Chapter 3

Novel operational matrix method for the numerical solution of nonlinear reaction-advection-diffusion equation of fractional order

3.1 Introduction

Fractional calculus theory is used to describe the mathematical modeling of real-life problems in the best way in Engineering, Physics [23], Biology [64], Finance [65, 66, 67, 68], etc. Fractional calculus also plays a vital role in partial differential equations(PDEs), mainly in transportation in porous media i.e., to formulate the transport problems in fractional order reaction-advection-diffusion equation (FRADE). In today's livelihood, the water pollution problem is a big issue for every living thing. Drinking polluted water can lead to bacterial infection and susceptibility to diseases like cholera, typhoid, and dysentery. Apart from this, drinking dirty water can also cause viral infection. Viral infections cause dangerous diseases like hepatitis A, flu, cholera, and jaundice. It can also be seen that the polluted water mixes with the groundwater through the porous media. Transportation of the groundwater is modeled by the advection-diffusion equation (ADE). Numerous researchers have studied groundwater modeling to analyze potential resources and predict the future environmental impact. It has been a tremendous contribution to the study of the transport of solutes in porous media that many scientists and engineers have focused on the possibility of contamination of the subsurface environment. Diffusion is the most common natural process. In such a process, molecules move from a region of high concentration to a region of low concentration. Diffusion of oxygen molecules into blood, contaminants into groundwater, tumor cells into healthy cells, and kidney filtration of chemicals and toxins are examples of diffusion processes. The advection process is defined as the movement of particles and substances by bulk motion. The difference between diffusion and advection can be understood by examples: from a tea bag immersed in hot water, tea mixing into the water is a diffusion process, while ink or some chemical put into a flowing river is both advection and diffusion.

This chapter aims to develop the new operational matrix method based on Vieta-Fibonacci polynomials and to validate the proposed method by applying it to find the approximate solution of the following nonlinear time FRADE given by

$$ {}_{0}^{c}D_{t}^{\alpha}u(x,t) + \lambda_1 \frac{\partial u}{\partial x} = \nabla^2 u + \lambda_2(1-u)u + f(x,t), \qquad 0 < \alpha \leq 1, \tag{3.1} $$

under the prescribed initial and boundary conditions as

$$ \begin{aligned} u(x,0) &= \rho_1(x), \\ u(0,t) &= \rho_2(t), \\ u(1,t) &= \rho_3(t), \end{aligned} \tag{3.2} $$

where $f(x,t)$ is forced term. Several special cases of the concerned model are listed as follows: (i) if we assume $\lambda_1 = 0$, $\lambda_2 = 0$ and $0 < \alpha < 1$, then we obtain the time-fractional diffusion equation, (ii) if we assume $\lambda_1 = 0$, $\lambda_2 = 0$ and $1 < \alpha \leq 2$, then we obtain the time-fractional diffusion wave equation, (iii) when we choose $\lambda_1 = 0$ and $0 < \alpha \leq 1$ then time FRDE is found, (iv) choosing $\lambda_1 = 1$, $\lambda_2 = 0$ and $1 < \alpha \leq 2$, the time-fractional Cattaneo equation is obtained. But in this chapter, we will only consider time-fractional order RADE $(0 < \alpha < 1, \lambda_1 \neq 0, \lambda_2 \neq 0)$. In the proposed model, the authors have the fractional order derivative in the Caputo sense, as it helps to describe the physical and engineering processes. These models are used for accurate modelling of the physical systems towards damping. In the last few years, many researchers are working to find the numerical solutions of fractional order PDEs using various methods. The operational matrix method is the best method among those, which provides the best approximate solution with less effort. Most of the matrices use in this method have many zero entries, which reduce the computational time. In recent years, many authors have developed an operational matrix for different polynomials such as authors of [57] have formed operational matrix with the help of Legendre polynomial to approximate fractional order differential equation (FDE) and also [69] employed Chebyshev polynomials to develop an operational matrix to find the numerical solutions of FDE. [70] have obtained numerical solution of two-dimensional FRADE with Lucas operational matrix and [46] have obtained the numerical solution of diffusion equation of fractional order with chebyshev collocation and finite difference methods, [71] have solved fractional fuzzy differential equation, [72] has defined new concepts of fractional quantum calculus. [63] have approximated FRADE using Genocchi operational matrix. [73] have used homotopy method for the numerical solution of fractional PDE and [60] has developed a scheme to find the exact solution of the nonlinear diffusion equation. [74] have obtained the numerical solution of FRADE and [75] gave the approximate solution of the integro-differential equation. [76] have introduced a new approach to solve fractional order differential equations with Caputo-Fabrizio derivative. [77] have used the sixth kind Chebyshev collocation approach for the numerical investigation of inverse reaction-diffusion-convection problem. [78] have derived an operational matrix for clique polynomial to solve the Klein-Gordon equation in a fractional order system. [79] have investigated the numerical solution of variable order nonlinear integro-differential equations. Approximate solutions of some fractional problems can be found in [80, 81, 44, 82, 83, 84, 85, 86, 87]. But to the best of the author's knowledge, the solution of the FRADE given in equation (1) under prescribed initial and boundary conditions using two operational matrices one for time fractional derivative and other for spatial integer order derivative, have not yet been done by any researcher. In this chapter, an endeavor has been made to solve the FRADE under given boundary conditions by using Vieta-Fibonacci polynomials through the construction of operational matrices. Vieta-Fibonacci polynomials are a sequence of orthogonal polynomials that can be defined recursively. These polynomials are of great importance in many areas of mathematics, particularly approximation theory. The Chebyshev polynomials of the first and second kinds are closely related to Vieta-Fibonacci polynomials. Therefore, Vieta-Fibonacci polynomials are the modified Chebyshev polynomials [88]. The collocation techniques based on orthogonal polynomials have been successfully applied to various areas of science and engineering. They have been proven to be efficient and robust and provide exponential convergence while applied to various physical systems. In this respect, a new class of orthogonal functions called Vieta-Fibonacci polynomials is useful as this technique is based on the truncation of the expansion of the function. This helps to solve the resulting matrix equation, and as a result, the unknown Vieta-Fibonacci coefficients can be determined.

Encouraging by the preceding studies, a drive has been taken to study the proposed FRADE for different values of fractional order parameters, advection coefficient and the reaction term. The key purpose of this chapter is to analyse the convergence of the proposed method. The following are the major contributions of the present chapter:

1. Appropriate use of the Vieta-Fibonacci polynomials to derive the operational matrices for fractional as well as integer order derivatives to find the numerical solution of the proposed nonlinear time fractional mathematical model.
2. Graphical presentations of damping nature of the concerned model due to the presence of sink term and advection terms for different particular cases.
3. The most important contribution is the showcasing of the accuracy of the proposed method through error analysis while comparing it with the numerical solutions of the existing models through tables and graphs.

3.1.1 Function approximation

If $f(x)$ is a square-integrable function of x defined over the interval $[0, 1]$, then $f(x)$ can be written in combinations of Vieta-Fibonacci polynomials as

$$f(x) = \sum_{j=0}^{\infty} u_{j+1} VF^*_{j+1}(x), \tag{3.3}$$

where the coefficients u_j's are unknown.

The first $(n+1)$ terms only will be considered in the above series. Thus the approximation of the function $f(x)$ is reduced to the following finite series

$$f_n(x) \simeq \sum_{j=0}^{n} u_{j+1} VF^*_{j+1}(x) = U^T \phi(x), \tag{3.4}$$

where unknowns $U = [u_1, u_2, \cdots, u_n]^T$ are calculated using the following formula

$$u_j = \frac{8}{\pi} \int_0^1 f_n(x) VF^*_j(x) \sqrt{x - x^2}, \;\; j = 1, 2, 3, \cdots, \tag{3.5}$$

and the vector $\phi(x)$ is given by

$$\phi(x) = [VF_1^*(x), VF_2^*(x), \cdots VF_{n+1}^*(x)]^T. \tag{3.6}$$

3.2 Formation of shifted Vieta-Fibonacci operational matrix

Theorem 3.1. *Let $\phi(x)$ be the vector defined in Eq.(3.6) and suppose that $\alpha > 0$, then*

$${}_0^cD_x^\alpha \phi(x) = P^{(\alpha)}\phi(x), \tag{3.7}$$

where $P^{(\alpha)}$ is the operational matrix of fractional order α in Caputo sense defined by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} S(\lceil\alpha\rceil, 1) & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} S(\lceil\alpha\rceil, 2) & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} S(\lceil\alpha\rceil, n+1) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} S(i, 1) & \sum_{k=\lceil\alpha\rceil}^{i} S(i, 2) & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} S(i, n+1) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{n+1} S(n+1, 1) & \sum_{k=\lceil\alpha\rceil}^{n+1} S(n+1, 2) & \cdots & \sum_{k=\lceil\alpha\rceil}^{n+1} S(n+1, n+1) \end{pmatrix},$$

where

$$S(i,j) = \frac{(-1)^{i-k-1}2^{2k+2}\Gamma(1+k+i)\Gamma(1+k)}{\sqrt{\pi}\Gamma(i-k)\Gamma(2k+2)\Gamma(k+1-\alpha)} \times \sum_{r=0}^{j} \frac{(-1)^{j-r-1}2^{2r}\Gamma(1+r+j)\Gamma(r+k-\alpha+\frac{3}{2})}{\Gamma(j-r))\Gamma(2r+2)\Gamma(k+r-\alpha+3)}. \tag{3.8}$$

Proof. : By using the analytical form of shifted Vieta-Fibonacci polynomials and the definition of Caputo derivative, we have

$$ {}_0^cD_x^\alpha VF_i^*(x) = \sum_{k=0}^{i} \frac{(-1)^{i-k-1}2^{2k}\Gamma(i+k+1)}{\Gamma(i-k)\Gamma(2k+2)} \, {}_0^cD_x^\alpha x^k. $$

By definition of Caputo derivative ${}_0^cD_x^\alpha x^k = 0$ for $k = 0, 1, \cdots, \lceil\alpha\rceil$.

Therefore for $k \geq \lceil\alpha\rceil$, we have

$$ {}_0^cD_x^\alpha VF_i^*(x) = \sum_{k=\rceil\alpha\lceil}^{i} \frac{(-1)^{i-k-1}2^{2k}\Gamma(i+k+1)}{\Gamma(i-k)\Gamma(2k+2)} \times \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}. \tag{3.9} $$

Now, $x^{k-\alpha}$ is approximated using $(n+1)$ terms of shifted Vieta-Fibonacci polynomials as

$$ x^{k-\alpha} \simeq \sum_{j=1}^{n+1} c_{kj} VF_j^*(x), $$

where

$$ \begin{aligned} c_{kj} &= \frac{8}{\pi}\int_0^1 \sqrt{x-x^2} \;\; x^{k-\alpha} VF_j^*(x)dx, \\ &= \frac{8}{\pi}\int_0^1 \sqrt{x-x^2} \;\; x^{k-\alpha} \sum_{r=0}^{j} \frac{(-1)^{r-j-1}2^{2r}\Gamma(j+r+1)}{\Gamma(j-r)\Gamma(2r+2)} x^r dx, \\ &= \frac{8}{\pi}\sum_{r=0}^{j} \frac{(-1)^{r-j-1}2^{2r}\Gamma(j+r+1)}{\Gamma(j-r)\Gamma(2r+2)} \int_0^1 x^{k+r-\alpha}\sqrt{(x-x^2)}dx. \end{aligned} $$

After some manipulation, we get

$$ c_{kj} = \frac{4}{\sqrt{\pi}} \sum_{r=0}^{j} \frac{(-1)^{j-r-1}2^{2r}\Gamma(j+r+1)\Gamma(k+r-\alpha+\frac{3}{2})}{\Gamma(j-r)\Gamma(2r+2)\Gamma(k+r-\alpha-3)}, $$

$$x^{k-\alpha} \simeq \sum_{j=0}^{n+1} \frac{4}{\sqrt{\pi}} \sum_{r=0}^{j} \frac{(-1)^{j-r-1} 2^{2r} \Gamma(1+j+r)\Gamma(r+k-\alpha+\frac{3}{2})}{\Gamma(j-r)\Gamma(2r+2)\Gamma(k+r-\alpha-3)} VF_j^*(x).$$

Now, with the help of Eq.(3.9), we have

$$\begin{aligned} {}_0^c D_x^\alpha VF_i^*(x) = \sum_{j=0}^{n+1} \Bigg(\sum_{k=\lceil\alpha\rceil}^{i} & \frac{(-1)^{i-k-1} 2^{2k+2} \Gamma(1+k+i)\Gamma(1+k)}{\sqrt{\pi}\Gamma(i-k)\Gamma(2k+2)\Gamma(k+1-\alpha)} \\ \times \sum_{r=0}^{j} & \frac{(-1)^{j-r-1} 2^{2r} \Gamma(j+r+1)\Gamma(k+r-\alpha+\frac{3}{2})}{\Gamma(j-r)\Gamma(2r+2)\Gamma(k+r-\alpha+3)} \Bigg) VF_j^*(x). \end{aligned}$$

$${}_0^c D_x^\alpha VF_i^*(x) = \sum_{j=1}^{n+1} \left(\sum_{k=\lceil\alpha\rceil}^{i} S(i,j) \right) VF_j^*(x),$$

where

$$\begin{aligned} S(i,j) = & \frac{(-1)^{i-k-1} 2^{2k+2} \Gamma(1+k+i)\Gamma(1+k)}{\sqrt{\pi}\Gamma(i-k)\Gamma(2k+2)\Gamma(k+1-\alpha)} \\ & \times \sum_{r=0}^{j} \frac{(-1)^{j-r-1} 2^{2r} \Gamma(r+j+1)\Gamma(r+k-\alpha+\frac{3}{2})}{\Gamma(j-r)\Gamma(2r+2)\Gamma(k+r-\alpha+3)}. \end{aligned}$$

Hence the proof is completed

□

Theorem 3.2. *First order differentiation of the vector defined by Eq.(3.6), will be*

$$\frac{d\phi(x)}{dx} = P^{(1)}\phi(x), \tag{3.10}$$

where

$$P^{(1)} = (p_{i,j}) = \begin{cases} 4j, & \textit{for} \ \ k = i-j \begin{cases} k = 1, 3, \cdots, \mathrm{n}-1, & \mathrm{n} \ \textit{is even}, \\ k = 1, 3, \cdots, \mathrm{n}, & \mathrm{n} \ \textit{is odd}, \end{cases} \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. : Using the condition of orthogonality of the Vieta-Fibonacci polynomials proof is straightforward. □

Remark: Integer order differentiation of m times is given by

$$\frac{d^m \phi(x)}{dx^m} = (P^{(1)})^m \phi(x).$$

3.3 Application of the proposed method

To approximate the problem, let us assume that

$$u(x,t) \simeq u_n(x,t) = \phi^T(x) U \phi(t), \tag{3.11}$$

where $\phi(x)$ is vector defined in Eq.(3.6) and $U = (u_{i,j})$ is an unknown square matrix of order $(n+1)$, whose elements are to be determined.

Now, we have

$${}_0^c D_t^{\alpha} u(x,t) \simeq \phi^T(x) U P^{(\alpha)} \phi(t), \tag{3.12}$$

also

$$\frac{\partial^2 u}{\partial x^2} \simeq \phi^T(x) (P^{(2)})^T U \phi(t), \tag{3.13}$$

where $P^{(\alpha)}$ ans $P^{(2)}$ can be calculated from the previous section.

Now, using the Eqs.(3.11), (3.12) and (3.13) in model (3.1), we get

$$\phi^T(x)UP^{(\alpha)}\phi(t) + \lambda_1\phi^T(x)(P^{(1)})^T U\phi(t) = \phi^T(x)(P^{(2)})^T U\phi(t) \\ + \lambda_2\phi^T(x)U\phi(t)(1 - \phi^T(x)U\phi(t)) + f(x,t), \tag{3.14}$$

and the prescribed conditions will be

$$\begin{aligned} \phi^T(x)U\phi(0) &= \rho_1(x), \\ \phi^T(0)U\phi(t) &= \rho_2(t), \\ \phi^T(x)U\phi(1) &= \rho_3(t). \end{aligned} \tag{3.15}$$

Collocating the Eqs.(3.14) and (3.15) at certain collocation points, we have

$$\phi^T(x_i)UP^{(\alpha)}\phi(t_j) + \lambda_1\phi^T(x_i)(P^{(1)})^T U\phi(t_j) = \phi^T(x_i)(P^{(2)})^T U\phi(t_j) \\ + \lambda_2\phi^T(x_i)U\phi(t_j)(1 - \phi^T(x_i)U\phi(t_j)) + f(x_i,t_j), \tag{3.16}$$

and the prescribed conditions become

$$\begin{aligned} \phi^T(x_i)U\phi(0) &= \rho_1(x_i), \\ \phi^T(0)U\phi(t_j) &= \rho_2(t_j), \\ \phi^T(1)U\phi(t_j) &= \rho_3(t_j). \end{aligned} \tag{3.17}$$

This generates a system of nonlinear algebraic equations. By solving this nonlinear system an approximate solution is obtained.

3.4 Convergence analysis

Theorem 3.3. *If the fourth partial order derivative of a continuous function $u(x,t)$ defined on $[0,1]\times[0,1]$ is bounded by M, then Vieta Fibonacci expansion of the function will converge uniformly*

Proof. : Let us assume that $u(x,t)$ can be expressed in a series of Vieta-Fibonacci polynomials as

$$u_n(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i+1j+1}VF^*_{i+1}(x)F^*_{j+1}(t),$$

where unknowns can be obtained as

$$c_{i+1j+1}=\frac{4}{\pi^2}\int_0^1\int_0^1 u(x,t)VF^*_{i+1}(x)w(x)F^*_{j+1}(t)w(t)dxdt. \tag{3.18}$$

Replacing $x=\frac{2\cos(\theta_1)+2}{4}$ and $t=\frac{2\cos(\theta_2)+2}{4}$ in the above equation, we have

$$c_{i+1j+1}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u\Big(\frac{2\cos(\theta_1)+2}{4},\frac{2\cos(\theta_2)+2}{4}\Big)\sin(i+1)\theta_1\sin\theta_1\ \ \sin(j+1)\theta_2\sin\theta_2\ \ d\theta_1 d\theta_2.$$

Now, integrating twice w.r. to θ_1, we get

$$c_{i+1j+1}=\frac{1}{4\pi^2}\int_0^\pi\int_0^\pi u''\Big(\frac{2\cos(\theta_1)+2}{4},\frac{2\cos(\theta_2)+2}{4}\Big)\sin\theta_1\Omega_i(\theta_1)\ \ \sin(j+1)\theta_2\sin\theta_2\ \ d\theta_1 d\theta_2, \tag{3.19}$$

where

$$\Omega_i(\theta_1)=\Big[-\frac{\sin(1+i)\theta_1}{i(1+i)}+\frac{\sin(-1+i)\theta_1}{i(-1+i)}\Big]+\Big[\frac{\sin(3+i)\theta_1}{(3+i)(2+i)}-\frac{\sin(1+i)\theta_1}{(1+i)(2+i)}\Big].$$

Again, integrating Eq.(3.19) twice with respect to θ_2, after changing the order of integration, we obtain

$$c_{i+1j+1} = \frac{1}{64\pi^2}\int_0^\pi\int_0^\pi u''''\Big(\frac{2\cos(\theta_1)+2}{4},\frac{2\cos(\theta_2)+2}{4}\Big)\sin\theta_1\Omega_i(\theta_1)\quad \sin\theta_2\Omega_j\ (\theta_2)\ \ d\theta_1 d\theta_2, \tag{3.20}$$

where

$$\Omega_j(\theta_2) = \Big[-\frac{\sin(1+j)\theta_2}{j(1+j)}+\frac{\sin(-1+j)\theta_2}{j(-1+j)}\Big]+\Big[\frac{\sin(3+j)\theta_2}{(3+j)(2+j)}-\frac{\sin(1+j)\theta_2}{(1+j)(2+j)}\Big].$$

Taking the absolute value of the Eq.(3.20), we achieve

$$\mid c_{i+1j+1}\mid = \Bigg|\frac{1}{64\pi^2}\int_0^\pi\int_0^\pi u''''\Big(\frac{2\cos(\theta_1)+2}{4},\frac{2\cos(\theta_2)+2}{4}\Big)\sin\theta_1\Omega_i(\theta_1)\quad \sin\theta_2\Omega_j\ (\theta_2)\ \ d\theta_1 d\theta_2\Bigg|.$$

As $|u''''(x,t)|\le M$, $|\sin(\theta_1)|\le 1$ and $|\sin(\theta_2)|\le 1$, therefore

$$|c_{i+1j+1}| \le \frac{M}{64\pi^2}\int_0^\pi\int_0^\pi |\Omega_i(\theta_1)\ \Omega_j(\theta_2)|\quad d\theta_1 d\theta_2.$$

After some mathematical manipulation, we get

$$|c_{i+1j+1}| \le \frac{9M}{4(3+i)(2+i)(i-1)i\ (3+j)(2+j)(j-1)j}\ .$$

Thus, the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i+1j+1}$ is absolutely and uniformly convergent, and the expansion of the function is uniform. □

Theorem 3.4. *If the fourth partial order derivative of a continuous function $u(x,t)$, defined on $[0,1]\times[0,1]$ is bounded by some constant M, then error bound will be*

$$\epsilon_n < \frac{\pi M}{32(n+1)^3(m+1)^3},$$

where

$$\epsilon_n = \left(\int_0^1 \int_0^1 \Big[f(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i+1j+1} VF^*_{i+1}(x) VF^*_{j+1}(t)\Big]^2 dx\ dt \right)^{\frac{1}{2}}.$$

Proof. :

$$\begin{aligned}
\epsilon_n^2 &= \int_0^1 \int_0^1 \Big[f(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i+1j+1} VF^*_{i+1}(x) VF^*_{j+1}(t)\Big]^2 dx\ dt,\\
&= \int_0^1 \int_0^1 \Big[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i+1j+1} VF^*_{i+1}(x) VF^*_{j+1}(t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i+1j+1} VF^*_{i+1}(x) VF^*_{j+1}(t)\Big]^2 dx\ dt,\\
&= \int_0^1 \int_0^1 \Big[\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} c_{i+1j+1} VF^*_{i+1}(x) VF^*_{j+1}(t)\Big]^2 dx\ dt,\\
&= \int_0^1 \int_0^1 \Big[\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} c^2_{i+1j+1} VF^{*^2}_{i+1}(x) VF^{*^2}_{j+1}(t)\Big] dx\ dt,\\
&= \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} c^2_{i+1j+1} \sum_{j=m+1}^{\infty} \int_0^1 VF^{*^2}_{i+1}(x)\ dx \int_0^1 VF^{*^2}_{j+1}(t)\ dt,\\
&= \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} c^2_{i+1j+1} \quad \frac{\pi}{8} \times \frac{\pi}{8}.
\end{aligned}$$

Now, from Theorem 3.3, we have

$$\epsilon_n^2 \leq \frac{\pi^2}{64} \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \left(\frac{9M}{4(3+i)(2+i)(i-1)i\ (3+j)(2+j)(j-1)j} \right)^2,$$

$$\begin{aligned}
\epsilon_n &\leq \frac{9\pi M}{32} \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{1}{(3+i)(2+i)(i-1)i\ (3+j)(2+j)(j-1)j},\\
&\leq \frac{9\pi M}{32} \sum_{i=n+1}^{\infty} \frac{1}{(3+i)(2+i)(i-1)i} \times \sum_{j=m+1}^{\infty} \frac{1}{(3+j)(2+j)(j-1)j}, \qquad i,j > 1,\\
\epsilon_n &< \frac{9\pi M}{32} \sum_{i=n+1}^{\infty} \frac{1}{i^4} \times \sum_{i=m+1}^{\infty} \frac{1}{j^4}.
\end{aligned}$$

Since both the series are positive and decreasing, therefore using the integral test, we get

$$
\begin{aligned}
&< \frac{9\pi M}{32}\int_{n+1}^{\infty}\frac{1}{x^4}dx \quad \int_{m+1}^{\infty}\frac{1}{y^4}dy,\\
\epsilon_n &< \frac{\pi M}{32(n+1)^3(m+1)^3}.
\end{aligned}
$$

Hence the proof is completed. □

3.5 Numerical applications

Example 3.1 Consider the following time fractional PDE as a special case of the considered model as

$$
{}_0^cD_t^{\alpha}u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{3.21}
$$

under the prescribed initial and boundary conditions

$$
u(x,0) = 0,
$$

and

$$
u(0,t) = 0, \qquad u(1,t) = 0,
$$

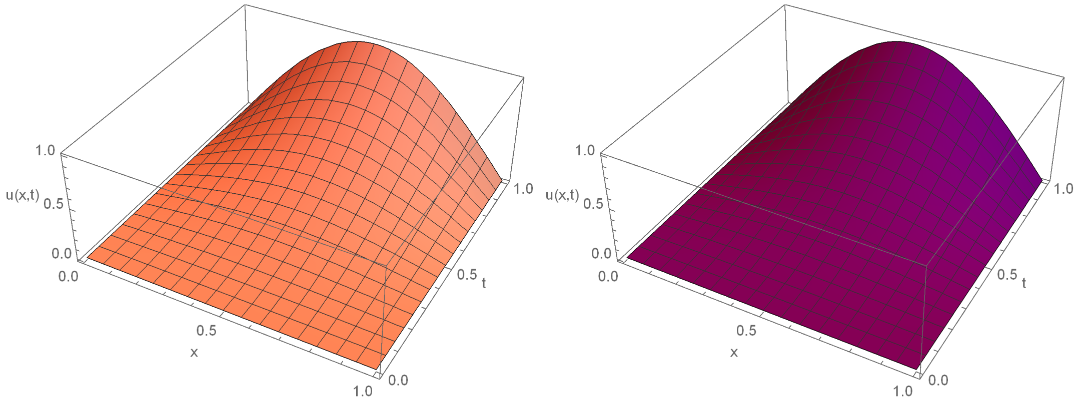
and the forcing term

$$
f(x,t) = \sin(x\pi)\Big[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + t^2\pi^2\Big],
$$

having an exact solution $u(x,t) = t^2\sin(x\pi)$. The maximum absolute error between approximate and analytical solutions for Example 3.1 is depicted through

TABLE 3.1: The maximum absolute errors at $t = 1$ for different values of α for Example 3.1

α	[89]	[89]	[89]	present method	CPU time (sec.)
$\frac{1}{4}$	1.0700×10^{-5}	6.4865×10^{-6}	3.5464×10^{-6}	3.4776×10^{-6}	6.486
$\frac{1}{2}$	6.1171×10^{-5}	3.8456×10^{-5}	1.8586×10^{-5}	5.6927×10^{-6}	5.172
$\frac{3}{4}$	2.8929×10^{-4}	1.9879×10^{-4}	1.1757×10^{-4}	9.1887×10^{-6}	7.642



(a) Exact solution (b) Numerical solution

FIGURE 3.1: Behaviour of exact and numerical solutions for $n = 8$ at $\alpha = 0.5$ for Example 3.1.

Table 3.1 for small value $n = 8$. It also compared the results obtained by the proposed method with the results obtained by different methods given in [89] through maximum absolute error using the formula

$$L_\infty = \max_{0\le x\le 1} |u_{approx}(x,1) - u_{exact}(x,1)|. \tag{3.22}$$

From the table, it is seen that the maximum absolute errors provided in [89] for $n = 80$ for all the three methods are higher than the method proposed here. The absolute error given by the proposed method is of order 10^{-6} for a small value of $n = 8$. The proposed method provides less error even for the small size of the operational matrix ($n = 8$). Thus, the method is more effective as compared to the existing methods given in [89], and it also takes less time to compute the solution even for a lower degree of approximations.

Example 3.2 Considering $\lambda_1 = 1$ and $\lambda_2 = 1$, the prescribed model reduces to the following nonlinear fractional order reaction-advection-diffusion equation

$$ {}_0^cD_t^\alpha u(x,t) + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + f(x,t). \tag{3.23}$$

The equation under the initial condition $u(x,0) = x$, and boundary conditions $u(0,t) = t$, $u(1,t) = 1+t$ and the forcing term

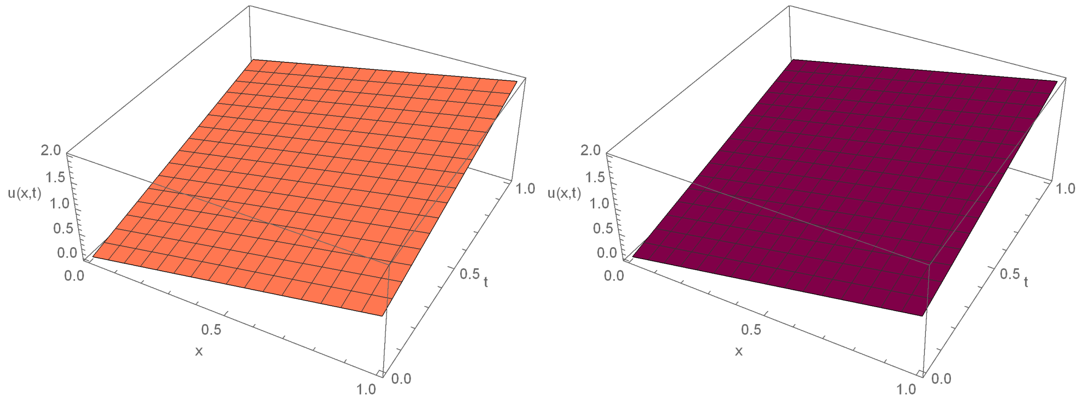
$$f(x,t) = \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 1 - (x+t)(1-(x+t)) \right),$$

has the exact solution as $u(x,t) = x+t$. The maximum absolute errors between

TABLE 3.2: The maximum absolute errors at $t = 0.5$ for $n = 6$ for Example 3.2

x	COM	present method	CPU time (sec.)
0.2	2.2607×10^{-4}	4.7198×10^{-5}	1.672
0.4	3.1930×10^{-4}	7.0136×10^{-5}	1.828
0.6	3.2813×10^{-4}	7.3044×10^{-5}	1.672
0.8	2.4792×10^{-4}	5.3593×10^{-5}	1.704

exact and approximate solutions for Example 3.2 are depicted through Table 3.2 for $n = 6$ and $\alpha = 0.7$. It is seen from Table 3.2 that the maximum absolute error obtained by the proposed method is much lesser as compared to using the Chebyshev operational matrix method (COM) given by [54]. Fig 3.2 shows the behaviour of exact and numerical solutions at $n = 3$. One can see that the approximate result is almost similar to the exact one.



(a) Exact solution (b) Numerical solution

FIGURE 3.2: Behaviour of exact and numerical solutions for $n = 3$ at $\alpha = 0.7$ for Example 3.2

3.6 Solution of the problem

After the validation of the efficiency of the proposed method with some existing problems, the method is applied to solve the considered mathematical model (3.1) under the prescribed initial and boundary conditions (3.2) with $\rho_1(x) = x$, $\rho_2(t) = t$ and $\rho_3(t) = t + 1$ and $f(x, t) = 0$ for different values of λ_1, λ_2 and α. Figs. 3.3-3.5 are drawn to show the variations of solute concentration $u(x, t)$ due to changes in α with advection term $\lambda_1 = 0.5$ for conservative and non-conservative systems at $t = 0.5$. From Fig 3.3, it is observed that on increasing the values of time derivative α, the solute profile decreases for the source term ($\lambda_2 = 1$). Figs. 3.4 and 3.5 are drawn for conservative ($\lambda_2 = 0$) and sink term ($\lambda_2 = -1$), respectively. It is seen that in both cases, the solute profile decreases on increasing α. Fig 3.6 depicts the effect of reaction term on solute concentration, which clearly shows that the solute profile of the conservative system is between the solute profiles for sink and source terms. The effect of advection term λ_1 on solute concentration for $\alpha = 1$ and $\lambda_2 = 1$ at $t = 0.5$ is shown through Fig 3.7. It is observed from the figure that on increasing the value of advection term λ_1, the concentration of the solute decreases.

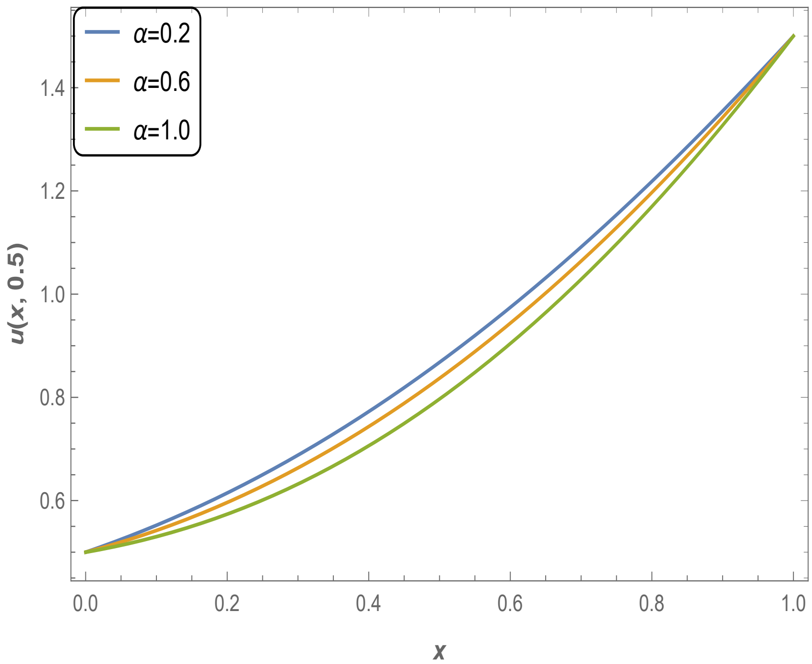


FIGURE 3.3: Variations of $u(x, 0.5)$ vs. x for various values of α for $\lambda_1 = 0.5,\ \ \lambda_2 = 1$ at $t = 0.5$.

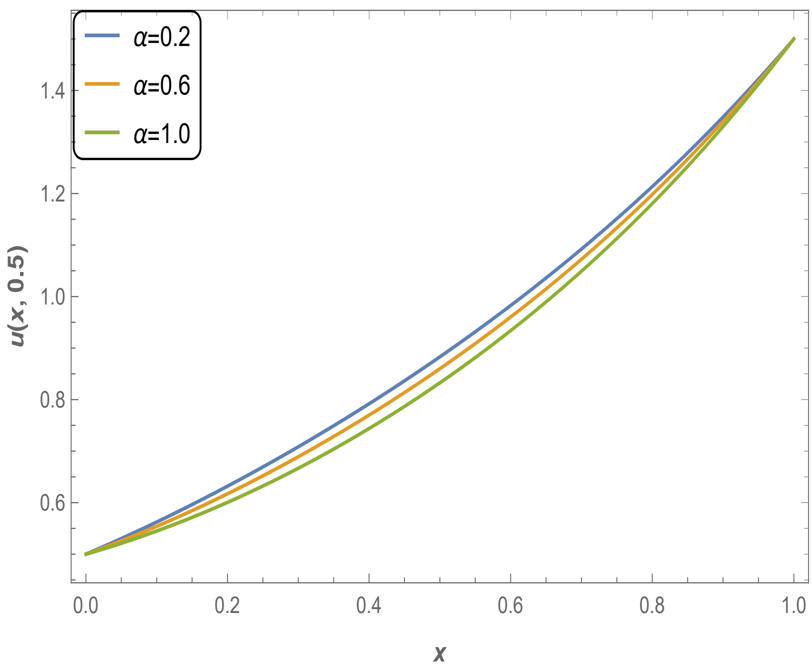


FIGURE 3.4: Variations of $u(x, 0.5)$ vs. x for various values of α for $\lambda_1 = 0.5,\ \ \lambda_2 = 0$ at $t = 0.5$.

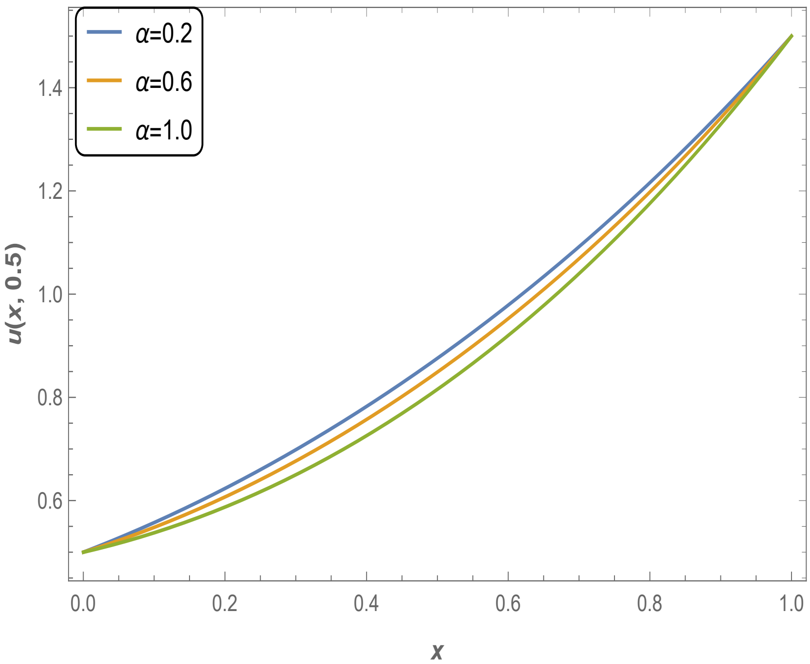


FIGURE 3.5: Variations of $u(x, 0.5)$ vs. x for various values of α for $\lambda_1 = 0.5$, $\lambda_2 = -1$ at $t = 0.5$.

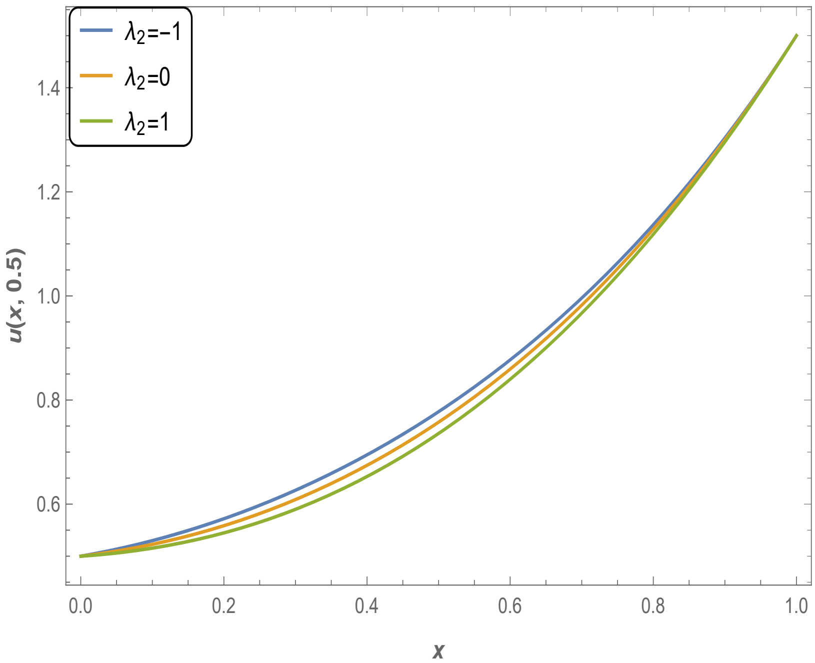


FIGURE 3.6: Variations of $u(x, 0.5)$ vs. x for various values of λ_2 for $\alpha = 1$, $\lambda_1 = 0.5$ at $t = 0.5$.

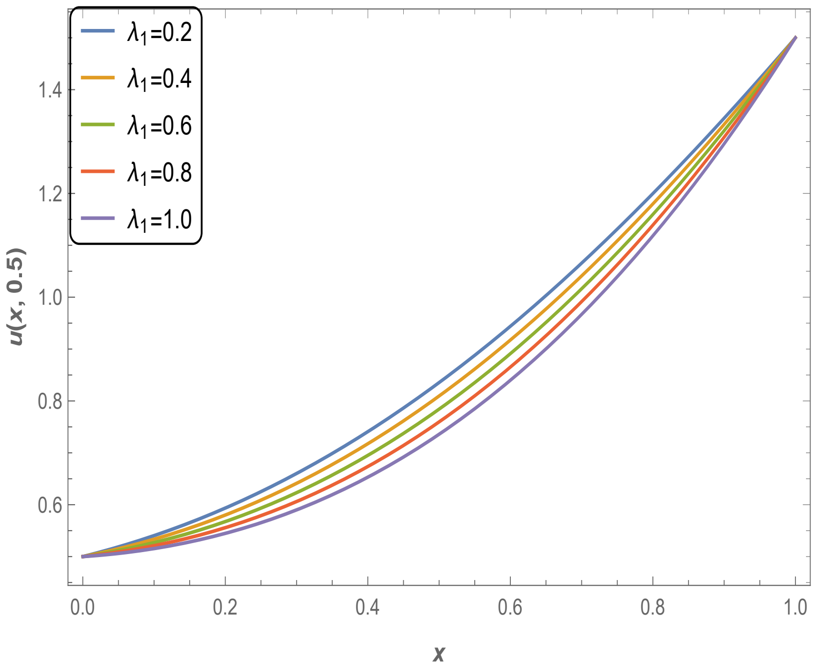


FIGURE 3.7: Variations of $u(x, 0.5)$ vs. x for various values of λ_1 for $\alpha = 1$ and $\lambda_2 = 1$ at $t = 0.5$.

3.7 Conclusion

In this chapter, the Vieta-Fibonacci polynomials with their shifted counterparts and some of their relevant properties have been discussed to investigate the approximate solution of the considered model using two operational matrices for fractional and integer order derivatives. The operational matrices reduce the nonlinear problem to the system of algebraic equations. By solving the system, an approximate solution to the problems is obtained. The efficiency of the method is validated by applying it to some existing problems, which clearly shows that the proposed method has excellent accuracy. The present method is assumed to be applied to solve time and space FRADE, even for two-dimensional and three-dimensional groundwater problems arising in porous media.

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