark and leptonic fields, $\psi^u_R, \psi^d_R, \psi^\ell_R$ are the right-handed up, down-type quarks and leptons, $\tilde{\varphi} = -i\sigma_2\varphi^*$, φ represent the SM Higgs doublet, and σ_2 is the second Pauli matrix. The couplings Y_{ij} stand for the effective Yukawa couplings such that $Y_{ij} = y_{ij}\varepsilon^{n_{ij}}$, where $\dfrac{\langle\chi\rangle}{\Lambda} = \dfrac{f}{\sqrt{2}\Lambda} = \varepsilon < 1$.

To establish a minimal form of the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry that can provide a simple set-up of the FN mechanism, we adopt the principle of minimum suppression (PMS). This principle posits that minimal suppression of the effective Yukawa couplings corresponds to a minimal form of the symmetry. For instance, if the top quark mass arises from a tree-level SM Yukawa operator, the bottom quark mass would require a suppression of order $y\varepsilon$, the charm quark mass a suppression of order $y\varepsilon^2$, the strange quark mass a suppression of order $y\varepsilon^3$, and the up and down quark masses suppression of at least $y\varepsilon^4$.

Furthermore, we need to count the number of hierarchical energy scales needed to account for the fermionic mass hierarchy in the SM. For instance, for the quark sector, we need three energy scales to explain the mass hierarchy among the three fermionic families. We note that only the second and the third quark families have intra-generational mass hierarchies, which require only two hierarchical energy scales to achieve an explanation for the mass hierarchy within the second and third quark families. These hierarchical energy scales emerge from the non-renormalizable operators of the flavon fields, as given in equation 2.1.

Since the top quark mass originates from renormalizable SM Yukawa operators, at least four hierarchical energy scales must arise from flavon operators to explain the overall hierarchical quark mass pattern. Therefore, the size of a minimal symmetry will be determined by this requirement and through the application of the PMS. The symmetry $\mathscr{Z}_N$ in the $\mathscr{Z}_2 \times \mathscr{Z}_N$ flavour symmetry is responsible for providing such non-renormalizable operators. Meanwhile, the $\mathscr{Z}_2$ symmetry distinguishes between up-type and down-type quarks, ensuring that identical non-renormalizable operators do not contribute simultaneously to the up and down quark mass matrices.

Finally, we must note that a minimal form of the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry should not only produce correct pattern of the charged fermion masses, it should be capable of explaining the quark mixing pattern, neutrino masses, and more importantly, it should predict correct pattern of the neutrino mixing angles. After taking into account above considerations, let us discuss the question of a minimal form of the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry that can provide a simple set-up of the FN mechanism.

2.2.1 $\mathscr{Z}_2 \times \mathscr{Z}_2$ flavour symmetry

The simplest choice is the $\mathscr{Z}_2 \times \mathscr{Z}_2$ flavour symmetry, which turns out to be a trivial selection since the only charges of the $\mathscr{Z}_2$ symmetry are ± 1, which are too trivial to provide four energy scales or equivalently non-trivial operators of the form given in equation 2.1. Hence, we conclude that this symmetry cannot be used to create a simple FN mechanism.

2.2.2 $\mathscr{Z}_2 \times \mathscr{Z}_3$ flavour symmetry

The first non-trivial form of the $\mathscr{Z}_2 \times \mathscr{Z}_N$ flavour symmetry, which may provide an implementation of the FN mechanism, is $\mathscr{Z}_2 \times \mathscr{Z}_3$. The symmetry $\mathscr{Z}_3$ has two non-trivial charges characterized by ω and ω^2, where ω is the cube root of unity. In the first scenario, following the PMS, we assign the charges to the SM and flavon fields as given in table 2.2.

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_3$
u_R, c_R, t_R	+	ω
$d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	1
$\psi^q_{L_1}$	+	ω^2
$\psi^q_{L_2}$	+	1
$\psi^q_{L_3}$	+	ω
$\psi^\ell_{L_1}$	+	ω^2
$\psi^\ell_{L_2}$	+	1
$\psi^\ell_{L_3}$	+	ω
χ	-	ω
φ	+	1

Table 2.2 The charges of left and right-handed fermions of three families of the SM, Higgs and the flavon field under the $\mathscr{Z}_2$ and $\mathscr{Z}_3$ product symmetry, where ω is the cube root of unity.

We observe that the masses of s and b quarks can be recovered from this charge assignment, for instance, mass of the s quark is of the order ε^3, and that of the

b quark is of the order ε. However, the mass of the u and d quarks are produced by the operators $y(\frac{\chi^\dagger}{\Lambda})^2\bar{\psi}\varphi u_R$ and $y(\frac{\chi^\dagger}{\Lambda})\bar{\psi}\varphi d_R$ instead of the operators $y(\frac{\chi}{\Lambda})^4\bar{\psi}\varphi u_R$ and $y(\frac{\chi}{\Lambda})^5\bar{\psi}\varphi d_R$. Any other charge assignment also does not reproduce masses of every quark.

As an additional check, we may assume that exactly identical diagonal operators for the u and d quarks in their mass matrices as given in table 2.3. This charge assignment is against the original sprite of the $\mathscr{Z}_2 \times \mathscr{Z}_N$ flavour symmetry, where the $\mathscr{Z}_2$ is exactly like the symmetry used in the type II 2HDM. It turns out that even in this case, the mass of the u quark is of the order ε. This conclusion does not change even if we provide different non-trivial charge assignments to the fermions and flavon fields under the $\mathscr{Z}_2 \times \mathscr{Z}_3$ flavour symmetry.

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_3$
c_R, t_R	+	ω
$u_R, d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	1
$\psi^q_{L_1}$	+	ω^2
$\psi^q_{L_2}$	+	1
$\psi^q_{L_3}$	+	ω
$\psi^\ell_{L_1}$	+	ω^2
$\psi^\ell_{L_2}$	+	1
$\psi^\ell_{L_3}$	+	ω
χ	-	ω
φ	+	1

Table 2.3 The charges of left and right-handed fermions of three families of the SM, Higgs and the flavon fields under the $\mathscr{Z}_2$ and $\mathscr{Z}_3$ product symmetry, where ω is the cube root of unity.

2.2.3 $\mathscr{Z}_2 \times \mathscr{Z}_4$ flavour symmetry

The charges of the symmetry $\mathscr{Z}_2 \times \mathscr{Z}_4$ are characterized by the fourth roots of unity, which are $1, \omega, \omega^2$ and ω^3 where $\omega = i$, $\omega^3 = \omega^*$ and $\omega^2 = -1$. We particularly note that minimally suppressed diagonal operator of the form $y(\frac{\chi}{\Lambda})^4 \bar{\psi}_L \varphi u_R$ is not the dominant operator no matter what charge we assign to the flavon and fermionic fields. This is because the tree-level SM Yukawa operator $y\bar{\psi}_L \varphi u_R$ is allowed for any charge assignment for the $y(\frac{\chi}{\Lambda})^4 \bar{\psi}_L \varphi u_R$ operator under the $\mathscr{Z}_2 \times \mathscr{Z}_4$ flavour symmetry.

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_4$
c_R, t_R	+	ω^2
$u_R, d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	ω
$\psi^q_{L_1}$	+	ω^2
$\psi^q_{L_2}$	+	1
$\psi^q_{L_3}$	+	ω^2
$\psi^\ell_{L_1}$	+	ω^2
$\psi^\ell_{L_2}$	+	1
$\psi^\ell_{L_3}$	+	ω^2
χ	-	ω
φ	+	1

Table 2.4 The charges of left and right-handed fermions of three families of the SM, Higgs and the flavon fields under the $\mathscr{Z}_2$ and $\mathscr{Z}_4$ product symmetry, where ω is the fourth root of unity.

Therefore, to produce the mass of the u-quark, we either choose a non-trivial transformation of the u_R-quark or the first family of the quarks under the $\mathscr{Z}_2$ symmetry, in addition to the next to the minimal suppressed operator of the order ε^5. One such charge assignment is given in table 2.4. In this case, the masses of the down-type quarks are produced correctly through the operators with minimal suppression. However, in the case of up-type quarks, still non-diagonal tree-level SM operators dominate the mass of the u-quark. Other alternative charge assignments also do not work for creating an FN mechanism through the $\mathscr{Z}_2 \times \mathscr{Z}_4$ symmetry.

2.2.4 $\mathscr{Z}_2 \times \mathscr{Z}_5$ flavour symmetry

We now impose the next flavour symmetry, that is, the $\mathscr{Z}_2 \times \mathscr{Z}_5$ symmetry, on the SM in a way that the various fields of the SM transform under this symmetry as given in table 2.5 [13]. As discussed earlier, we need to create at least four hierarchical energy scales as an origin of the quark mass spectrum. This means, for creating these energy scales, a non-trivial and a minimal $\mathscr{Z}_N$ symmetry should have at least four non-trivial charges. Thus, the symmetry $\mathscr{Z}_5$ could be such a symmetry. Moreover, we note that the transformation of fields under the $\mathscr{Z}_2 \times \mathscr{Z}_5$ symmetry is chosen such that the symmetry $\mathscr{Z}_2$ exactly acts like the way used in the type-II 2HDM [2].

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_5$
u_R, c_R, t_R	+	ω^2
$d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	ω
$\psi^q_{L_1}$	+	ω
$\psi^q_{L_2}$	+	ω^4
$\psi^q_{L_3}$	+	ω^2
$\psi^\ell_{L_1}$	+	ω
$\psi^\ell_{L_2}$	+	ω^4
$\psi^\ell_{L_3}$	+	ω^2
χ	-	ω
φ	+	1

Table 2.5 The charges of left and right-handed fermions of three families of the SM, right-handed neutrinos, Higgs, and singlet scalar fields under $\mathscr{Z}_2$ and $\mathscr{Z}_5$ symmetries, where ω is the fifth root of unity.

[2]Adding an additional Higgs doublet to this model such that it is odd under the $\mathscr{Z}_2$ symmetry will result in a type-I like 2HDM.

The $\mathscr{Z}_2 \times \mathscr{Z}_5$ flavour symmetry enables us to write the mass Lagrangian for charged fermions as,

$$
\begin{aligned}
-\mathscr{L}_{\rm Yukawa} &= \left(\frac{\chi}{\Lambda}\right)^4 y^u_{11}\bar{\Psi}^q_{L_1}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^4 y^u_{12}\bar{\Psi}^q_{L_1}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^4 y^u_{13}\bar{\Psi}^q_{L_1}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^2 y^u_{21}\bar{\Psi}^q_{L_2}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^2 y^u_{22}\bar{\Psi}^q_{L_2}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^2 y^u_{23}\bar{\Psi}^q_{L_2}\tilde{\varphi}t_R \\
&+ y^u_{31}\bar{\Psi}^q_{L_3}\tilde{\varphi}u_R + y^u_{32}\bar{\Psi}^q_{L_3}\tilde{\varphi}c_R + y^u_{33}\bar{\Psi}^q_{L_3}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^5 y^d_{11}\bar{\Psi}^q_{L_1}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{12}\bar{\Psi}^q_{L_1}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{13}\bar{\Psi}^q_{L_1}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^3 y^d_{21}\bar{\Psi}^q_{L_2}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^3 y^d_{22}\bar{\Psi}^q_{L_2}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^3 y^d_{23}\bar{\Psi}^q_{L_2}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right) y^d_{31}\bar{\Psi}^q_{L_3}\varphi d_R + \left(\frac{\chi}{\Lambda}\right) y^d_{32}\bar{\Psi}^q_{L_3}\varphi s_R + \left(\frac{\chi}{\Lambda}\right) y^d_{33}\bar{\Psi}^q_{L_3}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{11}\bar{\Psi}^\ell_{L_1}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{12}\bar{\Psi}^\ell_{L_1}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{13}\bar{\Psi}^\ell_{L_1}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{21}\bar{\Psi}^\ell_{L_2}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{22}\bar{\Psi}^\ell_{L_2}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{23}\bar{\Psi}^\ell_{L_2}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right) y^\ell_{31}\bar{\Psi}^\ell_{L_3}\varphi e_R + \left(\frac{\chi}{\Lambda}\right) y^\ell_{32}\bar{\Psi}^\ell_{L_3}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right) y^\ell_{33}\bar{\Psi}^\ell_{L_3}\varphi \tau_R \\
&+ \text{H.c.}
\end{aligned} \tag{2.6}
$$

The mass matrices for up- and down-type quarks and charged leptons read,

$$
\mathscr{M}^u = \frac{v}{\sqrt{2}}\begin{pmatrix} y^u_{11}\varepsilon^4 & y^u_{12}\varepsilon^4 & y^u_{13}\varepsilon^4 \\ y^u_{21}\varepsilon^2 & y^u_{22}\varepsilon^2 & y^u_{23}\varepsilon^2 \\ y^u_{31} & y^u_{32} & y^u_{33} \end{pmatrix}, \quad \mathscr{M}^d = \frac{v}{\sqrt{2}}\begin{pmatrix} y^d_{11}\varepsilon^5 & y^d_{12}\varepsilon^5 & y^d_{13}\varepsilon^5 \\ y^d_{21}\varepsilon^3 & y^d_{22}\varepsilon^3 & y^d_{23}\varepsilon^3 \\ y^d_{31}\varepsilon & y^d_{32}\varepsilon & y^d_{33}\varepsilon \end{pmatrix},
$$
$$
\mathscr{M}^\ell = \frac{v}{\sqrt{2}}\begin{pmatrix} y^\ell_{11}\varepsilon^5 & y^\ell_{12}\varepsilon^5 & y^\ell_{13}\varepsilon^5 \\ y^\ell_{21}\varepsilon^3 & y^\ell_{22}\varepsilon^3 & y^\ell_{23}\varepsilon^3 \\ y^\ell_{31}\varepsilon & y^\ell_{32}\varepsilon & y^\ell_{33}\varepsilon \end{pmatrix}. \tag{2.7}
$$

where $\varepsilon = 0.1$ can produce the required masses of the fermions.

The masses of the charged fermions can be written as[137],

$$\{m_t, m_c, m_u\} \simeq \Bigg\{ |y^u_{33}|,\ \left| y^u_{22} - \frac{y^u_{23} y^u_{32}}{|y^u_{33}|} \right| \varepsilon^2, \\ \left| y^u_{11} - \frac{y^u_{12} y^u_{21}}{|y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}|} - \frac{y^u_{13} |y^u_{31} y^u_{22} - y^u_{21} y^u_{32}| - y^u_{31} y^u_{12} y^u_{23}}{|y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}| |y^u_{33}|} \right| \varepsilon^4 \Bigg\} \frac{v}{\sqrt{2}}, \tag{2.8}$$

$$\{m_b, m_s, m_d\} \simeq \Bigg\{ |y^d_{33}|\varepsilon,\ \left| y^d_{22} - \frac{y^d_{23} y^d_{32}}{|y^d_{33}|} \right| \varepsilon^3, \\ \left| y^d_{11} - \frac{y^d_{12} y^d_{21}}{|y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}|} - \frac{y^d_{13} |y^d_{31} y^d_{22} - y^d_{21} y^d_{32}| - y^d_{31} y^d_{12} y^d_{23}}{|y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}| |y^d_{33}|} \right| \varepsilon^5 \Bigg\} \frac{v}{\sqrt{2}}, \tag{2.9}$$

$$\{m_\tau, m_\mu, m_e\} \simeq \Bigg\{ |y^l_{33}|\varepsilon,\ \left| y^l_{22} - \frac{y^l_{23} y^l_{32}}{|y^l_{33}|} \right| \varepsilon^3, \\ \left| y^l_{11} - \frac{y^l_{12} y^l_{21}}{|y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}|} - \frac{y^l_{13} |y^l_{31} y^l_{22} - y^l_{21} y^l_{32}| - y^l_{31} y^l_{12} y^l_{23}}{|y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}| |y^l_{33}|} \right| \varepsilon^5 \Bigg\} \frac{v}{\sqrt{2}}. \tag{2.10}$$

The quark mixing angles are [138],

$$\begin{aligned} \sin\theta_{12} &\simeq |V_{us}| \simeq \left| \frac{y^d_{12}}{y^d_{22}} - \frac{y^u_{12}}{y^u_{22}} \right| \varepsilon^2, \\ \sin\theta_{23} &\simeq |V_{cb}| \simeq \left| \frac{y^d_{23}}{y^d_{33}} - \frac{y^u_{23}}{y^u_{33}} \right| \varepsilon^2, \\ \sin\theta_{13} &\simeq |V_{ub}| \simeq \left| \frac{y^d_{13}}{y^d_{33}} - \frac{y^u_{12} y^d_{23}}{y^u_{22} y^d_{33}} - \frac{y^u_{13}}{y^u_{33}} \right| \varepsilon^4. \end{aligned} \tag{2.11}$$

Neutrino masses and mixing

The neutrino masses are obtained by adding three right-handed neutrinos as shown in table 2.5. The Lagrangian for the tree-level Majorana mass is,

$$\mathscr{L}_{\rm M_R} = c_{ij}\left[\frac{\chi^\dagger}{\Lambda}\right]^5 \chi^\dagger \bar{\nu}^c{}_{i,R}\nu_{j,R}, \tag{2.12}$$

where i, j are flavour indices.

The Majorana mass matrices $\mathscr{M}_R$ is,

$$\mathscr{M}_R = M\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \tag{2.13}$$

where $M = \langle\chi\rangle\left[\frac{\langle\chi\rangle}{\Lambda}\right]^5 = \frac{f}{\sqrt{2}}\varepsilon^5$.

The Dirac mass Lagrangian for neutrinos can be written as,

$$\begin{aligned} -\mathscr{L}^{\nu}_{\rm Yukawa} = {} & y^{\nu}_{11}\bar{\Psi}^{\ell}_{L_1}H\nu_{e_R}\left[\frac{\chi}{\Lambda}\right]^3 + y^{\nu}_{12}\bar{\Psi}^{\ell}_{L_1}H\nu_{\mu_R}\left[\frac{\chi}{\Lambda}\right]^3 + y^{\nu}_{13}\bar{\Psi}^{\ell}_{L_1}H\nu_{\tau_R}\left[\frac{\chi}{\Lambda}\right]^3 \\ & + y^{\nu}_{21}\bar{\Psi}^{\ell}_{L_2}H\nu_{e_R}\left[\frac{\chi}{\Lambda}\right] + y^{\nu}_{22}\bar{\Psi}^{\ell}_{L_2}H\nu_{\mu_R}\left[\frac{\chi}{\Lambda}\right] + y^{\nu}_{23}\bar{\Psi}^{\ell}_{L_2}H\nu_{\tau_R}\left[\frac{\chi}{\Lambda}\right] \\ & + y^{\nu}_{31}\bar{\Psi}^{\ell}_{L_3}H\nu_{e_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + y^{\nu}_{32}\bar{\Psi}^{\ell}_{L_3}H\nu_{\mu_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + y^{\nu}_{33}\bar{\Psi}^{\ell}_{L_3}H\nu_{\tau_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + \text{H.c.}. \end{aligned} \tag{2.14}$$

The Dirac mass matrix is given by,

$$\mathscr{M}_{\mathscr{D}} = \frac{v}{\sqrt{2}}\begin{pmatrix} y^{\nu}_{11}\varepsilon^3 & y^{\nu}_{12}\varepsilon^3 & y^{\nu}_{13}\varepsilon^3 \\ y^{\nu}_{21}\varepsilon & y^{\nu}_{22}\varepsilon & y^{\nu}_{23}\varepsilon \\ y^{\nu}_{31}\varepsilon & y^{\nu}_{32}\varepsilon & y^{\nu}_{33}\varepsilon \end{pmatrix}. \tag{2.15}$$

The mass matrix of neutrinos after including the Majorana mass terms can be written as,

$$\mathscr{M} = \begin{pmatrix} \mathscr{M}_L & \mathscr{M}_{\mathscr{D}} \\ \mathscr{M}_{\mathscr{D}}^T & \mathscr{M}_R \end{pmatrix}. \tag{2.16}$$

Since $v << f$, we ignore the contribution of the mass matrix $\mathscr{M}_L$ to the neutrino masses.[3] Now, we can use the type-I seesaw mechanism to determine the neutrino masses by assuming $\mathscr{M}_{\mathscr{D}} << \mathscr{M}_R$ [139]. Thus the light neutrino mass matrix is,

$$\mathscr{M} \approx -\mathscr{M}_{\mathscr{D}}\mathscr{M}_R^{-1}\mathscr{M}_{\mathscr{D}}^T, \tag{2.17}$$

$$\approx \frac{v}{\sqrt{2}}\varepsilon' \begin{pmatrix} -\frac{\varepsilon^4\left(c_{22}c_{33}y_{11}^{\nu 2}-2c_{22}y_{11}^{\nu}+c_{22}-2c_{33}y_{11}^{\nu}+c_{33}-y_{11}^{\nu 2}+4y_{11}^{\nu}-2\right)}{(c_{22}-1)(c_{33}-1)} & -\varepsilon^2 y_{11}^{\nu} & -\frac{\varepsilon^2\left(c_{33}y_{11}^{\nu}-y_{11}^{\nu}y_{33}^{\nu}+y_{33}^{\nu}-1\right)}{c_{33}-1} \\ -\varepsilon^2 y_{11}^{\nu} & -1 & -1 \\ -\frac{\varepsilon^2\left(c_{33}y_{11}^{\nu}-y_{11}^{\nu}y_{33}^{\nu}+y_{33}^{\nu}-1\right)}{c_{33}-1} & -1 & -\frac{\left(c_{33}+(y_{33}^{\nu}-2)y_{33}^{\nu}\right)}{c_{33}-1} \end{pmatrix},$$

where $\varepsilon' = \frac{v}{f\varepsilon^3}$, and we have assumed each and every coupling exactly one except those appearing in above equation.

[3] Alternatively, we can assume that it is forbidden by some discrete symmetry. For instance, if three left-handed fermionic doublets of quarks and leptons, and the Higgs doublet have a charge ω under a $\mathscr{Z}_3$ symmetry, the mass matrix $\mathscr{M}_L$ is forbidden.

We obtain two degenerate neutrino masses. The masses approximately are given by,

$$
\begin{aligned}
m_1 &\approx \frac{\left(-y_{11}^{\nu 2}+2y_{11}^{\nu}-1\right)}{c_{22}-1}\epsilon^4\epsilon' v/\sqrt{2}, \\
m_2 &\approx \frac{\left(-\sqrt{4c_{33}^2-8c_{33}+y_{33}^{\nu 4}-4y_{33}^{\nu 3}+6y_{33}^{\nu 2}-4y_{33}^{\nu}+5}-2y_{33}^{\nu}-y_{33}^{\nu 2}+2y_{33}^{\nu}+1\right)}{2(c_{33}-1)}\epsilon' v/\sqrt{2}, \\
m_3 &\approx \frac{\left(\sqrt{4c_{33}^2-8c_{33}+y_{33}^{\nu 4}-4y_{33}^{\nu 3}+6y_{33}^{\nu 2}-4y_{33}^{\nu}+5}-2c_{33}-y_{33}^{\nu 2}+2y_{33}^{\nu}+1\right)}{2(c_{33}-1)}\epsilon' v/\sqrt{2}.
\end{aligned}
\tag{2.18}
$$

This kind of approximate degenerate neutrino masses are well studied in literature, for instance, see references [140–143].

The leptonic mixing angles can be written as,

$$
\begin{aligned}
\sin\theta_{12} &\simeq \left|\frac{y_{12}^{\ell}}{y_{22}^{\ell}}-y_{11}^{\nu}\right|\epsilon^2, \\
\sin\theta_{23} &\simeq \left|\frac{1-c_{33}}{c_{33}+(y_{33}^{\nu}-2)y_{33}^{\nu}}\right|, \\
\sin\theta_{13} &\simeq \left|\frac{(c_{33}y_{11}^{\nu}-y_{11}^{\nu}y_{33}^{\nu}+y_{33}^{\nu}-1)}{c_{33}+(y_{33}^{\nu}-2)y_{33}^{\nu}}\right|\epsilon^2.
\end{aligned}
\tag{2.19}
$$

The remarkable observation is the pattern of the neutrino mixing angles. The mixing angle θ_{12} and θ_{13} are of the same order of magnitude, where θ_{13} is closer to the Cabibbo angle, and the mixing angle θ_{23} is completely unsuppressed.

2.2.5 $\mathscr{Z}_2\times\mathscr{Z}_9$ flavour symmetry

A non-minimal realization of the $\mathscr{Z}_{\rm N}\times\mathscr{Z}_{\rm M}$ flavour symmetry is the $\mathscr{Z}_2\times\mathscr{Z}_9$ flavour symmetry. The scalar and fermionic fields transform under this symmetry, as given in table 2.6.

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_9$
u_R, t_R	+	1
c_R	+	ω^4
$d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	ω^3
$\psi^q_{L_1}$	+	ω
$\psi^q_{L_2}$	+	ω^8
$\psi^q_{L_3}$	+	1
$\psi^\ell_{L_1}$	+	ω
$\psi^\ell_{L_2}$	+	ω^8
$\psi^\ell_{L_3}$	+	ω^6
χ	-	ω
φ	+	1

Table 2.6 The charges of left and right-handed fermions of three families of the SM, right-handed neutrinos, Higgs, and singlet scalar field under $\mathscr{Z}_2$ and $\mathscr{Z}_9$ symmetries, where ω is the ninth root of unity.

After imposing the $\mathscr{Z}_2 \times \mathscr{Z}_9$ flavour symmetry on the SM, the mass Lagrangian for charged fermions reads as,

$$
\begin{aligned}
-\mathscr{L}_{\text{Yukawa}} &= \left(\frac{\chi^\dagger}{\Lambda}\right)^8 y^u_{11}\bar{\psi}^q_{L_1}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^6 y^u_{12}\bar{\psi}^q_{L_1}\tilde{\varphi}c_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^8 y^u_{13}\bar{\psi}^q_{L_1}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^8 y^u_{21}\bar{\psi}^q_{L_2}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^4 y^u_{22}\bar{\psi}^q_{L_2}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^8 y^u_{23}\bar{\psi}^q_{L_2}\tilde{\varphi}t_R \\
&+ y^u_{31}\bar{\psi}^q_{L_3}\tilde{\varphi}u_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^4 y^u_{32}\bar{\psi}^q_{L_3}\tilde{\varphi}c_R + y^u_{33}\bar{\psi}^q_{L_3}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^7 y^d_{11}\bar{\psi}^q_{L_1}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^7 y^d_{12}\bar{\psi}^q_{L_1}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^7 y^d_{13}\bar{\psi}^q_{L_1}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^5 y^d_{21}\bar{\psi}^q_{L_2}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{22}\bar{\psi}^q_{L_2}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{23}\bar{\psi}^q_{L_2}\varphi b_R \\
&+ \left(\frac{\chi^\dagger}{\Lambda}\right)^3 y^d_{31}\bar{\psi}^q_{L_3}\varphi d_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^3 y^d_{32}\bar{\psi}^q_{L_3}\varphi s_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^3 y^d_{33}\bar{\psi}^q_{L_3}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^7 y^\ell_{11}\bar{\psi}^\ell_{L_1}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^7 y^\ell_{12}\bar{\psi}^\ell_{L_1}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^7 y^\ell_{13}\bar{\psi}^\ell_{L_1}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{21}\bar{\psi}^\ell_{L_2}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{22}\bar{\psi}^\ell_{L_2}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{23}\bar{\psi}^\ell_{L_2}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{31}\bar{\psi}^\ell_{L_3}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{32}\bar{\psi}^\ell_{L_3}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{33}\bar{\psi}^\ell_{L_3}\varphi \tau_R \\
&+ \text{H.c.} \qquad (2.20)
\end{aligned}
$$

The mass matrices for up and down-type quarks and charged leptons read,

$$\mathscr{M}^u = \frac{v}{\sqrt{2}} \begin{pmatrix} y^u_{11}\varepsilon^8 & y^u_{12}\varepsilon^6 & y^u_{13}\varepsilon^8 \\ y^u_{21}\varepsilon^8 & y^u_{22}\varepsilon^4 & y^u_{23}\varepsilon^8 \\ y^u_{31} & y^u_{32}\varepsilon^4 & y^u_{33} \end{pmatrix}, \quad \mathscr{M}^d = \frac{v}{\sqrt{2}} \begin{pmatrix} y^d_{11}\varepsilon^7 & y^d_{12}\varepsilon^7 & y^d_{13}\varepsilon^7 \\ y^d_{21}\varepsilon^5 & y^d_{22}\varepsilon^5 & y^d_{23}\varepsilon^5 \\ y^d_{31}\varepsilon^3 & y^d_{32}\varepsilon^3 & y^d_{33}\varepsilon^3 \end{pmatrix},$$
$$\mathscr{M}^\ell = \frac{v}{\sqrt{2}} \begin{pmatrix} y^\ell_{11}\varepsilon^7 & y^\ell_{12}\varepsilon^7 & y^\ell_{13}\varepsilon^7 \\ y^\ell_{21}\varepsilon^5 & y^\ell_{22}\varepsilon^5 & y^\ell_{23}\varepsilon^5 \\ y^\ell_{31}\varepsilon^3 & y^\ell_{32}\varepsilon^3 & y^\ell_{33}\varepsilon^3 \end{pmatrix}. \tag{2.21}$$

where $\varepsilon = 0.225$ is chosen to produce the masses of fermions.

The masses of charged fermions are approximately [137],

$$\{m_t, m_c, m_u\} \simeq \Big\{ |y^u_{33}|, \left| y^u_{22}\varepsilon^4 - \frac{y^u_{23}y^u_{32}}{|y^u_{33}|}\varepsilon^{12} \right|, \left| y^u_{11}\varepsilon^8 - \frac{y^u_{12}y^u_{21}}{|y^u_{22}|}\varepsilon^{10} - \frac{y^u_{13}|y^u_{31}y^u_{22} - y^u_{21}y^u_{32}|}{|y^u_{22}||y^u_{33}|}\varepsilon^8 \right| \Big\} v/\sqrt{2}, \tag{2.22}$$

$$\{m_b, m_s, m_d\} \simeq \Big\{ |y^d_{33}|\varepsilon^3, \left| y^d_{22} - \frac{y^d_{23}y^d_{32}}{|y^d_{33}|} \right| \varepsilon^5, \left| y^d_{11} - \frac{y^d_{12}y^d_{21}}{|y^d_{22} - y^d_{23}y^d_{32}/y^d_{33}|} - \frac{y^d_{13}|y^d_{31}y^d_{22} - y^d_{21}y^d_{32}| - y^d_{31}y^d_{12}y^d_{23}}{|y^d_{22} - y^d_{23}y^d_{32}/y^d_{33}||y^d_{33}|} \right| \varepsilon^7 \Big\} v/\sqrt{2}, \tag{2.23}$$

$$\{m_\tau, m_\mu, m_e\} \simeq \Big\{ |y^l_{33}|\varepsilon^3, \left| y^l_{22} - \frac{y^l_{23}y^l_{32}}{|y^l_{33}|} \right| \varepsilon^5, \left| y^l_{11} - \frac{y^l_{12}y^l_{21}}{|y^l_{22} - y^l_{23}y^l_{32}/y^l_{33}|} - \frac{y^l_{13}|y^l_{31}y^l_{22} - y^l_{21}y^l_{32}| - y^l_{31}y^l_{12}y^l_{23}}{|y^l_{22} - y^l_{23}y^l_{32}/y^l_{33}||y^l_{33}|} \right| \varepsilon^7 \Big\} v/\sqrt{2}. \tag{2.24}$$

The quark mixing angles are identical to that of the minimal $\mathscr{Z}_2 \times \mathscr{Z}_5$ flavour symmetry.

Neutrino masses and mixing

The masses and mixing of neutrinos in the non-minimal model is identical to that of the minimal model. Thus, we write the Majorana Lagrangian for right-handed neutrinos as,

$$\mathscr{L}_{\mathrm{M_R}} = c_{ij}\left[\frac{\chi}{\Lambda}\right]^3 \chi \bar{\nu^c}_{i,R}\nu_{j,R}, \tag{2.25}$$

where i, j are flavour indices.

The Majorana mass matrices $\mathscr{M}_R$ can be written as,

$$\mathscr{M}_R = M\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \tag{2.26}$$

where $M = \langle\chi\rangle\left[\frac{\langle\chi\rangle}{\Lambda}\right]^3 = \frac{f}{\sqrt{2}}\varepsilon^3$.

The Dirac mass Lagrangian for neutrinos is,

$$\begin{aligned} -\mathscr{L}^{\nu}_{\mathrm{Yukawa}} =& y^{\nu}_{11}\bar{\Psi}^{\ell}_{L_1}H\nu_{e_R}\left[\frac{\chi}{\Lambda}\right]^3 + y^{\nu}_{12}\bar{\Psi}^{\ell}_{L_1}H\nu_{\mu_R}\left[\frac{\chi}{\Lambda}\right]^3 + y^{\nu}_{13}\bar{\Psi}^{\ell}_{L_1}H\nu_{\tau_R}\left[\frac{\chi}{\Lambda}\right]^3 \\ &+ y^{\nu}_{21}\bar{\Psi}^{\ell}_{L_2}H\nu_{e_R}\left[\frac{\chi}{\Lambda}\right] + y^{\nu}_{22}\bar{\Psi}^{\ell}_{L_2}H\nu_{\mu_R}\left[\frac{\chi}{\Lambda}\right] + y^{\nu}_{23}\bar{\Psi}^{\ell}_{L_2}H\nu_{\tau_R}\left[\frac{\chi}{\Lambda}\right] \\ &+ y^{\nu}_{31}\bar{\Psi}^{\ell}_{L_3}H\nu_{e_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + y^{\nu}_{32}\bar{\Psi}^{\ell}_{L_3}H\nu_{\mu_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + y^{\nu}_{33}\bar{\Psi}^{\ell}_{L_3}H\nu_{\tau_R}\left[\frac{\chi^\dagger}{\Lambda}\right] + \mathrm{H.c.}. \end{aligned} \tag{2.27}$$

The Dirac mass matrix for neutrinos now reads,

$$\mathscr{M}_{\mathscr{D}} = \frac{v}{\sqrt{2}}\begin{pmatrix} y^{\nu}_{11}\varepsilon^3 & y^{\nu}_{12}\varepsilon^3 & y^{\nu}_{13}\varepsilon^3 \\ y^{\nu}_{21}\varepsilon & y^{\nu}_{22}\varepsilon & y^{\nu}_{23}\varepsilon \\ y^{\nu}_{31}\varepsilon & y^{\nu}_{32}\varepsilon & y^{\nu}_{33}\varepsilon \end{pmatrix}. \tag{2.28}$$

The mass matrix of neutrinos after including the Majorana mass term is,

$$\mathscr{M} = \begin{pmatrix} \mathscr{M}_L & \mathscr{M}_\mathscr{D} \\ \mathscr{M}_\mathscr{D}^T & \mathscr{M}_R \end{pmatrix}. \tag{2.29}$$

The light neutrino mass matrix is,

$$\mathscr{M} \approx -\mathscr{M}_\mathscr{D}\mathscr{M}_R^{-1}\mathscr{M}_\mathscr{D}^T, \tag{2.30}$$

$$\approx \frac{v}{\sqrt{2}}\epsilon' \begin{pmatrix} -\frac{\epsilon^4\left(c_{22}c_{33}y_{11}^{\nu 2}-2c_{22}y_{11}^\nu+c_{22}-2c_{33}y_{11}^\nu+c_{33}-y_{11}^{\nu 2}+4y_{11}^\nu-2\right)}{(c_{22}-1)(c_{33}-1)} & -\epsilon^2 y_{11}^\nu & -\frac{\epsilon^2(c_{33}y_{11}^\nu-y_{11}^\nu y_{33}^\nu+y_{33}^\nu-1)}{c_{33}-1} \\ -\epsilon^2 y_{11}^\nu & -1 & -1 \\ -\frac{\epsilon^2(c_{33}y_{11}^\nu-y_{11}^\nu y_{33}^\nu+y_{33}^\nu-1)}{c_{33}-1} & -1 & -\frac{(c_{33}+(y_{33}^\nu-2)y_{33}^\nu)}{c_{33}-1} \end{pmatrix},$$

where $\epsilon' = \frac{v}{f\epsilon}$, and we have again assumed each and every coupling exactly one except those appearing in above equation.

The neutrino masses approximately are,

$$m_1 \approx \frac{\left(-y_{11}^{\nu 2}+2y_{11}^\nu-1\right)}{c_{22}-1}\epsilon^4\epsilon' v/\sqrt{2}, \tag{2.31}$$

$$m_2 \approx \frac{\left(-\sqrt{4c_{33}^2-8c_{33}+y_{33}^{\nu 4}-4y_{33}^{\nu 3}+6y_{33}^{\nu 2}-4y_{33}^\nu+5}-2y_{33}^\nu-y_{33}^{\nu 2}+2y_{33}^\nu+1\right)}{2(c_{33}-1)}\epsilon' v/\sqrt{2},$$

$$m_3 \approx \frac{\left(\sqrt{4c_{33}^2-8c_{33}+y_{33}^{\nu 4}-4y_{33}^{\nu 3}+6y_{33}^{\nu 2}-4y_{33}^\nu+5}-2c_{33}-y_{33}^{\nu 2}+2y_{33}^\nu+1\right)}{2(c_{33}-1)}\epsilon' v/\sqrt{2}.$$

The neutrino mixing angles are,

$$\sin\theta_{12} \simeq \left|\frac{y^{\ell}_{12}}{y^{\ell}_{22}} - y^{\nu}_{11}\right| \varepsilon^2, \tag{2.32}$$

$$\sin\theta_{23} \simeq \left|\frac{1-c_{33}}{c_{33}+(y^{\nu}_{33}-2)y^{\nu}_{33}}\right|,$$

$$\sin\theta_{13} \simeq \left|\frac{(c_{33}y^{\nu}_{11} - y^{\nu}_{11}y^{\nu}_{33} + y^{\nu}_{33} - 1)}{c_{33}+(y^{\nu}_{33}-2)y^{\nu}_{33}}\right| \varepsilon^2.$$

2.2.6 $\mathcal{Z}_2 \times \mathcal{Z}_{11}$ flavour symmetry

The next non-minimal realization of the $\mathcal{Z}_{\rm N} \times \mathcal{Z}_{\rm M}$ flavour symmetry is the $\mathcal{Z}_2 \times \mathcal{Z}_{11}$ flavour symmetry. The $\mathcal{Z}_2 \times \mathcal{Z}_{11}$ flavour symmetry, in this work, is chosen to probe a relatively large value of the parameter ε, which is 0.28. The scalar and fermionic fields transform under this symmetry, as shown in table 2.7.

Fields	$\mathcal{Z}_2$	$\mathcal{Z}_{11}$
u_R, c_R	$+$	ω^2
t_R	$+$	1
$d_R, s_R, b_R, e_R, \mu_R, \tau_R$	-	ω^3
ψ^1_L	$+$	ω
ψ^2_L	$+$	ω^7
ψ^3_L	$+$	1
χ	-	ω
φ	$+$	1
σ	$+$	ω^4

Table 2.7 The charges of left and right-handed fermions of three families of the SM, right-handed neutrinos, Higgs, and singlet scalar fields under $\mathcal{Z}_2$ and $\mathcal{Z}_{11}$ symmetries, where ω is the eleventh root of unity.

The mass Lagrangian for charged fermions after imposing $\mathscr{Z}_2\times\mathscr{Z}_{11}$ flavour symmetry on the SM reads,

$$
\begin{aligned}
-\mathscr{L}_{\rm Yukawa} = & \left(\frac{\chi}{\Lambda}\right)^{10} y^u_{11}\bar{\Psi}^q_{L_1}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^{10} y^u_{12}\bar{\Psi}^q_{L_1}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^{12} y^u_{13}\bar{\Psi}^q_{L_1}\tilde{\varphi}t_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{6} y^u_{21}\bar{\Psi}^q_{L_2}\tilde{\varphi}u_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{6} y^u_{22}\bar{\Psi}^q_{L_2}\tilde{\varphi}c_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{4} y^u_{23}\bar{\Psi}^q_{L_2}\tilde{\varphi}t_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{2} y^u_{31}\bar{\Psi}^q_{L_3}\tilde{\varphi}u_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{2} y^u_{32}\bar{\Psi}^q_{L_3}\tilde{\varphi}c_R + y^u_{33}\bar{\Psi}^q_{L_3}\tilde{\varphi}t_R \\
& + \left(\frac{\chi}{\Lambda}\right)^{9} y^d_{11}\bar{\Psi}^q_{L_1}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^{9} y^d_{12}\bar{\Psi}^q_{L_1}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^{9} y^d_{13}\bar{\Psi}^q_{L_1}\varphi b_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^d_{21}\bar{\Psi}^q_{L_2}\varphi d_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^d_{22}\bar{\Psi}^q_{L_2}\varphi s_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^d_{23}\bar{\Psi}^q_{L_2}\varphi b_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^d_{31}\bar{\Psi}^q_{L_3}\varphi d_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^d_{32}\bar{\Psi}^q_{L_3}\varphi s_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^d_{33}\bar{\Psi}^q_{L_3}\varphi b_R \\
& + \left(\frac{\chi}{\Lambda}\right)^{9} y^\ell_{11}\bar{\Psi}^\ell_{L_1}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^{9} y^\ell_{12}\bar{\Psi}^\ell_{L_1}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^{9} y^\ell_{13}\bar{\Psi}^\ell_{L_1}\varphi \tau_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^\ell_{21}\bar{\Psi}^\ell_{L_2}\varphi e_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^\ell_{22}\bar{\Psi}^\ell_{L_2}\varphi \mu_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{7} y^\ell_{23}\bar{\Psi}^\ell_{L_2}\varphi \tau_R \\
& + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^\ell_{31}\bar{\Psi}^\ell_{L_3}\varphi e_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^\ell_{32}\bar{\Psi}^\ell_{L_3}\varphi \mu_R + \left(\frac{\chi^\dagger}{\Lambda}\right)^{3} y^\ell_{33}\bar{\Psi}^\ell_{L_3}\varphi \tau_R + \text{H.c.}
\end{aligned}
\tag{2.33}
$$

The mass matrices for up- and down-type quarks and charged leptons turn out to be,

$$\mathscr{M}^u = \frac{v}{\sqrt{2}} \begin{pmatrix} y^u_{11}\varepsilon^{10} & y^u_{12}\varepsilon^{10} & y^u_{13}\varepsilon^{12} \\ y^u_{21}\varepsilon^{6} & y^u_{22}\varepsilon^{6} & y^u_{23}\varepsilon^{4} \\ y^u_{31}\varepsilon^{2} & y^u_{32}\varepsilon^{2} & y^u_{33} \end{pmatrix}, \quad \mathscr{M}^d = \frac{v}{\sqrt{2}} \begin{pmatrix} y^d_{11}\varepsilon^{9} & y^d_{12}\varepsilon^{9} & y^d_{13}\varepsilon^{9} \\ y^d_{21}\varepsilon^{7} & y^d_{22}\varepsilon^{7} & y^d_{23}\varepsilon^{7} \\ y^d_{31}\varepsilon^{3} & y^d_{32}\varepsilon^{3} & y^d_{33}\varepsilon^{3} \end{pmatrix},$$
$$\mathscr{M}^\ell = \frac{v}{\sqrt{2}} \begin{pmatrix} y^\ell_{11}\varepsilon^{9} & y^\ell_{12}\varepsilon^{9} & y^\ell_{13}\varepsilon^{9} \\ y^\ell_{21}\varepsilon^{7} & y^\ell_{22}\varepsilon^{7} & y^\ell_{23}\varepsilon^{7} \\ y^\ell_{31}\varepsilon^{3} & y^\ell_{32}\varepsilon^{3} & y^\ell_{33}\varepsilon^{3} \end{pmatrix}. \tag{2.34}$$

The masses of quarks and charged leptons are approximately [137],

$$\{m_t, m_c, m_u\} \simeq \left\{ |y^u_{33}|, \left| y^u_{22} - \frac{y^u_{23} y^u_{32}}{|y^u_{33}|} \right| \varepsilon^6, \right.$$
$$\left. \left| y^u_{11} - \frac{y^u_{12} y^u_{21}}{|y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}|} - \frac{y^u_{13}|y^u_{31} y^u_{22} - y^u_{21} y^u_{32}| - y^u_{31} y^u_{12} y^u_{23}}{|y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}||y^u_{33}|} \right| \varepsilon^{10} \right\} v/\sqrt{2}, \tag{2.35}$$

$$\{m_b, m_s, m_d\} \simeq \left\{ |y^d_{33}|\varepsilon^3, \left| y^d_{22} - \frac{y^d_{23} y^d_{32}}{|y^d_{33}|} \right| \varepsilon^7, \right.$$
$$\left. \left| y^d_{11} - \frac{y^d_{12} y^d_{21}}{|y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}|} - \frac{y^d_{13}|y^d_{31} y^d_{22} - y^d_{21} y^d_{32}| - y^d_{31} y^d_{12} y^d_{23}}{|y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}||y^d_{33}|} \right| \varepsilon^{9} \right\} v/\sqrt{2}, \tag{2.36}$$

$$\{m_\tau, m_\mu, m_e\} \simeq \left\{ |y^l_{33}|\varepsilon,^3 \left| y^l_{22} - \frac{y^l_{23} y^l_{32}}{|y^l_{33}|} \right| \varepsilon^7, \right.$$
$$\left. \left| y^l_{11} - \frac{y^l_{12} y^l_{21}}{|y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}|} - \frac{y^l_{13}|y^l_{31} y^l_{22} - y^l_{21} y^l_{32}| - y^l_{31} y^l_{12} y^l_{23}}{|y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}||y^l_{33}|} \right| \varepsilon^{9} \right\} v/\sqrt{2}. \tag{2.37}$$

The mixing angles of quarks are found to be [138],

$$
\begin{aligned}
\sin\theta_{12} \simeq |V_{us}| &\simeq \left|\frac{y^d_{12}}{y^d_{22}}\varepsilon^2 - \frac{y^u_{12}}{y^u_{22}}\varepsilon^4\right|, \sin\theta_{23} \simeq |V_{cb}| \simeq \left|\frac{y^d_{23}}{y^d_{33}} - \frac{y^u_{23}}{y^u_{33}}\right|\varepsilon^4, \\
\sin\theta_{13} \simeq |V_{ub}| &\simeq \left|\frac{y^d_{13}}{y^d_{33}}\varepsilon^6 - \frac{y^u_{12}y^d_{23}}{y^u_{22}y^d_{33}}\varepsilon^8\right|.
\end{aligned}
\tag{2.38}
$$

The mixing angles here have a consistent hierarchical pattern of the order (ε^2, ε^4, ε^6). However, to produce the observed values of the fermionic masses and mixing angles, we need a relatively large value of the parameter ε.

2.2.7 The $\mathscr{Z}_8 \times \mathscr{Z}_{22}$ flavour symmetry

This symmetry can provide the so-called flavonic dark matter [15], and is inspired by the hierarchical VEVs model[22, 24], where the mass of the top quark originates from the dimension-5 operator. The transformations of fermionic and scalar fields under this symmetry are given in table 2.8.

Fields	$\mathscr{Z}_8$	$\mathscr{Z}_{22}$	Fields	$\mathscr{Z}_8$	$\mathscr{Z}_{22}$	Fields	$\mathscr{Z}_8$	$\mathscr{Z}_{22}$	Fields	$\mathscr{Z}_8$	$\mathscr{Z}_{22}$	Fields	$\mathscr{Z}_8$	$\mathscr{Z}_{22}$
u_R	ω^2	ω^2	c_R	ω^5	ω^5	t_R	ω^6	ω^6	d_R	ω^3	ω^3	s_R	ω^4	ω^4
b_R	ω^4	ω^4	$\psi^q_{L,1}$	ω^2	ω^{10}	$\psi^q_{L,2}$	ω	ω^9	$\psi^q_{L,3}$	ω^7	ω^7	$\psi^\ell_{L,1}$	ω^3	ω^3
$\psi^\ell_{L,2}$	ω^2	ω^2	$\psi^\ell_{L,3}$	ω^2	ω^2	e_R	ω^2	ω^{16}	μ_R	ω^5	ω^{19}	τ_R	ω^7	ω^{21}
ν_{e_R}	ω^2	1	ν_{μ_R}	ω^5	ω^3	ν_{τ_R}	ω^6	ω^4	χ	ω	ω	φ	1	1

Table 2.8 The charges of the SM and the flavon fields under the $\mathscr{Z}_8 \times \mathscr{Z}_{22}$ symmetry, where ω is the 8th and 22nd root of unity.

The $\mathscr{Z}_8 \times \mathscr{Z}_{22}$ symmetry allows us to write mass Lagrangian for the charged fermions as,

$$\begin{aligned}
-\mathscr{L}_{\text{Yukawa}} &= \left(\frac{\chi}{\Lambda}\right)^8 y^u_{11}\bar{\Psi}^q_{L_1}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^5 y^u_{12}\bar{\Psi}^q_{L_1}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^4 y^u_{13}\bar{\Psi}^q_{L_1}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^7 y^u_{21}\bar{\Psi}^q_{L_2}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^4 y^u_{22}\bar{\Psi}^q_{L_2}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right)^3 y^u_{23}\bar{\Psi}^q_{L_2}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^5 y^u_{31}\bar{\Psi}^q_{L_3}\tilde{\varphi}u_R + \left(\frac{\chi}{\Lambda}\right)^2 y^u_{32}\bar{\Psi}^q_{L_3}\tilde{\varphi}c_R + \left(\frac{\chi}{\Lambda}\right) y^u_{33}\bar{\Psi}^q_{L_3}\tilde{\varphi}t_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^7 y^d_{11}\bar{\Psi}^q_{L_1}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^6 y^d_{12}\bar{\Psi}^q_{L_1}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^6 y^d_{13}\bar{\Psi}^q_{L_1}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^6 y^d_{21}\bar{\Psi}^q_{L_2}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{22}\bar{\Psi}^q_{L_2}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^5 y^d_{23}\bar{\Psi}^q_{L_2}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^4 y^d_{31}\bar{\Psi}^q_{L_3}\varphi d_R + \left(\frac{\chi}{\Lambda}\right)^3 y^d_{32}\bar{\Psi}^q_{L_3}\varphi s_R + \left(\frac{\chi}{\Lambda}\right)^3 y^d_{33}\bar{\Psi}^q_{L_3}\varphi b_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^9 y^\ell_{11}\bar{\Psi}^\ell_{L_1}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^6 y^\ell_{12}\bar{\Psi}^\ell_{L_1}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^4 y^\ell_{13}\bar{\Psi}^\ell_{L_1}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^8 y^\ell_{21}\bar{\Psi}^\ell_{L_2}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{22}\bar{\Psi}^\ell_{L_2}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{23}\bar{\Psi}^\ell_{L_2}\varphi \tau_R \\
&+ \left(\frac{\chi}{\Lambda}\right)^8 y^\ell_{31}\bar{\Psi}^\ell_{L_3}\varphi e_R + \left(\frac{\chi}{\Lambda}\right)^5 y^\ell_{32}\bar{\Psi}^\ell_{L_3}\varphi \mu_R + \left(\frac{\chi}{\Lambda}\right)^3 y^\ell_{33}\bar{\Psi}^\ell_{L_3}\varphi \tau_R \\
&+ \text{H.c.}
\end{aligned} \tag{2.39}$$

The mass matrices of the up and down-type quarks and charged leptons take the forms,

$$\begin{gathered}
\mathscr{M}^u = \frac{v}{\sqrt{2}}\begin{pmatrix} y^u_{11}\varepsilon^8 & y^u_{12}\varepsilon^5 & y^u_{13}\varepsilon^4 \\ y^u_{21}\varepsilon^7 & y^u_{22}\varepsilon^4 & y^u_{23}\varepsilon^3 \\ y^u_{31}\varepsilon^5 & y^u_{32}\varepsilon^2 & y^u_{33}\varepsilon \end{pmatrix}, \quad \mathscr{M}^d = \frac{v}{\sqrt{2}}\begin{pmatrix} y^d_{11}\varepsilon^7 & y^d_{12}\varepsilon^6 & y^d_{13}\varepsilon^6 \\ y^d_{21}\varepsilon^6 & y^d_{22}\varepsilon^5 & y^d_{23}\varepsilon^5 \\ y^d_{31}\varepsilon^4 & y^d_{32}\varepsilon^3 & y^d_{33}\varepsilon^3 \end{pmatrix}, \\
\mathscr{M}^\ell = \frac{v}{\sqrt{2}}\begin{pmatrix} y^\ell_{11}\varepsilon^9 & y^\ell_{12}\varepsilon^6 & y^\ell_{13}\varepsilon^4 \\ y^\ell_{21}\varepsilon^8 & y^\ell_{22}\varepsilon^5 & y^\ell_{23}\varepsilon^3 \\ y^\ell_{31}\varepsilon^8 & y^\ell_{32}\varepsilon^5 & y^\ell_{33}\varepsilon^3 \end{pmatrix}.
\end{gathered} \tag{2.40}$$

where $\varepsilon = 0.225$ produces the masses and mixing of fermions.

The masses of charged fermions are given by [137],

$$\{m_t, m_c, m_u\} \simeq \left\{ |y^u_{33}|\varepsilon,\ \left| y^u_{22} - \frac{y^u_{23} y^u_{32}}{y^u_{33}} \right| \varepsilon^4, \right. \\ \left. \left| y^u_{11} - \frac{y^u_{12} y^u_{21}}{y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}} - \frac{y^u_{13}(y^u_{31} y^u_{22} - y^u_{21} y^u_{32}) - y^u_{31} y^u_{12} y^u_{23}}{(y^u_{22} - y^u_{23} y^u_{32}/y^u_{33}) y^u_{33}} \right| \varepsilon^8 \right\} v/\sqrt{2}, \tag{2.41}$$

$$\{m_b, m_s, m_d\} \simeq \left\{ |y^d_{33}|\varepsilon^3,\ \left| y^d_{22} - \frac{y^d_{23} y^d_{32}}{y^d_{33}} \right| \varepsilon^5, \right. \\ \left. \left| y^d_{11} - \frac{y^d_{12} y^d_{21}}{y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}} - \frac{y^d_{13}(y^d_{31} y^d_{22} - y^d_{21} y^d_{32}) - y^d_{31} y^d_{12} y^d_{23}}{(y^d_{22} - y^d_{23} y^d_{32}/y^d_{33}) y^d_{33}} \right| \varepsilon^7 \right\} v/\sqrt{2}, \tag{2.42}$$

$$\{m_\tau, m_\mu, m_e\} \simeq \left\{ |y^l_{33}|\varepsilon^3,\ \left| y^l_{22} - \frac{y^l_{23} y^l_{32}}{y^l_{33}} \right| \varepsilon^5, \right. \\ \left. \left| y^l_{11} - \frac{y^l_{12} y^l_{21}}{y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}} - \frac{y^l_{13}\left(y^l_{31} y^l_{22} - y^l_{21} y^l_{32}\right) - y^l_{31} y^l_{12} y^l_{23}}{\left(y^l_{22} - y^l_{23} y^l_{32}/y^l_{33}\right) y^l_{33}} \right| \varepsilon^9 \right\} v/\sqrt{2}. \tag{2.43}$$

The mixing angles of quarks can be written as[137],

$$\sin\theta_{12} \simeq |V_{us}| \simeq \left| \frac{y^d_{12}}{y^d_{22}} - \frac{y^u_{12}}{y^u_{22}} \right| \varepsilon,\ \sin\theta_{23} \simeq |V_{cb}| \simeq \left| \frac{y^d_{23}}{y^d_{33}} - \frac{y^u_{23}}{y^u_{33}} \right| \varepsilon^2, \tag{2.44}$$
$$\sin\theta_{13} \simeq |V_{ub}| \simeq \left| \frac{y^d_{13}}{y^d_{33}} - \frac{y^u_{12} y^d_{23}}{y^u_{22} y^d_{33}} - \frac{y^u_{13}}{y^u_{33}} \right| \varepsilon^3.$$

We show, in table 2.9, the values of the parameter ε, and the minimum values of the Yukawa couplings $|y_{ij}|_{\rm min}$ for all four $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetries, we discussed. We notice that the models based on the symmetries $\mathscr{Z}_2 \times \mathscr{Z}_9$ and $\mathscr{Z}_8 \times \mathscr{Z}_{22}$ are described by ideally an order-one couplings providing the best description of fermion masses and mixing.

Model	$\boldsymbol{\varepsilon}$	$\lvert y_{ij} \rvert_{\mathbf{min}}$
$\mathscr{Z}_2 \times \mathscr{Z}_5$	0.1	≈ 0.1
$\mathscr{Z}_2 \times \mathscr{Z}_9$	0.23	≈ 1
$\mathscr{Z}_2 \times \mathscr{Z}_{11}$	0.28	≈ 0.7
$\mathscr{Z}_8 \times \mathscr{Z}_{22}$	0.23	≈ 1

Table 2.9 Values of ε and $|y_{ij}|_{\text{min}}$ for each $\mathscr{Z}_{\text{N}} \times \mathscr{Z}_{\text{M}}$ flavour symmetry.

2.3 The scalar potential for the $\mathscr{Z}_{\text{N}} \times \mathscr{Z}_{\text{M}}$ flavour symmetry

The scalar potential of the model acquires the following form,

$$-\mathscr{L}_{\text{potential}} = -\mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2 - \mu_\chi^2 \chi^* \chi + \lambda_\chi (\chi^* \chi)^2 + \lambda_{\varphi\chi} (\chi^* \chi)(\varphi^\dagger \varphi). \quad (2.45)$$

In the phenomenological investigation, we assume $\lambda_{\varphi\chi} = 0$, i.e., no Higgs-flavon mixing. The effects of this coupling are explored in reference [98]. The coupling $\lambda_{\phi\chi}$ may regenerate itself at one-loop level through the flavon-Higgs interactions. This can occur through box and triangle diagrams, where the dominant contribution involves the heaviest fermion in the loop, which can be the bottom quark for $\mathscr{Z}_2 \times \mathscr{Z}_{5,9,11}$ or top quark in the case of $\mathscr{Z}_8 \times \mathscr{Z}_{22}$ flavour symmetry.

For box diagrams, the leading correction is proportional to $\frac{1}{(4\pi)^2} \frac{m_f^4}{v^2 f^2}$, while for triangle diagrams, it is of the order $\frac{1}{(4\pi)^2} \frac{m_f^3}{v^2 f}$, where m_f is the mass of the fermion in the loop. For the flavon VEV $f \sim 1$ TeV, these contributions remain significantly suppressed. Thus, at leading order, the assumption $\lambda_{\phi\chi} = 0$ remains a reasonable approximation.

We parametrize the scalar fields as,

$$\chi = \frac{f + s + i a}{\sqrt{2}}, \ \varphi = \begin{pmatrix} G^+ \\ \frac{v+h+iG^0}{\sqrt{2}} \end{pmatrix}. \tag{2.46}$$

We have the SM VEV given by $\langle \phi \rangle = v(\equiv v_{SM}) \simeq 246$ GeV and $\langle \chi \rangle = \frac{f}{\sqrt{2}}$. Here, G^+, G^0 are the Goldstone modes, which become the longitudinal components of the gauge bosons and give them mass after the electroweak symmetry is spontaneously broken.

2.3.1 Softly broken scalar potential

In the conventional approach [14, 95, 102, 144, 145], we can provide mass to the pseudoscalar component of the flavon field by adding the following term to the scalar potential, which breaks the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M}$ flavour symmetry softly,

$$V_\rho = \rho \ \chi^2 + \text{H.c.}. \tag{2.47}$$

The parameter ρ is complex, and its phase can be rotated away by redefining the flavon field χ resulting in a real value of the VEV of the field χ.

The minimization conditions produce the masses of scalar and pseudoscalar flavons,

$$m_s = \sqrt{\lambda_\chi} f \qquad \text{and} \qquad m_a = \sqrt{2\rho}. \tag{2.48}$$

The mass of the pseudoscalar flavon depends on the soft-breaking parameter ρ, and is a free parameter of the model. The flavour and collider phenomenology of flavon based on the softly broken scalar potential can be found in references [14, 95, 102, 144, 145].

2.3.2 Symmetry-conserving scalar potential

The $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry presents a novel scenario that simultaneously addresses the flavour problem and dark matter within a unified and generic framework [15]. This is achieved by formulating the flavon potential as,

$$V_\lambda = -\lambda \frac{\chi^{\tilde{N}}}{\Lambda^{\tilde{N}-4}} + \text{H.c.}, \tag{2.49}$$

where $\tilde{N}$ is the least common multiple of N and M.

Upon the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry breaking by the VEV $\langle\chi\rangle$, the flavonic Goldstone boson (a) receives the potential,

$$V_\lambda = -|\lambda| \left(\frac{f}{\sqrt{2}}\right)^{\tilde{N}} \frac{1}{\Lambda^{\tilde{N}-4}} e^{i\left(\frac{\tilde{N}a}{f}+\alpha\right)}.$$

where $\lambda = |\lambda| e^{i\alpha}$. Retaining only the real part,

$$\begin{aligned} V_\lambda &= -\frac{1}{4}|\lambda|\varepsilon^{\tilde{N}-4} f^4 \cos\left(\tilde{N}\frac{a}{f}+\alpha\right) \\ &= -\frac{1}{4}|\lambda|\varepsilon^{\tilde{N}-4} f^4 \left[1 - \frac{1}{2}\left(\frac{\tilde{N}^2 a^2}{f^2} + \frac{2\tilde{N}a\alpha}{f} + \alpha^2\right) + \dots\right], \end{aligned} \tag{2.50}$$

where $\varepsilon = \frac{f}{\sqrt{2}\Lambda}$.

The axial flavon mass turns out to be,

$$m_a^2 = \frac{1}{8}|\lambda|\tilde{N}^2 \varepsilon^{\tilde{N}-4} f^2, \tag{2.51}$$

which depends on the symmetry breaking scale and is no longer an arbitrary free parameter of the model. We notice that for a large value of the $\tilde{N}$, the axial flavon can be the so-called flavonic dark matter, a new class of dark matter particles [15].

In addition to the approach discussed above, the mass term for the axial flavon can be extracted using the binomial expansion as follows. By employing the flavon parametrization given in equation 2.46, and retaining only the real part, the flavon potential in equation 2.49 can be expressed as,

$$V_\lambda = -|\lambda| \left(\frac{f}{\sqrt{2}}\right)^{\tilde{N}} \frac{1}{\Lambda^{\tilde{N}-4}} \left(1 + \frac{ia}{f}\right)^{\tilde{N}} \cos\alpha. \tag{2.52}$$

Now, using binomial expansion

$$V_\lambda = -\frac{1}{4}|\lambda|\varepsilon^{\tilde{N}-4} f^4 \left[1 + i\frac{\tilde{N}a}{f} - \frac{\tilde{N}(\tilde{N}-1)}{2}\frac{a^2}{f^2} + \dots\right] \cos\alpha.$$

From here, the mass term for the axial flavon can be read as,

$$m_a^2 = \frac{1}{8}|\lambda|\varepsilon^{\tilde{N}-4} f^2 \tilde{N}(\tilde{N}-1)\cos\alpha. \tag{2.53}$$

The parameter $\tilde{N}$ is the least common multiple of N and M in the $\mathscr{Z}_N \times \mathscr{Z}_M$ symmetry, and it is typically large. For example, in the minimal $\mathscr{Z}_2 \times \mathscr{Z}_5$ model, we have $\tilde{N} = 10$. Therefore, for large $\tilde{N}$, we can approximate $(\tilde{N}-1) \approx \tilde{N}$. Additionally, for small α, we can take $\cos\alpha \sim \mathscr{O}(1)$. With these approximations, the above expression simplifies to,

$$m_a^2 = \frac{1}{8}|\lambda|\varepsilon^{\tilde{N}-4} f^2 \tilde{N}^2, \tag{2.54}$$

which is identical to the expression given in 2.51.

2.3.3 Couplings of flavon to fermions

Now, using equation 2.46, we can write

$$\frac{\chi}{\Lambda} = \varepsilon[1 + \frac{s+ia}{f}]. \tag{2.55}$$

For obtaining the couplings of the scalar and pseudoscalar couplings to fermionic pair, we write the effective Yukawa couplings in the following form,

$$\begin{aligned} Y_{ij}^f \left(\frac{v+h}{\sqrt{2}}\right) &= y_{ij}^f \left(\frac{\chi}{\Lambda}\right)^{n_{ij}^f} \left(\frac{v+h}{\sqrt{2}}\right) \cong y_{ij}^f \varepsilon^{n_{ij}^f} \frac{v}{\sqrt{2}} \left[1 + \frac{n_{ij}^f(s+ia)}{f} + \frac{h}{v}\right] \\ &= \mathscr{M}_{ij}^f \left[1 + \frac{n_{ij}^f(s+ia)}{f} + \frac{h}{v}\right]. \end{aligned} \tag{2.56}$$

where $f = u, d, \ell$, and n_{ij}^f denote the power of the ε, which appears in the mass matrices $\mathscr{M}^f$.

We keep only the linear terms of the flavon field components s and a in equation 2.56 in our phenomenological analysis. The higher-order terms are not relevant for the analysis given in this work. The couplings of the scalar and pseudoscalar components s and a of the flavon field to the fermionic pair are derived from the matrix $n_{ij}^f \mathscr{M}_{ij}^f$, which cannot be diagonalized. This results in the non-diagonal flavour-changing and CP-violating interactions of the flavon field.

The couplings of the scalar component of the flavon field are obtained as,

$$y_{ij} = y_{s f_{iL} f_{iR}} = -i y_{a f_{iL} f_{iR}}. \tag{2.57}$$

2.4 The Standard Hierarchical VEVs model (SHVM)

In this section, we turn up to another approach to address the flavour problem that stands in stark contrast to the FN mechanism relying on $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry, discussed in the previous section, but shares the use of discrete symmetries. The hierarchical VEVs models (HVM) present a radical approach to solve the flavour problem by introducing the six gauge singlet scalar fields χ_i whose VEVs account for the fermionic mass spectrum and flavour mixing [22–25]. The HVM redefines the flavour problem in terms of the VEVs hierarchy of the scalar fields χ_i instead of the hierarchical Yukawa couplings and then solves it [22–26].

Apparently, this does not seem a solution to the flavour problem, since the hierarchy of the Yukawa couplings is replaced by the hierarchy of the VEVs, which are still free parameters of the theory. However, we hope that nature plays a subtle role in the case of flavour puzzle. The loftiness of nature may be revealed in an underlying theory where all the different hierarchical VEVs are multi-fermions chiral condensates of a dark-technicolour dynamics, and can be parameterized in terms of a single parameter of the dark-technicolour interactions. Such a scenario will be discussed in section 2.6 of this chapter.

In this section, we discuss what we call the "standard HVM" (SHVM) scenario, which is capable of predicting precise leptonic mixing angles in terms of the Cabibbo angle and the mass of the charm quark [23]. The SHVM is constructed by imposing a generic $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M} \times \mathscr{Z}_\mathrm{P}$ flavour symmetry on the SM, and on the gauge singlet scalar fields χ_i. The fields χ_i are assumed to be the bound states of strong dark-technicolor dynamics [22–25]. The $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M} \times \mathscr{Z}_\mathrm{P}$ flavour symmetry naturally emerges in the dark-technicolor paradigm through the breaking of three axial $U(1)_A$ symmetries [24, 25]. One can see section 2.6 for more details.

The transformations of the composite scalar fields χ_i under the $\mathscr{G}_{\rm SM} \equiv SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ symmetry of the SM are given as,

$$\chi_i : (1,1,0), \tag{2.58}$$

where $i = 1-6$.

After imposing the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M} \times \mathscr{Z}_{\rm P}$ flavour symmetry on the SM and the scalar fields χ_i, we can recover the masses of the charged fermions through the dimension-5 operators as given below [22, 23],

$$\mathscr{L} = \frac{1}{\Lambda_{\rm F}}\left[y_{ij}^u \bar{\psi}_{L_i}^q \tilde{\varphi} \psi_{R_j}^u \chi_r + y_{ij}^d \bar{\psi}_{L_i}^q \varphi \psi_{R_j}^d \chi_r + y_{ij}^\ell \bar{\psi}_{L_i}^\ell \varphi \psi_{R_j}^\ell \chi_r\right] + \text{H.c..} \tag{2.59}$$

where i and j denote the family indices, ψ_L^q, ψ_L^ℓ are the quark and lepton left-handed doublets, $\psi_R^u, \psi_R^d, \psi_R^\ell$ show the right-handed up, down type quarks and leptons, φ and $\tilde{\varphi} = -i\sigma_2\varphi^*$ show the SM Higgs field and its conjugate, and σ_2 stands for the second Pauli matrix. We produce the required pattern of the charged fermions masses for the SHVM by assigning the following generic charges under the $\mathscr{Z}_{\rm N} \times \mathscr{Z}_{\rm M} \times \mathscr{Z}_{\rm P}$ flavour symmetry [23],

$$\begin{aligned}
&\psi_{L_1}^q : (+,1,\omega_{14}^{\rm P-1}),\ \psi_{L_2}^q : (+,1,\omega_{14}),\ \psi_{L_3}^q : (+,1,\omega_{14}^2), \\
&u_R : (-,1,\omega_{14}^{\rm P-3}),\ c_R : (+,1,\omega_{14}^6),\ t_R : (+,1,\omega_{14}^4), \\
&d_R : (-,1,\omega_{14}^{12}),\ s_R : (+,1,\omega_{14}^{12}),\ b_R : (+,1,\omega_{14}^{12}), \\
&\psi_{L_1}^\ell : (+,\omega_4^3,\omega_{14}^{12}),\ \psi_{L_2}^\ell : (+,\omega_4^3,\omega_{14}^{10}),\ \psi_{L_3}^\ell : (+,\omega_4^3,\omega_{14}^6), \\
&e_R : (-,\omega_4^3,\omega_{14}^{10}),\ \mu_R : (+,\omega_4^3,\omega_{14}^{P-1}),\ \tau_R : (+,\omega_4^3,\omega_{14}), \\
&\chi_1 : (-,1,\omega_{14}^2),\ \chi_2 : (+,1,\omega_{14}^5),\ \chi_3 : (+,1,\omega_{14}^2), \\
&\chi_4 : (+,1,\omega_{14}^{13}),\ \chi_5 : (+,1,\omega_{14}^{11}),\ \chi_6 : (+,1,\omega_{14}^{10}),
\end{aligned} \tag{2.60}$$

where ω_4 is the fourth and ω_{14} is the fourteenth root of unity corresponding to the symmetries $\mathscr{Z}_4$ and $\mathscr{Z}_{14}$, respectively. Moreover, it turns out that we must have $\mathrm{N} = 2$ to recover the desired flavour structure of the charged fermions in the standard HVM. Also, $\mathrm{M} \geq 4$, and $\mathrm{P} \geq 14$ are needed.

We obtain the neutrino masses and mixing by adding three right-handed neutrinos ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$ to the SM through the following dimension-6 Lagrangian[4]

,

$$-\mathscr{L}^{\nu}_{\text{Yukawa}} = y^{\nu}_{ij} \bar{\psi}^{\ell}_{L_i} \tilde{\varphi} \nu_{f_R} \left[\frac{\chi_r \chi_7 (\text{or} \chi_r \chi_7^{\dagger})}{\Lambda_{\text{F}}^2} \right] + \text{H.c.}, \tag{2.61}$$

where χ_7 is a gauge singlet scalar field, which is a bound state of a strong and dark QCD-like dynamics of vector-like fermions. This is elaborated in the section 2.6.

The equation 2.61 places a severe constraint on the symmetry $\mathscr{Z}_{\text{P}}$ allowing only $\mathrm{P} = 14$, making it a magic number. We show the charge assignments of the fields for the normal ordering of neutrino masses as given in table 2.10 under the $\mathscr{Z}_2 \times \mathscr{Z}_4 \times \mathscr{Z}_{14}$ flavour symmetry. The flavour problem of the SM can be addressed in terms of the VEVs pattern $\langle\chi_4\rangle > \langle\chi_1\rangle$, $\langle\chi_2\rangle >> \langle\chi_5\rangle$, $\langle\chi_3\rangle >> \langle\chi_6\rangle$, $\langle\chi_3\rangle >> \langle\chi_2\rangle >> \langle\chi_1\rangle$, and $\langle\chi_6\rangle >> \langle\chi_5\rangle >> \langle\chi_4\rangle$.

Fields	$\mathscr{Z}_2$	$\mathscr{Z}_4$	$\mathscr{Z}_{14}$	Fields	$\mathscr{Z}_2$	$\mathscr{Z}_4$	$\mathscr{Z}_{14}$	Fields	$\mathscr{Z}_2$	$\mathscr{Z}_4$	$\mathscr{Z}_{14}$	Fields	$\mathscr{Z}_2$	$\mathscr{Z}_4$	$\mathscr{Z}_{14}$
u_R	-	1	ω_{14}^{11}	d_R, s_R, b_R	+	1	ω_{14}^{12}	$\psi^q_{L_3}$	+	1	ω_{14}^{2}	τ_R	+	ω_4^3	ω_{14}
c_R	+	1	ω_{14}^{6}	χ_4	+	1	ω_{14}^{13}	$\psi^\ell_{L_1}$	+	ω_4^3	ω_{14}^{12}	ν_{e_R}	+	ω_4	ω_{14}^{8}
t_R	+	1	ω_{14}^{4}	χ_5	+	1	ω_{14}^{11}	$\psi^\ell_{L_2}$	+	ω_4^3	ω_{14}^{10}	ν_{μ_R}	-	ω_4	ω_{14}^{3}
χ_1	-	1	ω_{14}^{2}	χ_6	+	1	ω_{14}^{10}	$\psi^\ell_{L_3}$	+	ω_4^3	ω_{14}^{6}	ν_{τ_R}	-	ω_4	ω_{14}^{3}
χ_2	+	1	ω_{14}^{5}	$\psi^q_{L_1}$	+	1	ω_{14}^{13}	e_R	-	ω_4^3	ω_{14}^{10}	χ_7	-	ω_4^2	ω_{14}^{8}
χ_3	+	1	ω_{14}^{2}	$\psi^q_{L_2}$	+	1	ω_{14}	μ_R	+	ω_4^3	ω_{14}^{13}	φ	+	1	1

Table 2.10 The charges of left- and right-handed fermions and scalar fields under $\mathscr{Z}_2$, $\mathscr{Z}_4$, and $\mathscr{Z}_{14}$ symmetries for the normal mass ordering. Here, ω is the 4th and 14th root of unity corresponding to the symmetries $\mathscr{Z}_4$ and $\mathscr{Z}_{14}$.

[4]Here, we have adopted the notation χ_r, where $r = 1, \ldots, 6$, for the flavon fields to distinguish the flavon subindex i in χ_i from the family indices i, j in equations 2.59 and 2.61.

The Lagrangian providing masses to charged fermions is now given as,

$$\begin{aligned}\mathscr{L}_f = \frac{1}{\Lambda}\Big[& y^u_{11}\bar{\Psi}^q_{L_1}\tilde{\varphi}u_R\chi_1 + y^u_{13}\bar{\Psi}^q_{L_1}\tilde{\varphi}t_R\chi_2^\dagger + y^u_{22}\bar{\Psi}^q_{L_2}\tilde{\varphi}c_R\chi_2^\dagger + y^u_{23}\bar{\Psi}^q_{L_2}\tilde{\varphi}t_R\chi_5 \\ &+ y^u_{32}\bar{\Psi}^q_{L_3}\tilde{\varphi}c_R\chi_6 + y^u_{33}\bar{\Psi}^q_{L_3}\tilde{\varphi}t_R\chi_3^\dagger + y^d_{11}\bar{\Psi}^q_{L_1}\varphi d_R\chi_4^\dagger + y^d_{12}\bar{\Psi}^q_{L_1}\varphi s_R\chi_4^\dagger \\ &+ y^d_{13}\bar{\Psi}^q_{L_1}\varphi b_R\chi_4^\dagger + y^d_{21}\bar{\Psi}^q_{L_2}\varphi d_R\chi_5^\dagger + y^d_{22}\bar{\Psi}^q_{L_2}\varphi s_R\chi_5^\dagger + y^d_{23}\bar{\Psi}^q_{L_2}\varphi b_R\chi_5^\dagger \\ &+ y^d_{31}\bar{\Psi}^q_{L_3}\varphi d_R\chi_6^\dagger + y^d_{32}\bar{\Psi}^q_{L_3}\varphi s_R\chi_6^\dagger + y^d_{33}\bar{\Psi}^q_{L_3}\varphi b_R\chi_6^\dagger + y^\ell_{11}\bar{\Psi}^\ell_{L_1}\varphi e_R\chi_1 \\ &+ y^\ell_{12}\bar{\Psi}^\ell_{L_1}\varphi \mu_R\chi_4 + y^\ell_{13}\bar{\Psi}^\ell_{L_1}\varphi \tau_R\chi_5 + y^\ell_{22}\bar{\Psi}^\ell_{L_2}\varphi \mu_R\chi_5 + y^\ell_{23}\bar{\Psi}^\ell_{L_2}\varphi \tau_R\chi_2^\dagger \\ &+ y^\ell_{33}\bar{\Psi}^\ell_{L_3}\varphi \tau_R\chi_2 + \text{H.c.}\Big]. \end{aligned} \tag{2.62}$$

The mass matrices of the charged fermions can be written as,

$$\mathscr{M}^u = \frac{v}{\sqrt{2}}\begin{pmatrix} y^u_{11}\varepsilon_1 & 0 & y^u_{13}\varepsilon_2 \\ 0 & y^u_{22}\varepsilon_2 & y^u_{23}\varepsilon_5 \\ 0 & y^u_{32}\varepsilon_6 & y^u_{33}\varepsilon_3 \end{pmatrix}, \quad \mathscr{M}^d = \frac{v}{\sqrt{2}}\begin{pmatrix} y^d_{11}\varepsilon_4 & y^d_{12}\varepsilon_4 & y^d_{13}\varepsilon_4 \\ y^d_{21}\varepsilon_5 & y^d_{22}\varepsilon_5 & y^d_{23}\varepsilon_5 \\ y^d_{31}\varepsilon_6 & y^d_{32}\varepsilon_6 & y^d_{33}\varepsilon_6 \end{pmatrix},$$
$$\mathscr{M}^\ell = \frac{v}{\sqrt{2}}\begin{pmatrix} y^\ell_{11}\varepsilon_1 & y^\ell_{12}\varepsilon_4 & y^\ell_{13}\varepsilon_5 \\ 0 & y^\ell_{22}\varepsilon_5 & y^\ell_{23}\varepsilon_2 \\ 0 & 0 & y^\ell_{33}\varepsilon_2 \end{pmatrix}. \tag{2.63}$$

where $\varepsilon_i = \dfrac{\langle\chi_i\rangle}{\Lambda}$ and $\varepsilon_i < 1$, The zero entries in the mass matrices correspond to terms that are not invariant with the charges assigned under $\mathscr{Z}_2 \times \mathscr{Z}_4 \times \mathscr{Z}_{14}$ flavour symmetry, as specified in table 2.10.

The Lagrangian generating neutrino masses in the normal ordering is,

$$\begin{aligned}\mathscr{L}_f = \frac{1}{\Lambda^2}[& y^{\nu}_{11}\bar{\Psi}^{\ell}_{L_1}\tilde{\varphi}\psi^{\nu}_{R_1}\chi^{\dagger}_1\chi^{\dagger}_7 + y^{\nu}_{12}\bar{\Psi}^{\ell}_{L_1}\tilde{\varphi}\psi^{\nu}_{R_2}\chi^{\dagger}_4\chi_7 + y^{\nu}_{13}\bar{\Psi}^{\ell}_{L_1}\tilde{\varphi}\psi^{\nu}_{R_3}\chi^{\dagger}_4\chi_7 \\ & + y^{\nu}_{22}\bar{\Psi}^{\ell}_{L_2}\tilde{\varphi}\psi^{\nu}_{R_2}\chi_4\chi_7 + y^{\nu}_{23}\bar{\Psi}^{\ell}_{L_2}\tilde{\varphi}\psi^{\nu}_{R_3}\chi_4\chi_7 + y^{\nu}_{32}\bar{\Psi}^{\ell}_{L_3}\tilde{\varphi}\psi^{\nu}_{R_2}\chi_5\chi_7 \\ & + y^{\nu}_{33}\bar{\Psi}^{\ell}_{L_3}\tilde{\varphi}\psi^{\nu}_{R_3}\chi_5\chi_7 + \text{H.c.}].\end{aligned} \tag{2.64}$$

The Dirac mass matrix for neutrinos is given by,

$$\mathscr{M}^{\nu} = \frac{v}{\sqrt{2}}\begin{pmatrix} y^{\nu}_{11}\varepsilon_1\varepsilon_7 & y^{\nu}_{12}\varepsilon_4\varepsilon_7 & y^{\nu}_{13}\varepsilon_4\varepsilon_7 \\ 0 & y^{\nu}_{22}\varepsilon_4\varepsilon_7 & y^{\nu}_{23}\varepsilon_4\varepsilon_7 \\ 0 & y^{\nu}_{32}\varepsilon_5\varepsilon_7 & y^{\nu}_{33}\varepsilon_5\varepsilon_7 \end{pmatrix}, \tag{2.65}$$

where $\varepsilon_7 = \frac{\langle\chi_7\rangle}{\Lambda_F} < 1$.

The diagonalization of the fermionic mass matrices is performed through the bi-unitary transformations given by,

$$U^{\dagger}_{u,d,\ell,\nu}\mathscr{M}^{u,d,\ell,\nu}V_{u,d,\ell,\nu} = \mathscr{M}^{u,d,\ell,\nu}_{\text{dia}} \tag{2.66}$$

The numerical form of the coupling matrices, and bi-unitary transformation matrices are given in the appendix.

The masses of charged fermions are,

$$\begin{aligned}
m_t &= |y^u_{33}|\,\varepsilon_3 v/\sqrt{2},\ m_c = \left|y^u_{22}\varepsilon_2 - \frac{y^u_{23}y^u_{32}\varepsilon_5\varepsilon_6}{y^u_{33}\varepsilon_3}\right| v/\sqrt{2},\ m_u = |y^u_{11}|\,\varepsilon_1 v/\sqrt{2},\\
m_b &\approx |y^d_{33}|\varepsilon_6 v/\sqrt{2}, m_s \approx \left|y^d_{22} - \frac{y^d_{23}y^d_{32}}{y^d_{33}}\right| \varepsilon_5 v/\sqrt{2},\\
m_d &\approx \left|y^d_{11} - \frac{y^d_{12}y^d_{21}}{y^d_{22} - \frac{y^d_{23}y^d_{32}}{y^d_{33}}} - \frac{y^d_{13}(y^d_{31}y^d_{22} - y^d_{21}y^d_{32}) - y^d_{31}y^d_{12}y^d_{23}}{(y^d_{22} - \frac{y^d_{23}y^d_{32}}{y^d_{33}})y^d_{33}}\right| \varepsilon_4 v/\sqrt{2},\\
m_\tau &\approx |y^\ell_{33}|\varepsilon_2 v/\sqrt{2},\ m_\mu \approx |y^\ell_{22}|\varepsilon_5 v/\sqrt{2},\ m_e = |y^\ell_{11}|\,\varepsilon_1 v/\sqrt{2}.
\end{aligned} \tag{2.67}$$

Assuming almost all $|y^{u,d}_{ij}| = 1$, the quark mixing angles turn out to be,

$$\begin{aligned}
\sin\theta_{12} &\simeq \frac{\varepsilon_4}{\varepsilon_5},\\
\sin\theta_{23} &\simeq \frac{\varepsilon_5}{\varepsilon_6},\\
\sin\theta_{13} &\simeq \frac{\varepsilon_4}{\varepsilon_6} - \left|\frac{y^u_{13}}{y^u_{33}}\right| \frac{\varepsilon_2}{\varepsilon_3}.
\end{aligned} \tag{2.68}$$

This results in the prediction of the quark mixing angle θ_{13} given by,

$$\sin\theta_{13} \simeq \sin\theta_{12}\sin\theta_{23} - \left|\frac{y^u_{13}}{y^u_{33}}\right| \frac{\varepsilon_2}{\varepsilon_3}. \tag{2.69}$$

The neutrino masses can be written as,

$$\begin{aligned}
m_3 &\approx |y^\nu_{33}|\,\varepsilon_5\varepsilon_7 \frac{v}{\sqrt{2}},\\
m_2 &\approx \left|y^\nu_{22} - \frac{y^\nu_{23}y^\nu_{32}}{y^\nu_{33}}\right| \varepsilon_4\varepsilon_7 \frac{v}{\sqrt{2}},\\
m_1 &\approx |y^\nu_{11}|\,\varepsilon_1\varepsilon_7 \frac{v}{\sqrt{2}}.
\end{aligned} \tag{2.70}$$

The masses of neutrinos are $\{m_3, m_2, m_1\} = \{5.05 \times 10^{-2}, 8.67 \times 10^{-3}, 2.67 \times 10^{-4}\}$ eV for the numerical values of y^{ν}_{ij} given in the appendix, and for numerical values of ε_i, and ε_7 which are,

$$\begin{aligned} &\varepsilon_1 = 3.16 \times 10^{-6},\ \varepsilon_2 = 0.0031,\ \varepsilon_3 = 0.87, \varepsilon_4 = 0.000061, \\ &\varepsilon_5 = 0.000270,\ \varepsilon_6 = 0.0054, \varepsilon_7 = 7.18 \times 10^{-10}. \end{aligned} \tag{2.71}$$

The main achievement of the SHVM is the neutrino mixing angles. Assuming all the couplings $|y^{\nu}_{ij}| = 1$, they can be written as [23],

$$\begin{aligned} \sin\theta^{\ell}_{12} &\simeq 1 - 2\sin\theta_{12}, \\ \sin\theta^{\ell}_{23} &\simeq 1 - \sin\theta_{12}, \\ \sin\theta^{\ell}_{13} &\simeq \sin\theta_{12} - \frac{m_s}{m_c}. \end{aligned} \tag{2.72}$$

where $m_s/m_c = \varepsilon_5/\varepsilon_2$.

This results in very precise predictions of the leptonic mixing angles [23],

$$\sin\theta^{\ell}_{12} = 0.55 \pm 0.00134,\ \sin\theta^{\ell}_{23} = 0.775 \pm 0.00067,\ \sin\theta^{\ell}_{13} = 0.1413 - 0.1509, \tag{2.73}$$

which may be probed by next generation neutrino experiments such as as DUNE, Hyper-Kamiokande, and JUNO [146].

2.5 Scalar potential of the standard HVM

The scalar potential of the SHVM can be written as, [5]

$$\begin{aligned} V \;=\; & \mu^2\varphi^\dagger\varphi+\lambda(\varphi^\dagger\varphi)^2+\mu^2_{\chi_1}|\chi_1|^2+\mu^2_{\chi_2}|\chi_2|^2+\mu^2_{\chi_3}|\chi_3|^2+\mu^2_{\chi_4}|\chi_4|^2 \qquad (2.74)\\ & +\mu^2_{\chi_5}|\chi_5|^2+\mu^2_{\chi_6}|\chi_6|^2+\mu^2_{\chi_7}|\chi_7|^2+\lambda_{\chi_1}|\chi_1|^4+\lambda_{\chi_2}|\chi_2|^4+\lambda_{\chi_3}|\chi_3|^4+\lambda_{\chi_4}|\chi_4|^4\\ & +\lambda_{\chi_5}|\chi_5|^4+\lambda_{\chi_6}|\chi_6|^4+\lambda_{\chi_7}|\chi_7|^4+\lambda_{\varphi\chi}\varphi^\dagger\varphi\chi_i^\dagger\chi_j+\lambda_{\chi_{12}}|\chi_1|^2|\chi_2|^2+\lambda_{\chi_{13}}|\chi_1|^2|\chi_3|^2\\ & +\lambda_{\chi_{14}}|\chi_1|^2|\chi_4|^2+\lambda_{\chi_{15}}|\chi_1|^2|\chi_5|^2+\lambda_{\chi_{16}}|\chi_1|^2|\chi_6|^2+\lambda_{\chi_{17}}|\chi_1|^2|\chi_7|^2\\ & +\lambda_{\chi_{23}}|\chi_2|^2|\chi_3|^2+\lambda_{\chi_{24}}|\chi_2|^2|\chi_4|^2+\lambda_{\chi_{25}}|\chi_2|^2|\chi_5|^2+\lambda_{\chi_{26}}|\chi_2|^2|\chi_6|^2\\ & +\lambda_{\chi_{27}}|\chi_2|^2|\chi_7|^2+\lambda_{\chi_{34}}|\chi_2|^3|\chi_3|^4+\lambda_{\chi_{35}}|\chi_2|^3|\chi_5|^2+\lambda_{\chi_{36}}|\chi_3|^2|\chi_6|^2\\ & +\lambda_{\chi_{37}}|\chi_3|^2|\chi_7|^2+\lambda_{\chi_{45}}|\chi_4|^2|\chi_5|^2+\lambda_{\chi_{46}}|\chi_4|^2|\chi_6|^2+\lambda_{\chi_{47}}|\chi_4|^2|\chi_7|^2\\ & +\lambda_{\chi_{56}}|\chi_5|^2|\chi_6|^2+\lambda_{\chi_{57}}|\chi_5|^2|\chi_7|^2+\lambda_{\chi_{67}}|\chi_6|^2|\chi_7|^2+\lambda_{1122}\chi_1^2\chi_2^2+\lambda_{1133}\chi_1^2\chi_3^{\dagger 2}\\ & +\lambda_{2233}\chi_2^2\chi_3^2+\lambda_{2224}\chi_2^3\chi_4+\lambda_{4436}\chi_4^2\chi_3^\dagger\chi_6^\dagger+\lambda_{2555}\chi_2\chi_5^{\dagger 3}+\sigma_{16}\chi_1^2\chi_6+\sigma_{26}\chi_2^2\chi_6^\dagger\\ & +\sigma_{36}\chi_3^2\chi_6+\sigma_{43}\chi_4^2\chi_3+\sigma_{73}\chi_7^2\chi_3^\dagger+\sigma_{345}\chi_3\chi_4^\dagger\chi_5+\sigma_{235}\chi_2\chi_3^\dagger\chi_5+\sigma_{456}\chi_4\chi_5\chi_6^\dagger\\ & +\sigma_{246}\chi_2\chi_4\chi_6+\sigma_{345}\chi_3\chi_4^\dagger\chi_5+\text{H.c.}. \end{aligned}$$

We can parametrize the scalar fields as,

$$\chi_i(x)=\frac{v_i+s_i(x)+i\,a_i(x)}{\sqrt{2}},\ \varphi=\begin{pmatrix} G^+ \\ \frac{v+h+iG^0}{\sqrt{2}} \end{pmatrix}. \qquad (2.75)$$

[5] For simplification, we impose an extra $\mathscr{Z}_2'$ symmetry on the right-handed fermions such that right-handed fermions transform as $\psi_R : -$, the singlet scalar fields χ as $\chi_i : -$ where $i = 1-6$, and the field χ_7 as $\chi_7 : +$ under this new $\mathscr{Z}_2'$ symmetry. This will forbid all cubic terms in the scalar potential without affecting the flavour structure of the model.

2.5.1 Emergence of a multiple ALPs scenario

We use the following potential for providing masses to pseudoscalars [15],

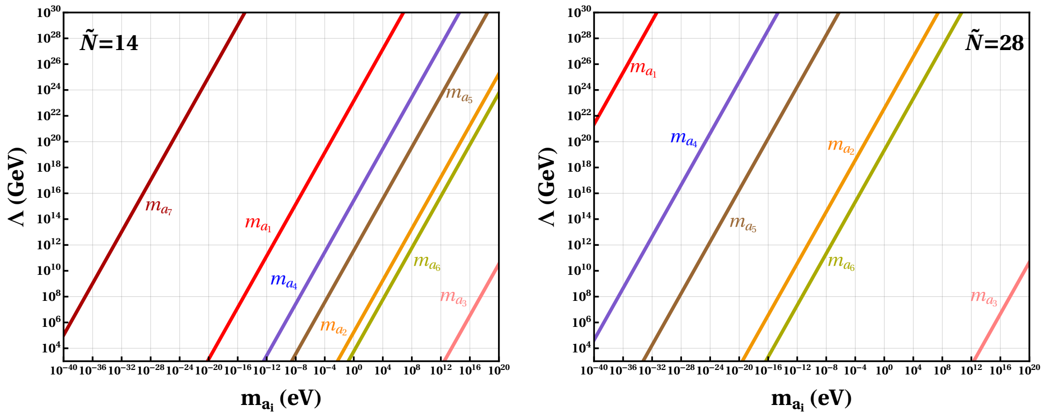
$$V_\lambda = -\lambda_{i,1}\frac{\chi_i^{\tilde{N}}}{\Lambda^{\tilde{N}-4}} + \text{H.c.}. \tag{2.76}$$

The potential V_λ can be written as,

$$V_\lambda = -|\lambda_{i,1}|\varepsilon_i^{\tilde{N}}\Lambda^4 \cos\left(\tilde{N}\frac{a_i}{v_i} + \theta_i\right). \tag{2.77}$$

The masses of pseudoscalars are,

$$m_{a_i}^2 = \frac{1}{4}\tilde{N}^2\varepsilon_i^{\tilde{N}-2}|\lambda_{i,1}|\Lambda^2. \tag{2.78}$$



(a) (b)

Fig. 2.2 Variation of the masses of the pseudoscalars m_{a_i} with respect to the scale Λ.

We show the masses of pseudoscalars with respect to the variation of the scale Λ in figure 2.2 for $\tilde{N} = 14$ and $\tilde{N} = 28$. We notice that except the pseudoscalar a_3, which is a heavy state, all other pseudoscalars can act like ALPs.

2.5.2 Masses of scalars

For the phenomenological analysis, we use a simple choice $\lambda_{\varphi\chi} = 0$ and $\lambda_{\chi_i} = \lambda_\chi$. The mass matrix square of the scalar particles can numerically be diagonalized by writing $v_i = \sqrt{2}\varepsilon_i \Lambda$, where ε_i are given in equation 2.71. Thus, we can write,

$$U^T \mathscr{M}_s^2 U = \mathscr{M}_{dia}^2, \tag{2.79}$$

where $\mathscr{M}_s^2$ is the mass matrix square of the scalars, $\mathscr{M}_{dia}$ is the diagonalized scalar mass matrix, and U is an orthogonal matrix. These matrices are given in appendix.

Now, the physical scalars are related to the un-physical scalars through the transformation,

$$\begin{pmatrix} s_7 \\ s_1 \\ s_4 \\ s_5 \\ s_2 \\ s_6 \\ s_3 \end{pmatrix} = U^T \begin{pmatrix} h_7 \\ h_1 \\ h_4 \\ h_5 \\ h_2 \\ h_6 \\ h_3 \end{pmatrix}, \tag{2.80}$$

The masses of scalars are given by,

$$m_{h_7}^2 = 4.72 \times 10^{-18} \lambda_\chi \Lambda^2,\ m_{h_1}^2 = 9.33 \times 10^{-11} \lambda_\chi \Lambda^2,\ m_{h_4}^2 = 3.57 \times 10^{-8} \lambda_\chi \Lambda^2,$$
$$m_{h_5}^2 = 7.3 \times 10^{-7} \lambda_\chi \Lambda^2,\ m_{h_2}^2 = 9.45 \times 10^{-5} \lambda_\chi \Lambda^2,\ m_{h_6}^2 = 3.67 \times 10^{-4} \lambda_\chi \Lambda^2,\ m_{h_3}^2 = 12 \lambda_\chi \Lambda^2. \tag{2.81}$$

2.5.3 Couplings of scalars and pseudoscalars to fermions

We can write the singlet fields χ_i as,

$$\frac{\chi_i}{\Lambda} = \varepsilon_i[1 + \frac{s_i + ia_i}{v_i}]. \tag{2.82}$$

The equation 2.59 can now be written in the following form,

$$y^f_{ij}\frac{\chi_i}{\Lambda}\varphi f\bar{f} \cong y^f_{ij}\varepsilon_i\frac{v}{\sqrt{2}}\left[1 + \frac{(s_i + ia_i)}{v_i} + \frac{h}{v}\right]f\bar{f}, \tag{2.83}$$

where $f = u, d, \ell$.

The couplings of Goldstone bosons a_i with fermions $f\bar{f}$ are now given by,

$$y^u_{a_i f_{iL} f_{jR}} \equiv y^u_{a_i ij} = i\frac{v}{\sqrt{2}}\begin{pmatrix} y^u_{11}\varepsilon_1/v_1 & 0 & y^u_{13}\varepsilon_2/v_2 \\ 0 & y^u_{22}\varepsilon_2/v_2 & y^u_{23}\varepsilon_5/v_5 \\ 0 & y^u_{32}\varepsilon_6/v_6 & y^u_{33}\varepsilon_3/v_3 \end{pmatrix}, \tag{2.84}$$

$$y^d_{a_i ij} = i\frac{v}{\sqrt{2}}\begin{pmatrix} y^d_{11}\varepsilon_4/v_4 & y^d_{12}\varepsilon_4/v_4 & y^d_{13}\varepsilon_4/v_4 \\ y^d_{21}\varepsilon_5/v_5 & y^d_{22}\varepsilon_5/v_5 & y^d_{23}\varepsilon_5/v_5 \\ y^d_{31}\varepsilon_6/v_6 & y^d_{32}\varepsilon_6/v_6 & y^d_{33}\varepsilon_6/v_6 \end{pmatrix},$$

$$y^\ell_{a_i ij} = i\frac{v}{\sqrt{2}}\begin{pmatrix} y^\ell_{11}\varepsilon_1/v_1 & y^\ell_{12}\varepsilon_4/v_4 & y^\ell_{13}\varepsilon_5/v_5 \\ 0 & y^\ell_{22}\varepsilon_5/v_5 & y^\ell_{23}\varepsilon_2/v_2 \\ 0 & 0 & y^\ell_{33}\varepsilon_2/v_2 \end{pmatrix}. \tag{2.85}$$

The numerical values of these dimensionless couplings $y^{u,d,\ell}_{a_i ij}$ are given in the appendix. We notice that due to the appearance of the factor $1/v_i$ in the coupling matrices, the FCNC interactions of scalars and pseudoscalars with fermions are unavoidable. The coupling of scalars to fermions are determined by using equations 2.80-2.83, and bi-unitary transformation matrices used for diagonalizing the fermionic mass matrices.

Now using $v_i = \sqrt{2}\varepsilon_i \Lambda$, the coupling matrices in equations 2.84, 2.85, can be written in a simpler form,

$$y^u_{a_i f_{iL} f_{jR}} \equiv y^u_{a_i ij} = i\frac{v}{2\Lambda}\begin{pmatrix} y^u_{11} & 0 & y^u_{13} \\ 0 & y^u_{22} & y^u_{23} \\ 0 & y^u_{32} & y^u_{33} \end{pmatrix}, \quad y^d_{a_i ij} = i\frac{v}{2\Lambda}\begin{pmatrix} y^d_{11} & y^d_{12} & y^d_{13} \\ y^d_{21} & y^d_{22} & y^d_{23} \\ y^d_{31} & y^d_{32} & y^d_{33} \end{pmatrix},$$
$$y^\ell_{a_i ij} = i\frac{v}{2\Lambda}\begin{pmatrix} y^\ell_{11} & y^\ell_{12} & y^\ell_{13} \\ 0 & y^\ell_{22} & y^\ell_{23} \\ 0 & 0 & y^\ell_{33} \end{pmatrix}. \tag{2.86}$$

2.6 The dark-technicolour paradigm

In this section, we discuss a common UV origin of the models based on the $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M}$ flavour symmetry, and the VEVs hierarchy that is based on a generic $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M} \times \mathscr{Z}_\mathrm{P}$ flavour symmetry. For this purpose, we use the technicolour scenario proposed in reference [24]. In this scenario, we have the $\mathscr{G} \equiv SU(\mathrm{N_{TC}}) \times \mathrm{SU}(\mathrm{N_{DTC}}) \times \mathrm{SU}(\mathrm{N_F})$ symmetry, where TC stands for technicolour, DTC for dark-technicolour, and F represents strong dynamics of vector-like fermions.

In the TC dynamics, there are $\mathrm{K_{TC}}$ flavours behaving under the $SU(3)_c \times SU(2)_L \times U(1)_Y \times \mathscr{G}$ as [24],

$$T^i_q \equiv \begin{pmatrix} T \\ B \end{pmatrix}_L : (1,2,0,\mathrm{N_{TC}},1,1),\ \mathrm{T^i_R} : (1,1,1,\mathrm{N_{TC}},1,1), \mathrm{B^i_R} : (1,1,-1,\mathrm{N_{TC}},1,1), \tag{2.87}$$

where $i = 1,2,3\cdots$, and the electric charge of T is $+\frac{1}{2}$ and $-\frac{1}{2}$ for B, and the first three quantum numbers refer to the SM.

The dark TC symmetry $SU(\mathrm{N_{DTC}})$ has $\mathrm{K_{DTC}}$ flavours transforming under $SU(3)_c \times SU(2)_L \times U(1)_Y \times \mathscr{G}$ as [24],

$$\mathscr{D}_q^i \equiv \mathscr{C}_{L,R}^i : (1,1,1,1,\mathrm{N_{DTC}},1),\ \mathscr{S}_{L,R}^i : (1,1,-1,1,\mathrm{N_{DTC}},1), \tag{2.88}$$

where $i = 1,2,3\cdots$, and electric charges are $+\frac{1}{2}$ for $\mathscr{C}$ and $-\frac{1}{2}$ for $\mathscr{S}$.

In a similar manner, there are $\mathrm{K_F}$ fermionic flavours corresponding to the $SU(N_\mathrm{F})$ symmetry transforming under $SU(3)_c \times SU(2)_L \times U(1)_Y \times \mathscr{G}$ as [24],

$$\begin{aligned} F_{L,R} &\equiv U_{L,R}^i \equiv (3,1,\frac{4}{3},1,1,\mathrm{N_F}), D_{L,R}^i \equiv (3,1,-\frac{2}{3},1,1,\mathrm{N_F}), \\ N_{L,R}^i &\equiv (1,1,0,1,1,\mathrm{N_F}), E_{L,R}^i \equiv (1,1,-2,1,1,\mathrm{N_F}), \end{aligned} \tag{2.89}$$

where $i = 1,2,3\cdots$.

2.6.1 UV completion of the SHVM

Now we assume the existence of an extended technicolour dynamics (ETC) mediating the TC fermions, the left-handed SM fermions, and the F_R fermions, and an extended-DTC (EDTC) symmetry containing the DTC fermions, the right-handed SM singlet fermions, and the F_L fermions. We notice that the $SU(\mathrm{N})_\mathrm{F}$ symmetry acts like a connecting bridge between the TC and DTC dynamics and results in a suppression of the mixing between the TC and the DTC dynamics by the factor $1/\Lambda$. This may explain the SM-like behaviour of the discovered Higgs.

There are three axial $U(1)_\mathrm{A}$ symmetries, namely, $U(1)_\mathrm{A}^\mathrm{TC}$, $U(1)_\mathrm{A}^\mathrm{DTC}$ and $U(1)_\mathrm{A}^\mathrm{F}$ in this model. In general, an axial symmetry $U(1)_A$ in a QCD-like gauge theory is broken by instantons, resulting in a $2K$-fermion operator with a non-vanishing VEV and $2K$ conserved quantum number [295]. Thus we have,

$$U(1)_A \to \mathscr{Z}_{2K}, \tag{2.90}$$

where K shows the massless flavours of the gauge dynamics in the N-dimensional representation of the gauge group $SU(\mathrm{N})$. This provides the existence of a generic $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M} \times \mathscr{Z}_\mathrm{P}$ flavour symmetry, where $\mathrm{N} = 2\mathrm{K}_\mathrm{TC}$, $\mathrm{M} = 2\mathrm{K}_\mathrm{DTC}$, and $\mathrm{P} = 2\mathrm{K}_\mathrm{F}$. After the breaking, we have certain conserved axial charges modulo $2K$. For achieving the SHVM, as discussed in reference [24], we can use multi-fermion chiral condensates to form the singlet scalar fields χ_r. These condensates or composite operators contain global axial charge X_{DTC}, which is conserved modulo $2K$ [295].

The masses of charged fermions can be realized through the interactions shown in the upper part of figure 2.3. In the lower part of figure 2.3, we show the formation of on-shell TC chiral condensate $\langle\varphi\rangle$ and on-shell DTC multifermion chiral condensates $\langle\chi_r\rangle$ responsible for providing masses to the charged SM fermions through the $\mathscr{Z}_\mathrm{N} \times \mathscr{Z}_\mathrm{M} \times \mathscr{Z}_\mathrm{P}$ flavour symmetry.

We notice that the electroweak scale can be generated by a strong conformal dynamics as recently shown in reference [296]. Therefore, we choose the $SU(\mathrm{N}_\mathrm{TC})$, which produces the electroweak symmetry breaking via the Higgs mechanism, to be a conformal strong dynamics. The other two dynamics, the DTC symmetry $SU(\mathrm{N}_\mathrm{DTC})$ and the dark-QCD $SU(\mathrm{N}_\mathrm{F})$ are assumed to be QCD-like theories.

Now, a multi-fermion condensate can be written as [297],

$$\langle(\bar{\psi}_L \psi_R)^n\rangle \sim \big(\Lambda \exp(k\Delta\chi)\big)^{3n}, \tag{2.91}$$

where $\Delta\chi$ is the chirality of an operator, k stands for a constant, and Λ denotes the scale of the underlying gauge dynamics.

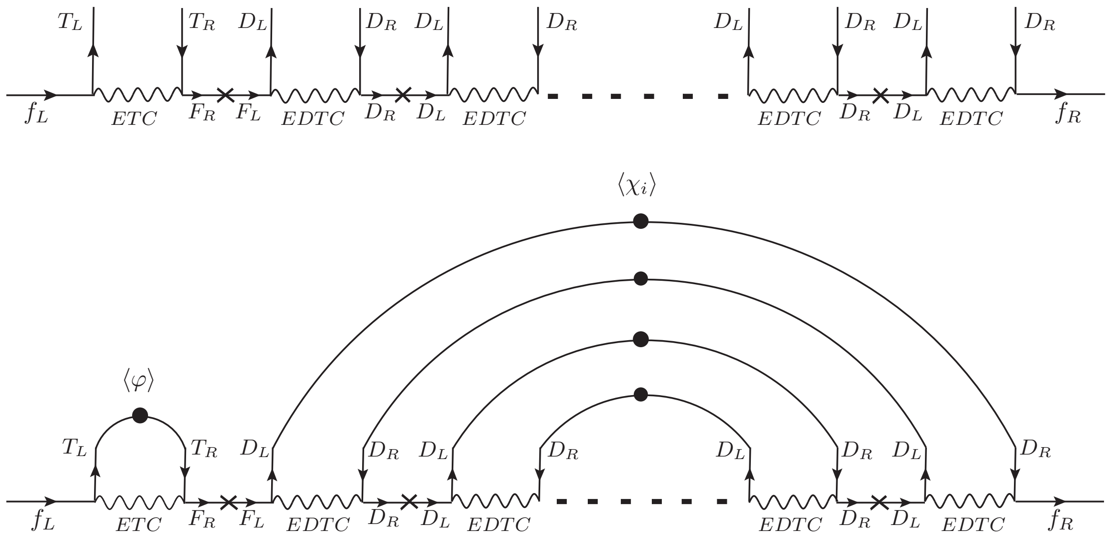


Fig. 2.3 The Feynman diagrams for the masses of the quarks and charged leptons in the dark-technicolour paradigm. On the top, the generic interactions of the SM, TC, F and DTC fermions are shown. In the bottom, we see the formations of a TC on shell chiral condensates, $\langle\varphi\rangle$ (circular blob), and a generic on shell multifermion chiral condensates $\langle\chi_r\rangle$(collection of circular blobs), and the resulting mass of the SM fermions.

For a large anomalous dimension, the condensates are given by [298],

$$\langle \bar{T}T\rangle_{\Lambda_{\rm ETC}} \approx -\frac{N_{TC}}{2\pi^2\gamma_m}(\Lambda_{TC})^{3-\gamma_m}(\Lambda_{ETC})^{\gamma_m}. \tag{2.92}$$

This condensate will be used for the conformal $SU(\mathrm{N_{TC}})$ symmetry, which provides the electroweak symmetry breaking.

For the DTC symmetry $SU(\mathrm{N_{DTC}})$ and for the dark-QCD $SU(\mathrm{N_F})$, the condensates are given by [298],

$$\begin{aligned}\langle \bar{D}D\rangle_{\Lambda_{\rm DETC}} &\approx -\frac{N_{DTC}}{4\pi^2}\Lambda^3_{DTC}\exp(k\Delta\chi), \\ \langle \bar{F}F\rangle_{\Lambda_{\rm GUT}} &\approx -\frac{N_F}{4\pi^2}\Lambda^3\exp(k_F\Delta\chi).\end{aligned} \tag{2.93}$$

The mass-matrices in equation 2.63 of the charged fermions can be approximately generated by the operators,

$$\mathscr{M}_{\mathscr{U},\mathscr{D},\ell} \quad = \quad y^f_{ij}\left[-\frac{g^2_{\rm ETC}}{\Lambda^2_{\rm ETC}}\langle\bar{T}T\rangle_{\Lambda_{\rm ETC}}\right]\frac{1}{\Lambda}\left[-\frac{g^{2n}_{\rm DETC}}{\Lambda^{3n-1}_{\rm DETC}}\left(\langle\bar{D}D\rangle_{\Lambda_{\rm DETC}}\right)^n\right]. \tag{2.94}$$

where $n = 1,2,3\cdots$ and $f = u,d,\ell$. We can further write the mass matrices in the following form using equations 2.91-2.93,

$$\mathscr{M}_{\mathscr{U},\mathscr{D},\ell} \quad = \quad y^f_{ij}\frac{N_{TC}}{2\pi^2}\frac{\Lambda^2_{\rm TC}}{\Lambda_{\rm ETC}}\frac{1}{\Lambda}\left[\frac{N_{DTC}}{4\pi^2}\right]^{n_i}\frac{\Lambda^{n_i+1}_{\rm DTC}}{\Lambda^{n_i}_{\rm EDTC}}\left[\exp(n_ik)\right]^{n_i/2}, \tag{2.95}$$

where we have assumed $\gamma_m = 1$, $g_{\rm ETC} = 1$, $g_{\rm DETC} = 1$, and $n_i = 2,4,6,\cdots 2n$ shows the number of fermions in a multi-fermion chiral condensate that acts like the VEV $\langle\chi_r\rangle$ [24], and $\Lambda_{\rm TC}$, $\Lambda_{\rm DTC}$ and Λ stand for the scale of the TC, DTC, and F dynamics, respectively.

Using equation 2.91, we observe the following from equation 2.95,

$$\varepsilon_r \propto \frac{1}{\Lambda}\left[\frac{N_{DTC}}{4\pi^2}\right]^{n_i}\frac{\Lambda^{n_i+1}_{\rm DTC}}{\Lambda^{n_i}_{\rm EDTC}}\left[\exp(n_ik)\right]^{n_i/2}. \tag{2.96}$$

Thus, we can construct the masses of charged fermions given in equation 2.67 in terms of equation 2.96, and at the leading order the mass of a charged fermion is given by,

$$m_f \approx |y^f_{11}|\frac{N_{TC}}{2\pi^2}\frac{\Lambda^2_{\rm TC}}{\Lambda_{\rm ETC}}\frac{1}{\Lambda}\left[\frac{N_{DTC}}{4\pi^2}\right]^{n_i}\frac{\Lambda^{n_i+1}_{\rm DTC}}{\Lambda^{n_i}_{\rm EDTC}}\left[\exp(n_ik)\right]^{n_i/2}. \tag{2.97}$$

The neutrino masses and mixing are achieved by assuming that the ETC and EDTC symmetries eventually can be accommodated in a GUT theory providing the necessary dimension-6 operators responsible for neutrino masses given in equation 2.61. We show the required interactions in the upper part of figure 2.4 mediated by the GUT gauge bosons between the F_L and F_R fermions. The role of the VEV $\langle\chi_7\rangle$ is played by the chiral condensate $\langle\bar{F}_LF_R\rangle$. The formation of the dimension-6 operators responsible for neutrino masses is shown in the lower part of figure 2.4, where the $\langle\chi_7\rangle$ denotes the on-shell chiral condensate (circular blob).

The neutrino mass matrix given in equation 2.65 is recovered as,

$$\mathscr{M}_{\mathscr{N}} = y^{\nu}_{ij}\frac{N_{TC}}{2\pi^2}\frac{\Lambda^2_{\text{TC}}}{\Lambda_{\text{ETC}}}\frac{1}{\Lambda}\left[\frac{N_{DTC}}{4\pi^2}\right]^{n_i}\frac{\Lambda^{n_i+1}_{\text{DTC}}}{\Lambda^{n_i}_{\text{EDTC}}}\left[\exp(n_i k)\right]^{n_i/2}\frac{1}{\Lambda}\frac{N_F}{4\pi^2}\frac{\Lambda^3}{\Lambda^2_{\text{GUT}}}\exp(2k_F), \quad (2.98)$$

where,

$$\varepsilon_7 \propto \frac{1}{\Lambda}\frac{\Lambda^3}{\Lambda^2_{\text{GUT}}}\exp(2k_F). \quad (2.99)$$

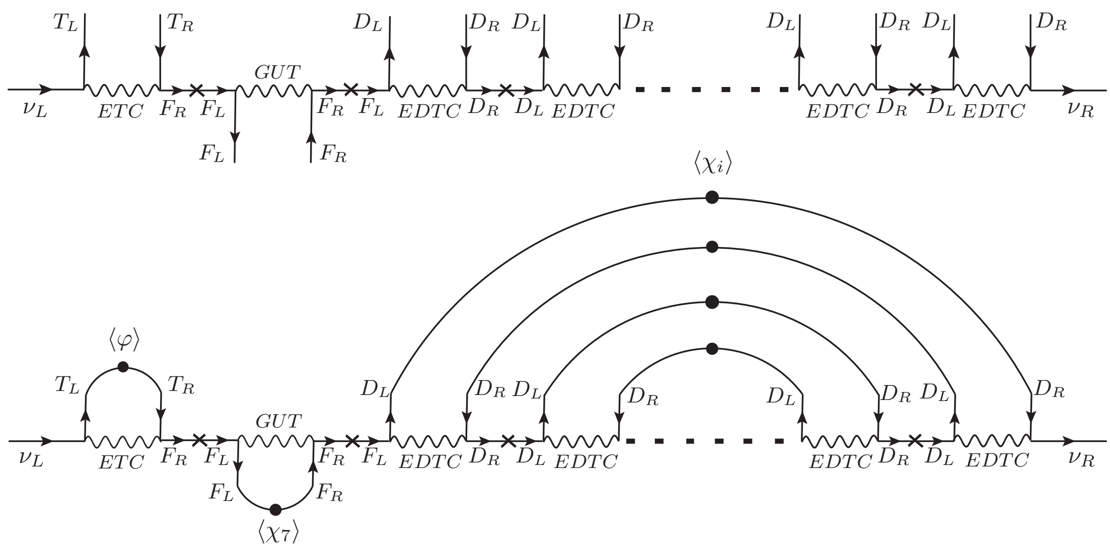


Fig. 2.4 The Feynman diagrams for the masses of neutrinos in the dark-technicolour paradigm. On the top, there are generic interactions involving the SM, TC, F and DTC gauge sectors mediated by ETC, EDTC and GUT gauge bosons. In the bottom, we show the generic Feynman diagram after the formation of the fermionic condensates.

The masses of neutrinos at the leading order are given by,

$$m_{\nu} = |y^{\nu}_{11}|\frac{N_{TC}}{2\pi^2}\frac{\Lambda^2_{\text{TC}}}{\Lambda_{\text{ETC}}}\frac{1}{\Lambda}\left[\frac{N_{DTC}}{4\pi^2}\right]^{n_i}\frac{\Lambda^{n_i+1}_{\text{DTC}}}{\Lambda^{n_i}_{\text{EDTC}}}\left[\exp(n_i k)\right]^{n_i/2}\frac{1}{\Lambda}\frac{N_F}{4\pi^2}\frac{\Lambda^3}{\Lambda^2_{\text{GUT}}}\exp(2k_F). \quad (2.100)$$

2.6.2 UV completion of the FN mechanism

For achieving the UV completion of the FN mechanism in the dark-technicolour paradigm, we need to assume that the SM, TC, and DTC dynamics are accommodated in a common extended technicolour symmetry. This condition is not necessary for the UV completion of the SHVM, and provides a crucial difference between the SHVM and the FN mechanism in the dark-technicolour paradigm. This leads to the required interactions for the charged fermions masses in the FN mechanism, which are shown in the upper part of figure 2.5. The mass generation through the FN mechanism is shown in the lower part of figure 2.5, where $\langle\varphi\rangle$ and $\langle\chi\rangle$ are the on-shell chiral condensate acting like the VEVs of the Higgs and flavon fields.

Now, the masses of the SM fermions in the FN mechanism originate as,

$$m_f \approx |y_{ii}^f| \frac{N_{TC}}{2\pi^2} \frac{\Lambda_{\rm TC}^2}{\Lambda_{\rm ETC}} \left(\frac{1}{\Lambda} \frac{N_{DTC}}{4\pi^2} \frac{\Lambda_{\rm DTC}^3}{\Lambda_{\rm ETC}^2} \exp(2k) \right)^{n_{ij}^f}, \tag{2.101}$$

where $f = u, d$, and the order parameter ε can be identified as,

$$\varepsilon \propto \frac{1}{\Lambda} \frac{N_{DTC}}{4\pi^2} \frac{\Lambda_{\rm DTC}^3}{\Lambda_{\rm ETC}^2} \exp(2k), \tag{2.102}$$

and the SM Higgs VEV can be written as,

$$\langle\varphi\rangle \propto \frac{N_{TC}}{2\pi^2} \frac{\Lambda_{\rm TC}^2}{\Lambda_{\rm ETC}}. \tag{2.103}$$

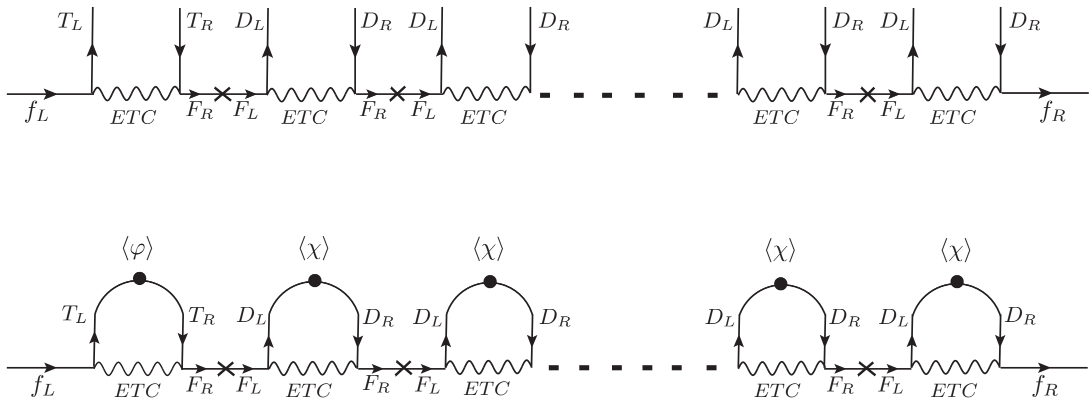


Fig. 2.5 Feynman diagrams for the masses of the quarks and charged leptons in the dark-technicolour paradigm. On the top, the generic interactions of the SM, TC, F and DTC fermions are shown. In the bottom, we see the formations of the condensate (circular blob), and the resulting mass of the SM fermions.

In the sections 2.2 and 2.4 of this chapter, we discussed the theoretical setup of the different realizations of the $\mathscr{Z}_N \times \mathscr{Z}_M$ flavour symmetry and the standard hierarchical VEVs model (SHVM) based on a generic $\mathscr{Z}_N \times \mathscr{Z}_{\mathrm{M}} \times \mathscr{Z}_P$ flavour symmetry, that are two unique and unconventional framework that can effectively address the flavour problem of the SM. We shall now discuss the phenomenological investigation of both of these frameworks in the subsequent chapters.