ABSTRACT

Let G be a separable locally compact group and π an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. An *inductive algebra* is a weakly closed abelian subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ that is normalized by $\pi(G)$, i.e., $\pi(g)\mathcal{A}\pi(g)^{-1} = \mathcal{A}$ for each $g \in G$. If we wish to emphasize the dependence on π, we will use the term π-inductive algebra. A *maximal inductive algebra* is a maximal element of the set of inductive algebras, partially ordered by inclusion.

The identification of inductive algebras can shed light on the possible realizations of $\mathcal{H}$ as a space of sections of a homogeneous vector bundle (see e.g. [5–8]). For self-adjoint maximal inductive algebras there is a precise result known as Mackey's Imprimitivity Theorem, as explained in the introduction to [5]. Inductive algebras have also found applications in operator theory (see e.g. [2, 10]).

In **Chapter 1**, we introduce inductive algebras and explain how they arise from sections of a homogeneous vector bundle. In addition, we provide a brief survey of the literature on inductive algebras.

The affine group is the semidirect product $k^\times \ltimes k$, where k is a field. In **Chapter 2**, we classify the maximal inductive algebras for the representations of the affine group of a finite field. We show that each irreducible representation of the affine group of a finite field has a unique maximal inductive algebra, and it is self-adjoint.

The following is the main result of this chapter:

- Recall that the affine group of a finite field has a unique irreducible representation other than the characters. It acts on the Hilbert space $\mathcal{H} = L^2(k^\times)$.

For each $\varphi \in \mathcal{H}$, let $m_\varphi : \mathcal{H} \to \mathcal{H}$ be defined by $m_\varphi(F) = \varphi F$. Then $\mathcal{B} = \{m_\varphi \mid \varphi \in \mathcal{H}\}$ is the only self-adjoint maximal $\pi-$inductive algebra.

In **Chapter 3**, we show that inductive algebras for a compact group are self-adjoint. This is significant because, in general, the classification of self-adjoint inductive algebras is easier than the classification of all inductive algebras. This is because the methods of spectral theory are available only in the former case.

The following are the main results of this chapter:

1. Let (X, μ) be a measure space. The algebra $L^\infty(X, \mu)$ is finite dimensional if and only if all of its subalgebras are self-adjoint.

2. Let G be a compact group and π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a π-inductive algebra, then $\mathcal{A}$ is self-adjoint.

The motion group of the plane is the semidirect product $M(2) = SO(2) \ltimes \mathbb{R}^2$, with the standard action of $SO(2)$ on $\mathbb{R}^2$. In **Chapter 4**, we classify the maximal inductive algebras for the representations of $M(2)$. We show that *each irreducible representation of the motion group of the plane has a unique maximal inductive algebra, and it is self-adjoint.*

References

[1] George W. Mackey, *The Theory of Unitary Group Representations*, The University of Chicago Press, Chicago, London, 1976.

[2] Amritanshu Prasad and M. K. Vemuri, *Inductive algebras and homogeneous shifts*, Complex Anal. Oper. Theory **4** (2010), no. 4, 1015–1027.

[3] Amritanshu Prasad and M. K. Vemuri, *Inductive algebras for finite Heisenberg groups*, Comm. Algebra **38** (2010), no. 2, 509–514.

[4] K. N. Raghavan, *Finite dimensional inductive algebras are trivial*, Comm. Algebra **33** (2005), no. 10, 3783–3785.

[5] Giovanni Stegel, *Inductive algebras for trees*, Pacific J. Math. **216** (2004), no. 1, 177–200.

[6] Giovanni Stegel *Even automorphisms of trees and inductive algebras*, Int. J. Pure Appl. Math. **29** (2006), no. 4, 521–552.

[7] Tim Steger and M. K. Vemuri, *Inductive algebras for* $\mathrm{SL}(2,\mathbb{R})$, Illinois J. Math. **49** (2005), no. 1, 139–151.

[8] M. K. Vemuri, *Realizations of the canonical representation*, Proc. Indian Acad. Sci. Math. Sci. **118** (2008), no. 1, 115–131.

[9] Mitsuo Sugiura, *Unitary representations and harmonic analysis*, Kodansha Ltd., Tokyo; Halstead Press [John Wiley & Sons], New York-London-Sydney, 1975, An introduction.

[10] Adam Korányi, *Homogeneous bilateral block shifts*, Proc. Indian Acad. Sci. Math. Sci. **124** (2014), no. 2, 225–233.

[11] Promod Sharma and M. K. Vemuri, *Inductive algebras for the affine group of a finite field*, Afr. Mat. **33** (2022), no. 2, Paper No. 46, 4.

[12] Promod Sharma and M. K. Vemuri, *Inductive algebras for the motion group of a plane*, arxiv, https://arxiv.org/abs/2212.03625.

[13] Promod Sharma and M. K. Vemuri, *Inductive algebras for compact groups*, Mathematische Nachrichten, 2023, to appear.