

Nonsmooth Analysis of Interval-Valued Functions and Optimization Methods for Set-Valued Maps



Thesis submitted in partial fulfillment
for the Award of Degree

Doctor of Philosophy

by

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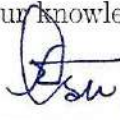
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Dedicated to

My Husband & My Parents

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List of Figures

2.1	Geometrical illustration of IVFs of Remark 2.5	51
2.2	The IVF \mathbf{F} of Note 2.1	55
2.3	The IVF \mathbf{F} of Example 2.2	58
2.4	The IVF \mathbf{F} of Note 2.3	64
3.1	IVFs \mathbf{F} and \mathbf{D} of Example 3.1	72
3.2	IVFs \mathbf{F} and \mathbf{D} of Example 3.2	74
3.3	The IVF \mathbf{F} of Example 3.3	83
3.4	The IVF \mathbf{V} of Example 3.4	89
4.1	The geometrical representation of two possible gH -Dini Hadamard ϵ -subgradients of \mathbf{F} of Example 4.1	103
4.2	The geometrical representation of two possible gH -Dini Hadamard ϵ -subgradients of \mathbf{F} of Remark 4.1	106
5.1	Obtained output of Algorithm 1 for Example 5.1	162
5.2	Obtained output of Algorithm 1 for Example 5.2	163
5.3	Obtained output of Algorithm 1 for Example 5.3	165
5.4	Obtained output of Algorithm 1 for Example 5.4	166
5.5	Obtained output of Algorithm 1 for Example 5.5	168
5.6	Obtained output of Algorithm 1 for Example 5.6	170
5.7	Obtained output of Algorithm 1 for Example 5.7	171
6.1	Obtained output of Algorithm 2 for Example 6.1	206
6.2	Obtained output of Algorithm 2 for Example 6.2	207
6.3	Obtained output of Algorithm 2 for Example 6.3	208
6.4	Obtained output of Algorithm 2 for Example 6.4	210
6.5	Obtained output of Algorithm 2 for Example 6.5	212
6.6	Obtained output of Algorithm 2 for Example 6.6	214

6.7 Obtained output of Algorithm 2 for Example 6.7 215

List of Tables

5.1	Performance of Algorithm 1 on Example 5.1	161
5.2	Performance of Algorithm 1 on Example 5.2	163
5.3	Performance of Algorithm 1 on Example 5.3	164
5.4	Performance of Algorithm 1 on Example 5.4	166
5.5	Performance of Algorithm 1 on Example 5.4 with $t_k \in (0, 1)$	167
5.6	Performance of SD method (see [1]) on Example 5.4 with $t_k \in (0, 1)$	167
5.7	Performance of Algorithm 1 on Example 5.5	168
5.8	Performance of Algorithm 1 on Example 5.5 with $t \in (0, 1)$	169
5.9	Performance of Algorithm 1 on Example 5.5 with $t_k = 1$	169
5.10	Performance of Algorithm 1 on Example 5.6	170
5.11	Performance of Algorithm 1 on Example 5.7	172
6.1	Performance of Algorithm 2 on Example 6.1	206
6.2	Output of Algorithm 2 on Example 6.1 with $x_0 = 2.3000$	206
6.3	Performance of Algorithm 2 on Example 6.2	207
6.4	Performance of Algorithm 2 on Example 6.3	209
6.5	Output of Algorithm 2 on Example 6.3 with $x_0 = (1.0000, -1.5000)^\top$	209
6.6	Performance of Algorithm 2 on Example 6.4	211
6.7	Performance of Algorithm 2 on Example 6.5	212
6.8	Output of Algorithm 2 on Example 6.5 with $x_0 = 4.0000$	213
6.9	Performance of Algorithm 2 on Example 6.6	213
6.10	Performance of Algorithm 2 on Example 6.7	216
6.11	Performance of Algorithm 2 on Example 6.7 with $x_0 = (-11, -1.5)^\top$	216

List of Symbols

\mathbb{R}	Set of all real numbers
\mathbb{R}_+	Set of all nonnegative real numbers
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty, +\infty\}$
\mathbb{R}^n	Euclidean space of dimension n
\mathcal{Y}	A finite-dimensional Banach space
$I(\mathbb{R})$	Set of all compact intervals
$\overline{I(\mathbb{R})}$	$I(\mathbb{R}) \cup \{+\infty, +\infty\}$
$\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \dots$	The elements of $I(\mathbb{R})$
$\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}, \widehat{\mathbf{Z}}, \dots$	The elements of $I(\mathbb{R})^n$
$\mathbf{0}$	$[0, 0]$
$\mathbf{F} : \mathcal{Y} \rightarrow I(\mathbb{R})$	An IVF on domain \mathcal{Y}
$\mathbf{\Gamma} : I(\mathbb{R})^n \rightarrow I(\mathbb{R})$	An IVF of interval-valued variables
$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$	A set-valued mapping from \mathbb{R}^n to \mathbb{R}^m
$\mathcal{P}(\mathbb{R}^m)$	The class of all nonempty subsets of \mathbb{R}^m
$\text{int}(A)$	The interior for $A \in \mathcal{P}(\mathbb{R}^m)$
$\text{cl}(A)$	The closure for $A \in \mathcal{P}(\mathbb{R}^m)$
$ A $	The cardinality for $A \in \mathcal{P}(\mathbb{R}^m)$
$\text{bd}(A)$	Boundary of $A \in \mathcal{P}(\mathbb{R}^m)$
$\text{conv}(A)$	Convex hull of $A \in \mathcal{P}(\mathbb{R}^m)$
$\ \cdot\ $	Euclidean norm of a vector
$[k]$	The set $\{1, 2, \dots, k\}$
$\text{Min}(A, K)$	The minimal set of A
$\text{WMin}(A, K)$	The weakly minimal set of A w.r.t. K
$K \subseteq \mathbb{R}^m$	A closed, convex, pointed, and solid cone
K^*	The dual cone of K
Ψ_e	The Gerstewitz function associated with $e \in \text{int}(K)$
\preceq_l	Lower set less relation

\bar{w}	Cardinality of a set of minimal elements at a given point \bar{x}
\prec_l	Strict lower set less relation
P_x	Partition set at x
\tilde{f}^a	A vector-valued function from \mathbb{R}^n to $\prod_{j=1}^{\bar{w}} \mathbb{R}^m$.

Abbreviations

Abbreviation	Description
IVF	Interval-Valued Function
IOP	Interval Optimization Problem
KKT	Karush-Kuhn-Tucker
SOP	Set Optimization Problem
VOP	Vector Optimization Problem