Chapter 3

Simple waves for anti-van der Waals modified Chaplygin gas in 2D magnetohydrodynamics

3.1 Introduction

This chapter focuses on the study of simple waves in a two-dimensional MHD system involving anti-van der Waals-modified Chaplygin gas (AWMCG). The AWMCG model, which incorporates both real and exotic gas behaviours, has significant applications in fields such as aerodynamics and cosmology, especially in modelling dark energy and matter interactions due to its unique property of exhibiting negative pressure. This characteristic makes AWMCG particularly interesting for studying wave propagation under non-ideal gas conditions.

We apply the Sufficient Condition of the characteristic decomposition method (CDM), developed by Hu and Sheng [27, 28], which simplifies the analysis of hyperbolic systems. This method allows us to establish the existence of simple waves in both steady and pseudo-steady MHD systems, even when the system is non-reducible. By employing CDM, we show that the physical variables remain constant along characteristic curves, confirming the presence of simple waves. These findings extend classical wave theories to more complex systems involving AWMCG dynamics and magnetohydrodynamic flows, enhancing their applicability to a broader range of physical and astrophysical phenomena. In this chapter, we employ CDM as a sufficient condition developed by Hu and Sheng [27, 28]. This approach simplifies the determination of the existence of a simple wave, making the numerical implementation straightforward compared with more complex methods that require extensive calculations, as used in [32, 33, 102]. This allows for efficient and effective analysis applicable to various gaseous models [24, 25, 38, 40, 92, 100].

Tsein [103] and von Karman [104] used pure Chaplygin gas and applied mathematical approximations to calculate the lift force on an aeroplane wing. Benaoum [105] introduced the concept of a modified Chaplygin gas to describe single-phase flow. Shah and Singh [106] studied the behaviour of steepened waves in two-phase Chaplygin flows using a source term. Recently, Chaudhary et al. [107] considered an equation of state for AWMCG [108], which is written as follows:

$$P(\rho) = \frac{\mathcal{A}(1-a\rho)^{\gamma}}{\rho^{\gamma}} - \frac{\mathcal{B}\rho}{1-b\rho}, \quad 0 \leq \gamma \leq 1 \,. \tag{3.1.1}$$

The absolute compressibility factors are denoted as $\mathcal{A}$ and $\mathcal{B}$, where a and b represent the omitted volume of exotic gas and excluded volume of real gas in the van der Waals model, respectively. The equation of state (3.1.1) combines the anti-van der Waals standard gas term $\frac{\mathcal{B}\rho}{1-b\rho}$ with the anti-van der Waals Chaplygin gas term $\frac{\mathcal{A}(1-a\rho)^{\gamma}}{\rho^{\gamma}}$, and is referred to as the AWMCG. We focused on pure anti-Chaplygin gas to explore the characteristics of AWMCG. Anti-Chaplygin gas is unique because of its negative pressure, distinguishing it from conventional gases. This unique behaviour makes its study of particular interest, as many scientists have investigated its potential effects on cosmic expansion and other phenomena, especially dark energy. Gases within dark matter exhibit non-ideal behaviour under high-pressure or low-temperature conditions. By incorporating the van der Waals excluded volumes into the equation of state, we obtain a more generalized equation for the anti-Chaplygin gas, known as AWMCG.

The concept of a modified van der Waals Chaplygin gas has been discussed in previous studies, such as Sharma and Singh [109]. Viala and Horedt [110] studied the properties of polytropes with negative polytropic exponents that correspond to anti-Chaplygin gases. Lipscombe [111] discovered that a gravastar composed of Chaplygin gas requires a thin layer of exotic matter made of anti-Chaplygin gas. This condition is supported by the theory of wiggly strings proposed by Carter and Vilenkin [112, 113], respectively. When $\mathcal{B} = 0$, equation (3.1.1) describes the generalized anti-van der Waals Chaplygin gas, and if we take $b = 1$ and $\mathcal{B} = 0$, then equation (3.1.1) represents the pure anti-van der Waals Chaplygin gas. This model extends the Chaplygin gas model, first introduced by Chaplygin [8] in 1904. Unlike dark-energy and high-energy physics, the modified Chaplygin gas features negative pressure, which explains the interaction between dark energy and dark matter through unique fluid dynamics. This investigation provides new insights into dark matter compared with the existing literature [9, 10, 12, 95, 114].

The goal of this study is to expand the well-known discovery of reducible equations [2]. Their analysis reveals that any hyperbolic state adjacent to a constant state must form a simple wave. Our research employs a strong and sufficient condition motivated by Hu and Sheng [28] to ensure the existence of characteristic decompositions in a typical two-by-two quasilinear system with strict hyperbolicity. This sufficient condition is given in Section 3.2. These decompositions enable us to prove that any wave adjacent to a constant state remains a simple wave, even when the coefficients vary with independent variables, that is, for a non-reducible system. As a result, we extended these conclusions to pseudo-steady Euler equations, demonstrating the same fundamental outcomes.

3.2 Methodology

Considering a standard 2×2 quasilinear hyperbolic system,

$$\begin{bmatrix} U \\ V \end{bmatrix}_x + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}_y = 0, \tag{3.2.1}$$

where the coefficients $c_{ij} = c_{ij}(x, y, U, V)$, $i, j = 1, 2$. According to Courant and Friedrichs [2], if differential equation (3.2.1) can be simplified such that the coefficients c_{ij} depend solely on U and V, then in a flow region where a characteristic section maintains constant values of U and V, the adjacent regions will exhibit a simple wave flow.

Hu and Sheng [27, 28] provided a broader, more comprehensive criterion for the presence of characteristic decompositions. Utilizing these decompositions, they expanded Courant and Friedrichs's [115] results from reducible to non-reducible equations. If coefficients c_{ij} $(i, j = 1, 2)$ satisfy

$$\begin{aligned} &c_{21}A_2 - c_{12}A_3 = 0, \\ &(c_{22} - c_{11})A_3 + c_{21}(A_1 - A_4) = 0, \\ &(c_{22} - c_{11})A_2 + c_{12}(A_1 - A_4) = 0, \end{aligned} \tag{3.2.2}$$

where

$$\begin{aligned} A_1 &= c_{11x} + c_{11}c_{11y} + c_{21}c_{12y}, \\ A_2 &= c_{12x} + c_{12}c_{11y} + c_{22}c_{12y}, \\ A_3 &= c_{21x} + c_{11}c_{21y} + c_{21}c_{22y}, \\ A_4 &= c_{22x} + c_{12}c_{21y} + c_{22}c_{22y} \ . \end{aligned}$$

The coefficients c_{ij}, where, i and j range from 1 to 2 in equation (3.2.2) vary with the parameters U and V. As a result, the flow in an area neighbouring a constant state must generate a simple wave, with the parameters (U, V) remaining unchanged through a set of characteristic curves, which may not necessarily be straight. Specifically, if the coefficients c_{ij} $(i, j = 1, 2)$ meet the conditions

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 = 0. \tag{3.2.3}$$

The flow then forms a simple wave, with the parameters (U, V) remaining unchanged through a family of straight characteristic lines.

3.3 Steady 2D MHD system

For AWMCG, the 2D Euler equation system in the steady-state scenario is denoted by

$$\begin{aligned} &\rho\left(U_x+V_y\right)+U\rho_x+V\rho_y=0,\\ &\rho\left(UU_x+VU_y\right)+\left(P_\rho+\mu h h_\rho\right)\rho_x=0,\\ &\rho\left(UV_x+VV_y\right)+\left(P_\rho+\mu h h_\rho\right)\rho_y=0,\end{aligned} \tag{3.3.1}$$

using the equation of state given in equation (3.1.1). Using the momentum equations of system (3.3.1), we obtain

$$U^2U_x+V^2U_y+UV(U_y+V_x)+\left(\frac{P_\rho+\mu h h_\rho}{\rho}\right)(U\rho_x+V\rho_y)=0 \tag{3.3.2}$$

The equation (3.3.2) can be written as

$$\left(s_m^2-U^2\right)U_x+\left(s_m^2-V^2\right)V_y-2UVU_y=0\ , \tag{3.3.3}$$

where $s_m=\sqrt{s_c^2+s_a^2}$ denotes the magnetoacoustic speed. Here, $s_c=\sqrt{-\left[\frac{\mathcal{A}\gamma(1-a\rho)^{\gamma-1}}{\rho^{\gamma+1}}+\frac{\mathcal{B}}{(1-b\rho)^2}\right]}$ is the local sound speed and $s_a=\sqrt{\mu\alpha_0^2\rho}$ is the Alfven speed. The flow is considered to be irrotational, and given in (2.2.6). The matrix representation of the system of equations (3.3.3) and (2.2.6) may be expressed as

$$\begin{bmatrix}U\\V\end{bmatrix}_x+\begin{bmatrix}\frac{-2UV}{s_m^2-U^2} & \frac{s_m^2-V^2}{s_m^2-U^2}\\ -1 & 0\end{bmatrix}\begin{bmatrix}U\\V\end{bmatrix}_y=0 \tag{3.3.4}$$

The Bernoulli law for system (3.3.2) is obtained by the last two equations of system (3.3.1) and is represented as

$$\frac{U^2+V^2}{2}+\int_{\rho_0}^{\rho}\frac{1}{\rho}P'(\rho)d\rho=Const. \tag{3.3.5}$$

The following theorem is directly obtained from [2] or [27, 28] because the system (3.3.4) is reducible.

Theorem 3.3.1. *For the 2D irrotational, isentropic and steady MHD system* (3.3.4) *with an AWMCG* (3.1.1)*, the flow in a region adjacent to the constant state is a simple wave in which the physical variables* (U, V, s_m) *are constant along a family of straight lines with wave characteristics.*

3.4 Pseudo-steady 2D MHD system

Here, we assume that the flow is smooth and isentropic, let $u = U - \xi$ and, $v = V - \eta$ represent the pseudo-flow velocity. The equations of the MHD system in the similarity plane $(\xi, \eta) = \left(\frac{x}{t}, \frac{y}{t}\right)$ are given as

$$\begin{aligned}
&(\rho u)_\xi + (\rho v)_\eta + 2\rho = 0,\\
&\frac{1}{\rho}\left(P_\rho + \mu h h_\rho\right)\rho_\xi + u u_\xi + v u_\eta + u = 0,\\
&\frac{1}{\rho}\left(P_\rho + \mu h h_\rho\right)\rho_\eta + u v_\xi + v v_\eta + v = 0.
\end{aligned} \tag{3.4.1}$$

By solving the momentum equations of system (3.4.1), we get

$$u^2 u_\xi + uv\left(u_\eta + v_\xi\right) + v^2 v_\eta + \left(\frac{P_\rho + \mu h h_\rho}{\rho}\right)\left(u\rho_\xi + v\rho_\eta\right) + \left(u^2+v^2\right) = 0. \tag{3.4.2}$$

The irrotationality relation in the similarity plane can be described as (2.3.5). Since the flow is irrotational, there exists a potential ϕ such that $\phi_\xi = u$ and $\phi_\eta = v$. Therefore, through the integration of the momentum equations of the system (3.4.1) gives the pseudo-Bernoulli law

$$\frac{u^2+v^2}{2}+\int_{\rho_0}^{\rho}\frac{1}{\rho}P'(\rho)d\rho+\phi=Const. \tag{3.4.3}$$

For irrotational smooth flow, equations (3.4.2) and (2.3.5) can be written as

$$\begin{cases} (s_m^2-u^2)\,U_\xi-uv\,(U_\eta+V_\xi)+(s_m^2-v^2)\,V_\eta=0, \\ V_\xi-U_\eta=0 \end{cases} \tag{3.4.4}$$

supplemented by the pseudo-Bernoulli's law (3.4.3), where $s_m=\sqrt{s_c^2+s_a^2}$ is the magneto-acoustic speed. The system (3.4.4) may be transformed in the following matrix form

$$\begin{pmatrix}U\\V\end{pmatrix}_\xi+\begin{pmatrix}\frac{-2uv}{s_m^2-u^2} & \frac{s_m^2-v^2}{s_m^2-u^2}\\ -1 & 0\end{pmatrix}\begin{pmatrix}U\\V\end{pmatrix}_\eta=0 \tag{3.4.5}$$

By treating U and V as variables in ρ, from equation (3.4.3), we obtain $\rho_\xi=0$ and $\rho_\eta=0$. Consequently, we derive

$$\begin{cases} A_1=\left(\frac{-2uv}{s_m^2-u^2}\right)_\xi+\frac{2uv}{s_m^2-u^2}\left(\frac{2uv}{s_m^2-u^2}\right)_\eta-\left(\frac{s_m^2-v^2}{s_m^2-u^2}\right)_\eta \\ \quad=\frac{2(V-\eta)\,(s_m^2-(U-\xi)^2)+4(U-\xi)^2(V-\eta)}{(s_m^2-(U-\xi)^2)^2}-\frac{4(U-\xi)^2(V-\eta)}{(s_m^2-(U-\xi)^2)^2} \\ \quad-\frac{2(V-\eta)}{s_m^2-(U-\xi)^2}=0 \\ A_2=\left(\frac{s_m^2-v^2}{s_m^2-u^2}\right)_\xi-\frac{s_m^2-v^2}{s_m^2-u^2}\left(\frac{2uv}{s_m^2-u^2}\right)_\eta \\ \quad=\frac{-2(U-\xi)\,(s_m^2-(U-\xi)^2)}{(s_m^2-(U-\xi)^2)^2}+\frac{s_m^2-(V-\eta)^2}{s_m^2-(U-\xi)^2}\cdot\frac{2(U-\xi)}{s_m^2-(U-\xi)^2}=0, \\ A_3=0. \\ A_4=0 \end{cases}$$

Hence, the relations (3.2.2) are satisfied by the coefficients in equations (3.4.5). The outcome is as follows.

Theorem 3.4.1. *For a pseudo-steady MHD system (3.4.4), the flow in the region forms a simple wave in the similarity plane provided that region is adjacent to a constant state, where the physical quantities* $(U, V,$ *and* $s_m)$ *remain unchanged along a set of wave characteristics that are straight lines.*

3.5 Full 2D MHD system

The continuity equation and the equations of motion for a full MHD system are given as

$$\begin{aligned}
(U-\xi)\rho_\xi + (V-\eta)\rho_\eta + \rho\,(U_\xi + V_\eta) &= 0, &(3.5.1)\\
(U-\xi)U_\xi + (V-\eta)U_\eta + \frac{P_\rho + \mu h h_\rho}{\rho}\rho_\xi &= 0, &(3.5.2)\\
(U-\xi)V_\xi + (V-\eta)V_\eta + \frac{P_\rho + \mu h h_\rho}{\rho}\rho_\eta &= 0, &(3.5.3)\\
(U-\xi)e_\xi + (V-\eta)e_\eta + \frac{P}{\rho}\rho_\xi &= 0, &(3.5.4)
\end{aligned}$$

where the internal energy e is expressed by

$$\begin{aligned}
e(\rho) &= \mathcal{B}\log\left(\frac{1-b\rho}{\rho}\right) - \frac{\mathcal{A}}{\gamma+1}\left(\frac{1-a\rho}{\rho}\right)^{\gamma+1}\\
&= -\frac{P}{\rho(\gamma+1)}\left(\frac{1-a\rho}{\rho}\right) + \mathcal{B}\left[\log\left(\frac{1-b\rho}{\rho}\right) - \frac{1}{\gamma+1}\left(\frac{1-a\rho}{1-b\rho}\right)\right], \qquad (3.5.5)
\end{aligned}$$

where $0 \leq \gamma \leq 1$. Now consider the pseudo-flow directions $\partial_s := (U-\xi)\partial_\xi + (V-\eta)\partial_\eta$, that are contrary to the remaining characteristic directions. In the similarity plane, equations (3.5.2)-(3.5.4) are expressed as

$$\begin{aligned}
\partial_S \rho + \rho\,(U_\xi + V_\eta) &= 0,\\
\partial_S U + \frac{P_\rho + \mu h h_\rho}{\rho}\rho_\xi &= 0,\\
\partial_S V + \frac{P_\rho + \mu h h_\rho}{\rho}\rho_\eta &= 0,\\
\frac{1}{\rho P_\rho}\partial_S P + U_\xi + V_\eta &= 0.
\end{aligned}$$

It can be readily verified that the entropy $\mathcal{S} = \frac{P(\rho)}{\left(\frac{(1-a\rho)^\gamma}{\rho^\gamma} - \frac{\rho}{1-b\rho}\right)}$ remains constant along the pseudo-flow directions, meaning that

$$\partial_{\mathcal{S}}\left(\frac{P(\rho)}{\left(\frac{(1-a\rho)^\gamma}{\rho^\gamma} - \frac{\rho}{1-b\rho}\right)}\right) = 0. \tag{3.5.6}$$

Consequently, the flow along pseudo-flow lines remains isentropic. Using the continuity and momentum equations of the system described in equations (2.1.3) and defining the vorticity as $\omega = V_x - U_y$, same as in Chapter 2, we get the following equation:

$$\omega_t + (V\omega)_y + (U\omega)_x + \left(\frac{P_y + \mu h h_\rho \rho_y}{\rho}\right)_x - \left(\frac{P_x + \mu h h_\rho \rho_x}{\rho}\right)_y = 0 \tag{3.5.7}$$

In the similarity plane, vorticity satisfies the equation (2.4.5). Consequently, vorticity disappears when pseudo-flow characteristics emanate from a constant state. As a result, this region is devoid of rotational flow and isentropic. Accordingly, by adopting the results obtained from the pseudo-steady scenario, we are ready to formulate our main result.

Theorem 3.5.1. *In an irrotational and isentropic flow directed by a 2D full MHD system for an AWMCG in the self-similar plane adjacent to a constant-state region,* *a wave must be a simple wave. In addition, the family of wave characteristics will be straight lines, and the flow variables* (U, V, ω, p, ρ) *will remain constant.*

3.6 Conclusion

The current analysis has been prepared with the objective of exploring the general sufficient condition of the characteristic decomposition method for analysing complex fluid systems, specifically focusing on a 2D MHD system with a special type of gas. We discovered that in certain situations, the flow characteristics remain straight lines and the flow variables remain constant even when the system coefficients vary with independent variables. This result extends classical findings about simple waves in hyperbolic systems provided by Courant and Friedrichs [2]. The phenomenal observations obtained from the current investigation can be condensed as follows:

- In steady MHD and pseudo-MHD systems, any hyperbolic state adjacent to a constant state must be a simple wave, even in more complicated settings.
- The characteristic decomposition method simplifies the analysis of the full MHD equations by preserving entropy and vorticity along these characteristics.
- This work not only improves our understanding of MHD flows with anti-van der Waals modified Chaplygin gas but also provides a solid method for addressing similar problems in other fluid systems.

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