Chapter 5

The concentration and cavitation in the Riemann solution for non-homogeneous logarithmic equation of state with magnetic field. *

"The only way to learn mathematics
is to do mathematics."

–Paul Halmos

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5.1 Introduction

In the present chapter, we are interested in the hyperbolic conservation laws governing the 1-Dimensional unsteady simple flow of an inviscid, isentropic, and compressible fluid subject to a transverse magnetic field with the logarithmic equation of state (LEoS) and friction term (see [154], [155]).

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + \left(\rho\nu^2 + \frac{B^2}{2\mu} + P(\rho)\right)_x = \kappa\rho, \end{cases} \tag{5.1}$$

where $\rho > 0$ and $\nu > 0$, represent the density and velocity, $B \geq 0$, $\mu > 0$ and $P(\rho)$ is the transverse magnetic field, magnetic permeability and pressure respectively, and κ is non-zero constant. t and x are independent variables, which denote time and space respectively. We consider the magnetic field B and pressure P as

$$B(\rho) = c_1\rho, \qquad P(\rho) = c_2 \ln\rho, \tag{5.2}$$

where $c_1 > 0$ and $c_2 > 0$, are small parameters. We have analyzed the Riemann Problem (RP) for the non-homogeneous system with the magnetic field and LEoS. Following piecewise discontinuous initial conditions are considered for the problem:

$$\rho(x,0) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0, \end{cases} \qquad \nu(x,0) = \begin{cases} \nu_-, & x < 0, \\ \nu_+, & x > 0, \end{cases} \tag{5.3}$$

where ρ_-, ρ_+, ν_-, and ν_+ are given positive constant states. If we take $\kappa = 0$ and pressure vanishes, for $c_1 = 0, c_2 = 1$, the system (5.1) is reduced into homogeneous system as given below (see [138])

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2 + c_2 \ln\rho)_x = 0. \end{cases} \tag{5.4}$$

This is referred to as an isentropic Eulers equation with LEoS. [46] studied the cavitation and concentration processes when the pressure has vanished from the Riemann solution of (5.4). In astrophysics, the concept was presented to explain the unique features of molecular clouds that could not be explained by isothermal distribution. The logotropic dark fluid as a unification of dark energy and dark matter has been researched using the LEoS by [122], [123] and [124]. The Logotropic model, which is based on the Euler system with the LEoS, exhibits more fascinating cosmological characteristics than the various modified structures of the "Chaplygin gas model". As $c_1, c_2 \to 0$, the equation (5.1) becomes to vanishing pressure gas dynamics equation with the source term given below

$$\begin{cases} (\rho)_t + (\rho\nu)_x = 0, \\ (\rho\nu)_t + (\rho\nu^2)_x = \kappa\rho. \end{cases} \tag{5.5}$$

This can also be obtained by assuming the pressure as constant and the force as gravity, with κ representing the gravity constant [156]. The system (5.5) illustrates how free particles collide and stay together to build large-scale structures in the universe. ([157], [158]) have shown that the Jacobian matrix of system (5.5) is non-strictly hyperbolic, with repeated eigenvalues. [139] investigated the RP for system (5.5) and found the Riemann solution, which contains a delta shock wave and a vacuum state. ([130], [159]) have mentioned the homogeneous vanishing pressure gas dynamics model of RP.

The RP of a one-dimensional strictly hyperbolic system of conservation laws is known to have a solution, at least for slightly varying initial data (see [5, 14, 125, 127]), which is not the case for weak Hyperbolic systems. Consequently, it's predictable that some of them would come up with non-classical solutions. Delta shock waves are a possible non-classical solution. Delta shock waves are a novel type of discontinuity that is a generalisation of regular shock waves. At least one of the state variables, in the form of a discontinuous weighted Dirac delta function, can produce extreme concentration. Here is a brief description of some of the consequences of delta-shock waves. Construction of delta-shock waves and vacuum states for zero pressure Euler equations have been studied for isentropic [42], non-isentropic [49], and isothermal [129] fluids using a vanishing pressure approach. The analysis of the RP for non-linear hyperbolic conservation laws is a fascinating problem. Recently, the concentration and cavitation phenomena in the Riemann solutions for the isentropic and non-isentropic pressureless gasdynamics with several equations of state have been studied in [46, 49, 131, 133]. [138] has examined the limiting behavior of Riemann solutions to isentropic Euler equations for LEoS with source term. [128] has investigated the conservation laws of mass, momentum and energy for zero-pressure gas flow and obtained the delta shock in the solution. Riemann problem for non-homogeneous hyperbolic system is also very interesting topic amongst researchers of the mathematics and physics areas. Many authors have investigated the solution to the Riemann problem for the non-homogeneous hyperbolic system (see [134, 135, 139, 140, 155, 160]). The authors in [42, 53, 121, 139] have discussed the delta shocks in Riemann solutions by utilizing the concept of vanishing pressure method. [141] have studied the elementary wave solutions of RP and derived their properties explicitly. The existence and uniqueness of Riemann solutions in magnetogasdynamics have been obtained constructively using the characteristics technique in [143, 144, 145, 161, 162]. The Riemann solution generally determines a general solution's extensive time asymptotic behavior ([163]).

In this chapter, we have studied the stability of the solution of the RP for the non-homogeneous (friction term) hyperbolic system with the magnetic field and LEoS corresponding to the transport equations. As per our assumptions, the stability of the Riemann solution for a non-homogeneous hyperbolic system with the magnetic field and logarithmic equation has not been investigated by any researcher yet. The study of such type of problem is of great interest due to its extensive applications in the area of aerodynamics, cosmology, engineering, and astrophysics.

The following sections comprise this chapter: The non - homogeneous system (5.1) is changed into a homogeneous conservative system employing new state variables in Sect.5.2, and the general properties of the modified system are determined. For the modified system, the classical Riemann solution is also studied. The Riemann invariants for these characteristic fields have been computed. In sect.5.3, the Riemann solution for the transport equations, where pressure and magnetic field vanish in the gas dynamics model with friction term, is obtained. In Sect.5.4, the limiting behavior of Riemann solutions for the system (5.1) and (5.2) with the initial condition as given in (5.3) when $c_1, c_2 \to 0$ is discussed. We observed that how a delta shock wave emerges from a solution containing two shock waves when $\nu_- > \nu_+$, how a vacuum state arises from a solution containing two rarefaction waves when $\nu_- < \nu_+$, and when $\nu_- = \nu_+$, the contact discontinuity is formed from a solution containing either a rarefaction wave followed by a shock wave and vice versa. Finally, sect.5.5 is the conclusion of this discussion.

5.2 Riemann problem for modified system

Here, the Euler equations for the magnetogasdynamics with (LEoS) having a source term ($\kappa \neq 0$), like relaxation effect, friction, and damping, are obtained. Introducing the new velocity variable as [164],

$$u(x,t) = \nu(x,t) - \kappa t, \tag{5.6}$$

the equation (5.1) can be rewritten in the conservative form as:

$$\begin{cases} (\rho)_t + (\rho(u+\kappa t))_x = 0, \\ (\rho u)_t + \left(\rho u(u+\kappa t) + \frac{B^2}{2\mu} + c_2 \ln \rho\right)_x = 0. \end{cases} \tag{5.7}$$

The RP for the reformulated conservative system (5.7) has same initial condition,

$$\rho(x,0) = \begin{cases} \rho_-, x < 0, \\ \rho_+, x > 0, \end{cases}, \quad u(x,0) = \begin{cases} \nu_-, x < 0, \\ \nu_+, x > 0. \end{cases} \tag{5.8}$$

By translating in terms of new variables, $(\rho,\nu)(x,t) = (\rho, u+\kappa t)(x,t)$, the Riemann solution for equation (5.1) and (5.2) with the initial condition (5.3) can be obtained by comparing the equivalent ones for the equation (5.7) and (5.8). Equation (5.7) can be expressed in the quasi-linear form as

$$PW_t + QW_x = 0, \tag{5.9}$$

where

$$W = \begin{bmatrix} \rho \\ \nu \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ \nu & \rho \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \nu + \kappa t & \rho \\ \frac{c_1^2}{\mu} + \frac{c_2}{\rho^2} & \nu + \kappa t \end{bmatrix}. \tag{5.10}$$

We obtain λ_1 and λ_2, two distinct eigenvalues of system (5.10),

$$\lambda_1 = \nu + \kappa t - \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, \quad \lambda_2 = \nu + \kappa t + \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, \tag{5.11}$$

which infers that system (5.7) is strictly hyperbolic.

The right eigen vectors for both eigen values are

$$r_1 = \begin{pmatrix} -\rho \\ \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} \rho \\ \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} \end{pmatrix}, \tag{5.12}$$

respectively. Since for $c_1, c_2 > 0$, $\nabla\lambda_i \cdot r_i \neq 0$, where $i = 1, 2$. Here, $\nabla = (\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\nu})$, which shows that the characteristics roots λ_1 and λ_2 are genuinely non-linear. Hence, the waves in this situation are either rarefaction or shock waves, denoted by R and S. The Riemann invariants for these characteristics fields are

$$\begin{aligned} &1. \quad w = \nu - \int \frac{\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}}{\rho} d\rho, \\ &2. \quad z = \nu + \int \frac{\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}}{\rho} d\rho. \end{aligned} \tag{5.13}$$

5.2.1 Rarefaction waves

Since the equation (5.1) and (5.2) with the initial condition (5.3) are invarient under the new stretching coordinates: $(x, t) \to (\kappa x, \kappa t)$ (κ-constant). We have the self-similar solution $(\rho, u)(x, t) = (\rho, u)(\sigma), \sigma = \frac{x}{t}$, then the system (5.1),(5.2)and with initial data (5.3) are simplified as below

$$\begin{cases} -\sigma(\rho)_\sigma + (\rho(u + \kappa t))_\sigma = 0, \\ -\sigma(\rho u)_\sigma + (\rho u(u + \kappa t) + \frac{c_1^2\rho^2}{2\mu} + c_2 \ln\rho))_\sigma = 0, \end{cases} \tag{5.14}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

We consider a smooth solution, the equation (5.14) can be written as

$$\begin{bmatrix} -\sigma + (u+\kappa t) & \rho \\ -\sigma u + u(u+\kappa t) + \frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho} & -\sigma\rho + \rho(2u+\kappa t) \end{bmatrix} \begin{bmatrix} \rho_\sigma \\ u_\sigma \end{bmatrix} = 0. \tag{5.15}$$

The general solution (constant states) of (5.15) is

$$(\rho, u)(\sigma) = \text{ constant } (\rho > 0),$$

and the singular solution forms 1-rarefaction wave and 2-rarefaction wave,

$$R_1(\rho_-, \nu_-) : \begin{cases} \sigma = \lambda_1 = u + \kappa t - \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, \\ u = \nu_- - \int \frac{\sqrt{\frac{c_1^2\rho}{\mu}+\frac{c_2}{\rho}}}{\rho} d\rho, \\ \rho < \rho_-, \end{cases} \tag{5.16}$$

and

$$R_2(\rho_-, \nu_-) : \begin{cases} \sigma = \lambda_2 = u + \kappa t + \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, \\ u = \nu_- + \int \frac{\sqrt{\frac{c_1^2\rho}{\mu}+\frac{c_2}{\rho}}}{\rho} d\rho, \\ \rho > \rho_-. \end{cases} \tag{5.17}$$

Theorem 5.1. *The curve $R_1(\rho_-, \nu_-)$ is monotonically decreasing and convex.*

Proof. From (5.16), the curve $R_1(\rho_-, \nu_-)$ of 1-rarefaction wave is given by

$$u = \nu_- - \int \frac{\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}}{\rho} d\rho. \tag{5.18}$$

Differentiating above equation w.r.t. ρ, we get

$$\frac{du}{d\rho} = -\frac{\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}}{\rho} < 0, \text{ and } \frac{d^2u}{d\rho^2} = \frac{\frac{c_1^2\rho}{\mu} + \frac{3c_2}{\rho}}{2\rho^2\sqrt{\frac{c_1}{\rho} + \frac{c_2^2\rho}{\mu}}} > 0.$$

From the above calculation, we can conclude that the curve is monotonically decreasing and convex. Also, we can prove $\frac{du}{d\rho} > 0$ and $\frac{d^2u}{d\rho^2} < 0$; that is, the curve $R_2(\rho_-, \nu_-)$ is monotonically increasing and concave in the plane-(ρ, ν) $(\rho > 0)$.

Solving $(5.16)_2$ and $(5.17)_2$ (taking integration with limit ρ_- to ρ) we obtain

$$u = \nu_- + 2\left(\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right),$$

and

$$u = \nu_- - 2\left(\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right),$$

respectively. It can be seen from (5.16) that $\lim_{\rho\to 0^+} u = +\infty$, for the curve $R_1(\rho, \nu)$ which indicates that the asymptotic line of the curve $R_1(\rho, \nu)$ is the positive u-axis and for the curve $R_2(\rho, \nu)$, (5.17) is $\lim_{\rho\to+\infty} u = -\infty$.

5.2.2 Shock waves

For a bounded discontinuity, the Rankine-Hugoniot (R-H) conditions are given as

$$\begin{cases} -\sigma(t)[\rho] + [\rho u(u + \kappa t)] = 0, \\ -\sigma(t)[\rho u] + [\rho u(u + \kappa t) + \frac{c_1^2\rho^2}{2\mu} + c_2 \ln \rho] = 0, \end{cases} \tag{5.19}$$

where $\sigma(t)$ is the velocity of discontinuity and $[\rho] = \rho - \rho_-$. On solving (5.19), we obtain two shocks S_1 (1-shock) and S_2 (2-shock)

$$S_1(\rho_-, \nu_-): \begin{cases} \sigma_1(t) = \frac{(\rho u - \rho u_-)}{\rho - \rho_-} + \kappa t, \\ u = \nu_- - \sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1^2(\rho+\rho_-)}{2\mu} + \frac{c_2(ln\rho - ln\rho_-)}{(\rho-\rho_-)}\right)}(\rho - \rho_-), \\ \rho_- < \rho, u_- > u, \end{cases} \tag{5.20}$$

and

$$S_2(\rho_-, \nu_-): \begin{cases} \sigma_1(t) = \frac{(\rho u - \rho u_-)}{\rho - \rho_-} + \kappa t, \\ u = \nu_- + \sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1^2(\rho+\rho_-)}{2\mu} + \frac{c_2(ln\rho - ln\rho_-)}{(\rho-\rho_-)}\right)}(\rho - \rho_-), \\ \rho_- > \rho, u_- > u. \end{cases} \tag{5.21}$$

Differentiating $(5.20)_2$ with respect to ρ we have

$$\frac{du}{d\rho} = -\frac{\left(\frac{c_1^2}{\rho}\left(\frac{\rho^2}{2\mu\rho_-} - \frac{\rho_-^2}{2\mu\rho}\right) + \frac{c_2}{\rho^2}(\ln\rho - \ln\rho_-) + \left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\frac{c_2}{\rho}\right)}{2\sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1^2(\rho+\rho_-)}{2\mu} + \frac{c_2(ln\rho - ln\rho_-)}{(\rho-\rho_-)}\right)}(\rho - \rho_-)} < 0 for \rho > \rho_-. \tag{5.22}$$

Similarly differentiating $(5.21)_2$ with respect to ρ we have

$$\frac{du}{d\rho} = \frac{\left(\frac{c_1^2}{\rho}\left(\frac{\rho^2}{2\mu\rho_-} - \frac{\rho_-^2}{2\mu\rho}\right) + \frac{c_2}{\rho^2}(\ln\rho - \ln\rho_-) + \left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\frac{c_2}{\rho}\right)}{2\sqrt{\frac{1}{\rho\rho_-}\left(\frac{c_1^2(\rho+\rho_-)}{2\mu} + \frac{c_2(ln\rho - ln\rho_-)}{(\rho-\rho_-)}\right)}(\rho - \rho_-)} > 0 for \rho < \rho_-. \tag{5.23}$$

For $(\rho > \rho_-)$, we conclude that the curve S_1 (1-shock) is monotonically decreasing in the (ρ, ν)-plane. Similarly, we find $\nu_\rho > 0$ for $(\rho < \rho_-)$ from $(5.21)_2$, which shows that the curve S_2 (2- shock) is monotonic increasing in the (ρ, ν)-plane. From (5.20) and (5.21) we can clearly conclude that $\lim_{\rho\to+\infty} u = -\infty$ for curve S_1 (1-shock) and $\lim_{\rho\to 0^+} u = -\infty$ for curve S_2 (2-shock) respectively.

We conclude from the above two equations that 1-shock curve and (2-shock curve) are concave (convex) respectively, which is similar to 1-rarefaction (2-rarefaction) wave curve. There are four possible states, which consist of 1-shock curve $S_1(\rho_-, \nu_-)$, 1-rarefaction curve $R_1(\rho_-, \nu_-)$, 2-shock curve $S_2(\rho_-, \nu_-)$ and the 2-rarefaction curve $R_2(\rho_-, \nu_-)$. Hence, for the left state (ρ_-, ν_-), the curves divide the phase plane into four parts, which is presented in (Fig.5.1) as I, II, III and IV. Now, the construction of unique Riemann solution of the system (5.7) and (5.8), connected with the state (ρ_-, ν_-) and (ρ_+, ν_+) is as follows

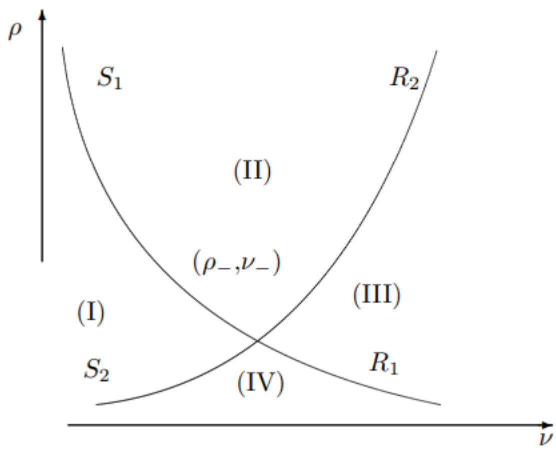

FIGURE 5.1: The (ρ, ν) phase plane for the model (5.1).

1. $(\rho_+, \nu_+) \in \mathrm{I}(\rho_-, \nu_-) : (\rho_-, \nu_-) + \mathrm{S}_1 + (\rho_*, \mathrm{u}_*) + \mathrm{S}_2 + (\rho_+, \nu_+)$,

2. $(\rho_+, \nu_+) \in \mathrm{II}(\rho_-, \nu_-) : (\rho_-, \nu_-) + \mathrm{S}_1 + (\rho_*, \mathrm{u}_*) + \mathrm{R}_2 + (\rho_+, \nu_+)$,

3. $(\rho_+, \nu_+) \in \mathrm{III}(\rho_-, \nu_-) : (\rho_-, \nu_-) + \mathrm{R}_1 + (\rho_*, \mathrm{u}_*) + \mathrm{R}_2 + (\rho_+, \nu_+)$,

4. $(\rho_+, \nu_+) \in \mathrm{IV}(\rho_-, \nu_-) : (\rho_-, \nu_-) + \mathrm{R}_1 + (\rho_*, \mathrm{u}_*) + \mathrm{S}_2 + (\rho_+, \nu_+)$,

where S_2, S_1, R_2, R_1 can be written in symbolically increasing, decreasing forms are $\overrightarrow{S_2}$, $\overleftarrow{S_1}$, $\overrightarrow{R_2}$, $\overleftarrow{R_1}$, respectively (see Fig. 5.1). Also, (ρ_*, u_*) represents the intermediate state. Using (5.6), the Riemann solution of (5.1) and (5.2) with initial condition (5.3) is given by

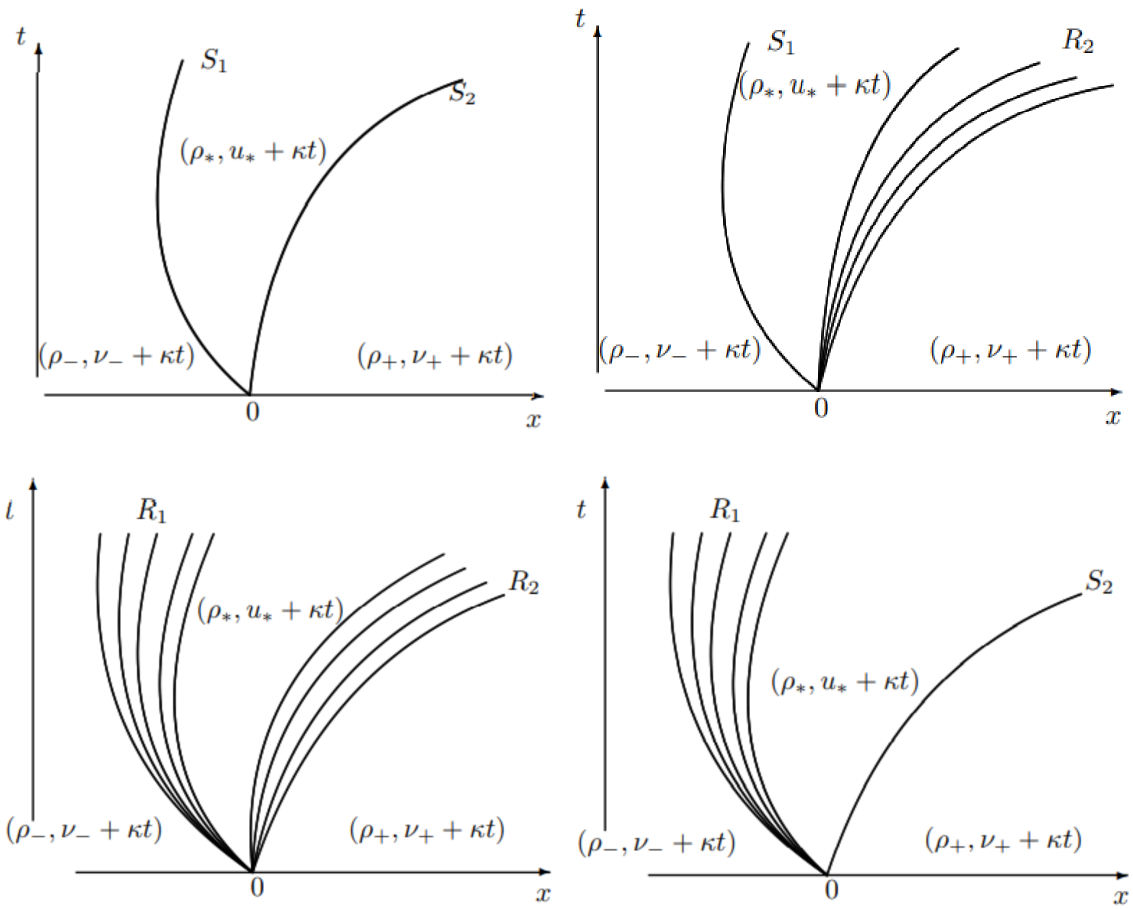

FIGURE 5.2: Solution construction of model (5.1) - (5.2) for case I, II, III and IV are shown in figure, respectively.

1. $(\rho_+, \nu_+) \in \mathrm{I}(\rho_-, \nu_-) : (\rho_-, \nu_- + \kappa \mathrm{t}) + \mathrm{S}_1 + (\rho_*, \mathrm{u}_* + \kappa \mathrm{t}) + \mathrm{S}_2 + (\rho_+, \nu_+ + \kappa \mathrm{t}),$

2. $(\rho_+, \nu_+) \in \mathrm{II}(\rho_-, \nu_-) : (\rho_-, \nu_- + \kappa \mathrm{t}) + \mathrm{S}_1 + (\rho_*, \mathrm{u}_* + \kappa \mathrm{t}) + \mathrm{R}_2 + (\rho_+, \nu_+ + \kappa \mathrm{t}),$

3. $(\rho_+, \nu_+) \in \mathrm{III}(\rho_-, \nu_-) : (\rho_-, \nu_- + \kappa \mathrm{t}) + \mathrm{R}_1 + (\rho_*, \mathrm{u}_* + \kappa \mathrm{t}) + \mathrm{R}_2 + (\rho_+, \nu_+ + \kappa \mathrm{t}),$

4. $(\rho_+, \nu_+) \in \mathrm{IV}(\rho_-, \nu_-) : (\rho_-, \nu_- + \kappa \mathrm{t}) + \mathrm{R}_1 + (\rho_*, \mathrm{u}_* + \kappa \mathrm{t}) + \mathrm{S}_2 + (\rho_+, \nu_+ + \kappa \mathrm{t}),$

5.3 Riemann solution of the transport equations (pressureless and vanishing magneto-gas dynamics model with friction term (5.5))

We briefly present some results on the Riemann problem to the system (5.5) (see [139]). Now using the transformation (5.6), system (5.5) is written as

$$\begin{cases} (\rho)_t + (\rho(u+\kappa t))_x = 0, \\ (\rho u)_t + (\rho u(u+\kappa t))_x = 0. \end{cases} \tag{5.24}$$

Assuming the same initial condition as before (5.3)

$$\rho(x,0) = \begin{cases} \rho_-, x<0, \\ \rho_+, x>0, \end{cases}, \quad u(x,0) = \begin{cases} \nu_-, x<0, \\ \nu_+, x>0. \end{cases} \tag{5.25}$$

The system (5.24) has two equal eigenvalues

$$\lambda_1 = \lambda_2 = u + \kappa t$$

and corresponding to eigenvalues only one right eigenvector which is given as

$$\overrightarrow{r_i} = (u,0)^T.$$

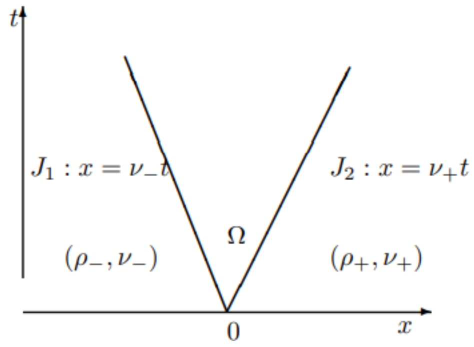

FIGURE 5.3: Characteristics overlapping domain.

Since $\nabla\lambda.\overrightarrow{r_i} = 0$, that is the characteristic fields corresponding to its characteristic roots are linearly degenerate. The self similar solutions are given as

$$(\rho, u)(x,t) = (\rho, u)(\chi), \quad \chi = \frac{x}{t}.$$

The system (5.24) can be reduced to the following boundary value problem

$$\begin{cases} -\chi(\rho)_\chi + (\rho u)_\chi = 0, \\ -\chi(\rho u)_\chi + (\rho u^2)_\chi = 0, \end{cases} \tag{5.26}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

We consider a smooth solution, the equation (5.26) can be written as

$$\begin{bmatrix} u-\chi & \rho \\ 0 & \rho(u-\chi) \end{bmatrix} \begin{bmatrix} \rho_\chi \\ u_\chi \end{bmatrix} = 0, \tag{5.27}$$

which ensures either singular solution

$$\begin{cases} \rho = 0, \\ u = \chi, \end{cases} \tag{5.28}$$

or the general solution which is constant

$$(\rho, u)(\chi) = k \quad (\rho \neq 0) \quad (k - constant).$$

This is known as vaccum state, where u is an arbitrary smooth function.

The R-H conditions hold for bounded discontinuity at $(\chi = \sigma(t))$:

$$\begin{cases} -\sigma(t)[\rho] + [\rho(u + \kappa t)] = 0, \\ -\sigma(t)[\rho u] + [\rho u(u + \kappa t)] = 0, \end{cases} \tag{5.29}$$

where $[\rho] = \rho_+ - \rho_-$ is jump discontinuity across ρ. Considering the equation (5.29) we have

$$J : \chi = \sigma(t) = (\nu_- + \kappa t)(= \lambda_-) = (\nu_+ + \kappa t)(= \lambda_+). \tag{5.30}$$

This is a contact discontinuity. It's a slip line, only the characteristics of the two-sided solution in the (x, t) plane.

The Riemann problems (5.24) and (5.25) can be solved by contact jumps, vacuums, or delta shock wave connecting two steady states $(\rho_\pm, \nu_\pm)$. There arises three cases.
Case-1: $\nu_- > \nu_+$, as shown in the figure, the characteristic curves starting from the origin overlap in the range of Ω (see Fig. 5.3). Therefore, the singularity should be displayed in Ω. Bounded jumps do not meet the R-H condition, so the singularity is not a finite-amplitude jump. In other words, there is no piecewise smooth and bordered solution. Motivated by [139], look for a solution with a delta distribution in the jump. The solution can be expressed by:

$$(\rho_-, \nu_-) + \delta S + (\rho_+, \nu_+).$$

Consider the solutions of (5.24) and (5.25) in the following forms

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, \nu_-), & x < \sigma(t), \\ (\omega(t)\delta(x-\sigma(t)), \nu_\delta), & x = \sigma(t), \\ (\rho_+, \nu_+), & x > \sigma(t), \end{cases} \tag{5.31}$$

where $\nu_\delta(t) = u_\delta + \kappa t$ and $\omega(t)$, are known as velocity of delta shock wave and weight, respectively and on this delta shock wave curve, ν_δ represents the intermediate variable. In case of the formation of a delta-shock wave, the generalized R-H conditions are of the form

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t), \\ \frac{d\omega(t)}{dt} = \sigma(t)[\rho] - [\rho\nu], \\ \frac{d(\omega(t)\sigma)}{dt} = \sigma(t)[\rho\nu] - [\rho\nu^2], \end{cases} \tag{5.32}$$

with the initial condition $(x(0), \omega(0)) = (0,0)$, to guarantee the unique solutions. The delta-shock must be satisfied by the entropy condition:

$$\nu_+ + \kappa t < \sigma < \nu_- + \kappa t.$$

Calculating the system (5.32) with the given initial data we obtain,

when $\rho_- = \rho_+$,

$$\sigma = \frac{(\nu_- + \nu_+)}{2}, \;\; x(t) = \frac{(\nu_- + \nu_+)}{2} t, \;\; \omega(t) = (\rho_-\nu_- - \rho_+\nu_+)t;$$

when $\rho_- \neq \rho_+$,

$$\sigma = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \;\; x(t) = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}\nu_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} t, \;\; \omega(t) = \sqrt{\rho_-\rho_+}(\nu_- - \nu_+)t.$$

Case-2: $\nu_- = \nu_+$, clearly, a contact discontinuity can be connected to certain states $(\rho_\pm, \nu_\pm)$. The solution can be expressed as (see [165])

$$(\rho_-, \nu_-) + J + (\rho_+, \nu_+).$$

that is

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, \nu_-), & -\infty < x < \nu_- t + \frac{1}{2}\kappa t^2, \\ (\rho_+, \nu_+), & \nu_- t + \frac{1}{2}\kappa t^2 < x < +\infty. \end{cases} \tag{5.33}$$

Case-3: $\nu_- < \nu_+$, there is no characteristics to pass through the region $\nu_- < \frac{x}{t} < \nu_+$, and the Riemann solution for (5.24) and (5.25) consists of two contact discontinuities plus a vacuum is created in this region. The solution can be written as

$$(\rho_-, \nu_-) + J_1 + Vac + J_2 + (\rho_+, \nu_+).$$

that is,

$$(\rho, \nu)(\chi) = \begin{cases} (\rho_-, \nu_-), & -\infty < x < \nu_- t + \frac{1}{2}\kappa t^2, \\ (0, \chi), & \nu_- t + \frac{1}{2}\kappa t^2 \le x \le \nu_+ t + \frac{1}{2}\kappa t^2, \\ (\rho_+, \nu_+), & \nu_+ t + \frac{1}{2}\kappa t^2 < x < +\infty. \end{cases} \tag{5.34}$$

To define the measure solution, the two-dimensional weighted delta function $\omega(s)\delta_\Gamma$ based on the smooth curve $\Gamma = \{(x(s), t(s)) : m_1 < s < m_2\}$ is defined as:

$$\langle \omega(s)\delta_\Gamma, \psi \rangle = \int_{m_1}^{m_2} \omega(s)\psi(x(s), t(s))ds, \tag{5.35}$$

for $\psi \in C_0^\infty(R \times R^+)$.

5.4 Riemann solution for Limiting behavior

Now we have to analyze the behavior of limit of the Riemann solutions of (5.1) and (5.2) with initial condition (5.3) as $c_1, c_2 \to 0$, which shows the formation of a delta-shock and a vacuum state. The following studies show a very nice representation of the limit.

5.4.1 Formation of delta-shock

In this section, we have discussed the concentration phenomena and the occurrence of delta shock wave in the Riemann solution of (5.1) and (5.2) with initial condition (5.3) as $c_1, c_2 \to 0$, when $\nu_+ < \nu_-$.

When $(\rho_+, \nu_+) \in I(\rho_-, \nu_-)$, for fixed $c_1, c_2 > 0$, let us consider the intermediate state (ρ_*, ν_*) connected with (ρ_-, ν_-) by a S_1 (1-shock) and (ρ_+, ν_+) by a S_2 (2-shock) with shock speed σ_1 and σ_2 respectively (see Fig. 5.4), are follows

$$S_1(\rho_-, \nu_-) : \begin{cases} \sigma_1(t) = \nu_- + \kappa t - \rho_* \sqrt{\frac{1}{\rho_* \rho_-} \left(\frac{c_1^2(\rho_* + \rho_-)}{2\mu} + \frac{c_2(ln\rho_* - ln\rho_-)}{(\rho_* - \rho_-)} \right)}, \\ u_* = \nu_- - \sqrt{\frac{1}{\rho_* \rho_-} \left(\frac{c_1^2(\rho_* + \rho_-)}{2\mu} + \frac{c_2(ln\rho_* - ln\rho_-)}{(\rho_* - \rho_-)} \right)} (\rho_* - \rho_-), \\ \rho_- < \rho_*, \end{cases} \tag{5.36}$$

and

$$S_2(\rho_-, \nu_-) : \begin{cases} \sigma_2(t) = u_* + \kappa t + \rho_+ \sqrt{\frac{1}{\rho_* \rho_+} \left(\frac{c_1^2(\rho_+ + \rho_*)}{2\mu} + \frac{c_2(ln\rho_+ - ln\rho_*)}{(\rho_+ - \rho_*)} \right)}, \\ \nu_+ = u_* + \sqrt{\frac{1}{\rho_* \rho_+} \left(\frac{c_1^2(\rho_+ + \rho_*)}{2\mu} + \frac{c_2(ln\rho_+ - ln\rho_*)}{(\rho_+ - \rho_*)} \right)} (\rho_+ - \rho_*), \\ \rho_+ < \rho_*, \end{cases} \tag{5.37}$$

Lemma 5.2.

$$\lim_{c_1,c_2\to 0}\rho_* = +\infty.$$

Proof From (5.35) and (5.36),we have

$$\begin{aligned}\nu_- - \nu_+ = &\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\right)\left(\frac{c_1^2(\rho_*^2-\rho_-^2)}{2\mu}+c_2(ln\rho_*-ln\rho_-)\right)}\\ &+\sqrt{\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\left(\frac{c_1^2(\rho_+^2-\rho_*^2)}{2\mu}+c_2(ln\rho_+-ln\rho_*)\right)}, \rho_*>\rho_\pm.\end{aligned} \tag{5.38}$$

If $\lim_{c_1,c_2\to 0}\rho_* = L \in (max(\rho_-,\rho_+),+\infty)$, then we get $\nu_- - \nu_+ = 0$, by choosing the limit of (5.38) as $c_1, c_2 \to 0$, which contradicts the statement that $\nu_- > \nu_+$. Hence $\lim_{c_1,c_2\to 0}\rho_* = +\infty$.

Lemma 5.3.

$$\lim_{c_1,c_2\to 0} c_1^2\rho_*^2 = \frac{2\mu\rho_-\rho_+(\nu_- - \nu_+)^2}{(\sqrt{\rho_-}+\sqrt{\rho_+})^2}.$$

Lemma 5.4.

$$\lim_{c_2\to 0} c_2\ln\rho_* = \frac{\rho_-\rho_+(\nu_- - \nu_+)^2}{(\sqrt{\rho_-}+\sqrt{\rho_+})^2}.$$

Proof Since $\lim_{c_1,c_2\to 0}\rho_* = +\infty$, on taking the limit $c_1, c_2 \to 0$ in system (5.37), we get

$$\nu_- - \nu_+ = \lim_{c_2\to 0}\sqrt{\frac{c_2\ln\rho_*}{\rho_-}} + \lim_{c_2\to 0}\sqrt{\frac{c_2\ln\rho_*}{\rho_+}} = \left(\frac{1}{\sqrt{\rho_-}}+\frac{1}{\sqrt{\rho_+}}\right)\lim_{c_2\to 0}\sqrt{c_2\ln\rho_*}$$

which impies that

$$\lim_{c_2\to 0} c_2\ln\rho_* = \frac{\rho_-\rho_+(\nu_- - \nu_+)^2}{(\sqrt{\rho_-}+\sqrt{\rho_+})^2}.$$

Lemma 5.5. *If $\nu_- > \nu_+$, then*

1. $\lim_{c_1,c_2\to 0}\int_{\sigma_1(t)}^{\sigma_2(t)}\rho_* dx = \nu_\delta(t)[\rho] - [\rho(u+\kappa t)]$,

2. $\lim_{c_1,c_2\to 0}\nu_* = \lim_{c_1,c_2\to 0}(u_*+\kappa t) = \lim_{c_1,c_2\to 0}\sigma_1(t) = \lim_{c_1,c_2\to 0}\sigma_2(t) = \nu_\delta(t)$,

3. $\lim_{c_1,c_2\to 0}\int_{\sigma_1(t)}^{\sigma_2(t)}\rho_* u_* dx = \nu_\delta(t)[\rho u] - [\rho u(u+\kappa t)]$,

where $\nu_\delta(t) = \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}(2\nu_+ - \nu_-)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} + \kappa t.$

Proof From system (5.35),(5.36), Lemma 5.2, Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned}
\lim_{c_1,c_2\to 0}\nu_* &= \lim_{c_1,c_2\to 0}(u_*+\kappa t)\\
&= \nu_- + \kappa t - \lim_{c_1,c_2\to 0}\sqrt{\frac{1}{\rho_*\rho_-}\left(\frac{c_1^2(\rho_*+\rho_-)}{2\mu} + \frac{c_2(\ln\rho_* - \ln\rho_-)}{(\rho_*-\rho_-)}\right)(\rho_*-\rho_-)},\\
&= \nu_- + \kappa t - \lim_{c_1,c_2\to 0}\sqrt{\frac{c_1^2\rho_*^2}{2\mu\rho_-}} - \lim_{c_1,c_2\to 0}\sqrt{\frac{c_2\ln\rho_*^2}{\rho_-}},\\
&= \nu_- + \kappa t - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})},\\
&= \nu_- - \frac{2\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} + \kappa t,\\
&= \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}(2\nu_+ - \nu_-)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} + \kappa t.
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{c_1,c_2\to 0}\sigma_1(t) &= \nu_- + \kappa t - \lim_{c_1,c_2\to 0}\sqrt{\frac{\rho_*}{\rho_-(\rho_*-\rho_-)}\left(\frac{c_1^2(\rho_*^2-\rho_-^2)}{2\mu} + c_2(\ln\rho_* - \ln\rho_-)\right)},\\
&= \nu_- + \kappa t - \lim_{c_1,c_2\to 0}\sqrt{\frac{c_1^2\rho_*^2}{2\mu\rho_-} + \frac{c_2\ln\rho_*}{\rho_-}},\\
&= \nu_- + \kappa t - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} - \frac{\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})},\\
&= \nu_- - \frac{2\sqrt{\rho_+}(\nu_- - \nu_+)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} + \kappa t,\\
&= \frac{\sqrt{\rho_-}\nu_- + \sqrt{\rho_+}(2\nu_+ - \nu_-)}{(\sqrt{\rho_-}+\sqrt{\rho_+})} + \kappa t.
\end{aligned}$$

Similarly,

$$\lim_{c_1,c_2\to 0}\sigma_2(t) = \lim_{c_1,c_2\to 0}\left(u_* + \kappa t + \sqrt{\frac{\rho_+}{\rho_*(\rho_* - \rho_+)}\left(\frac{c_1^2(\rho_+^2 - \rho_*^2)}{2\mu} + c_2(\ln\rho_+ - \ln\rho_*)\right)}\right),$$
$$= \lim_{c_1,c_2\to 0}(u_* + \kappa t) = \nu_\delta(t).$$

Hence,

$$\lim_{c_1,c_2\to 0}\nu_* = \lim_{c_1,c_2\to 0}(u_* + \kappa t) = \lim_{c_1,c_2\to 0}\sigma_1(t) = \lim_{c_1,c_2\to 0}\sigma_2(t) = \nu_\delta(t).$$

For S_1 and S_2, the first equation of R-H jump conditions (5.19), yields

$$\sigma_1(t)(\rho_* - \rho_-) = \rho_*(u_* + \kappa t) - \rho_-(\nu_- + \kappa t), \tag{5.39}$$

$$\sigma_2(t)(\rho_+ - \rho_*) = \rho_+(\nu_+ + \kappa t) - \rho_*(u_* + \kappa t). \tag{5.40}$$

From Equ. (5.39) and (5.40)

$$\rho_*(\sigma_2(t) - \sigma_1(t)) = \sigma_2(t)\rho_+ - \sigma_1(t)\rho_- - (\rho_+\nu_+ - \rho_-\nu_-) - (\rho_+ - \rho_-)\kappa t. \tag{5.41}$$

Taking the limit $(c_1, c_2 \to 0)$ on both sides of (5.41), we obtained

$$\lim_{c_1,c_2\to 0}\rho_*(\sigma_2(t) - \sigma_1(t)) = \nu_\delta(t)[\rho] - [\rho(u + \kappa t)],$$
$$\lim_{c_1,c_2\to 0}\int_{\sigma_1(t)}^{\sigma_2(t)}\rho_* dx = \nu_\delta(t)[\rho] - [\rho(u + \kappa t)].$$

For S_1 and S_2, the second equation of R-H jump conditions (5.19), yields

$$\sigma_1(t)(\rho_* u_* - \rho_-\nu_-) = \rho_* u_*(u_* + \kappa t) + \tfrac{c_1^2\rho_*^2}{2\mu} + c_2\ln\rho_* - \rho_-\nu_-(\nu_- + \kappa t) - \tfrac{c_1^2\rho_-^2}{2\mu} - c_2\ln\rho_-, \tag{5.42}$$

$$\sigma_2(t)(\rho_+\nu_+ - \rho_*\nu_*) = \rho_+\nu_+(\nu_+ + \kappa t) - \rho_* u_*(u_* + \kappa t) + \frac{c_1^2\rho_+^2}{2\mu} + c_2 \ln \rho_+ - \frac{c_1^2\rho_*^2}{2\mu} - c_2 \ln \rho_*. \quad (5.43)$$

From Equ. (5.42) and (5.43)

$$\rho_* u_*(\sigma_2(t) - \sigma_1(t)) = \sigma_2(t)\rho_+\nu_+ - \sigma_1(t)\rho_-\nu_- - \rho_+\nu_+(\nu_+ + \kappa t) + \rho_-\nu_-(\nu_- + \kappa t)$$
$$+ (\rho_+ - \rho_-)\kappa t + \frac{c_1^2\rho_+^2}{2\mu} + c_2 \ln \rho_+ - \frac{c_1^2\rho_*^2}{2\mu} - c_2 \ln \rho_*. \quad (5.44)$$

Taking the limit $(c_1, c_2 \to 0)$ on both sides of (5.44), we obtain

$$\lim_{c_1,c_2\to 0} \rho_* u_*(\sigma_2(t) - \sigma_1(t)) = \nu_\delta(t)[\rho u] - [\rho u(u + \kappa t) + \frac{c_1^2\rho^2}{2\mu} + c_2 \ln \rho],$$
$$\lim_{c_1,c_2\to 0} \int_{\sigma_1(t)}^{\sigma_2(t)} \rho_* u_* dx = \nu_\delta(t)[\rho u] - [\rho u(u + \kappa t) + \frac{c_1^2\rho^2}{2\mu} + c_2 \ln \rho].$$

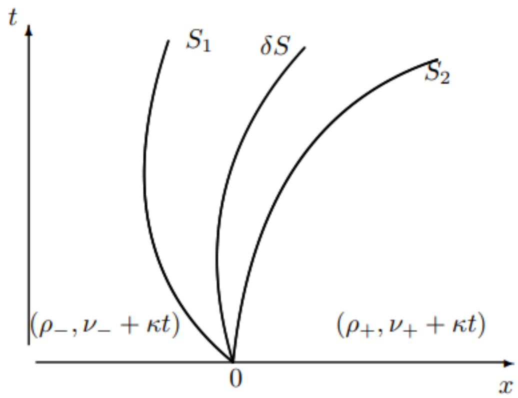

FIGURE 5.4: Riemann solution when $(\rho_+, \nu_+) \in \mathrm{S_1S_2}(\rho_-, \nu_-)$ in (x, t)-plane

Remark 5.6. In Lemma 5.2 and Lemma 5.5, we see that, as $c_1, c_2 \to 0$, both S_1 and S_2 curves overlap (see Figure 5.4), the density of the intermediate state ρ_* becomes singular, and the limit of ρ_* has a singularity, which is a weighted Dirac delta function with speed $\nu_\delta(t)$.

Based on the above analysis, we have obtained the following conclusion.

Theorem 5.7. *When $\nu_- > \nu_+$ and $c_1, c_2 \to 0$, then the Riemann solution of (5.1) and (5.2) with initial condition (5.3) having two shock waves converges to delta shock solution of equation (5.5).*

5.4.2 Formation of vaccum state

In this section, we discuss the cavitation phenomenon and occurrence of vacuum state in Riemann solution to (5.1)-(5.2) with condition (5.3) as $(c_1, c_2 \to 0)$, when $\nu_+ > \nu_-$ (see Fig. 5.5).

Lemma 5.8. *If $\nu_+ > \nu_-$, there exists a sufficiently small $c > 0$ such that $0 < c_1, c_2 < c$ and $(\rho_+, \nu_+) \in R_1R_2(\rho_-, \nu_-)$.*

Proof If $\rho_- = \rho_+$, then $(\rho_+, \nu_+) \in R_1R_2(\rho_-, \nu_-)$, for any $(c_1, c_2 > 0)$. Now we have to focus when $\rho_- \neq \rho_+$. From (5.16) and (5.17), it can be seen that the desirable states (ρ, u) that can be connected to the left state (ρ_-, ν_-) on the right by a R_1 or a R_2 must hold

$$
\begin{aligned}
R_1 : u &= \nu_- + 2\left(\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right), \rho < \rho_-, \\
R_2 : u &= \nu_- + 2\left(\sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}} - \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}\right), \rho > \rho_-.
\end{aligned}
\tag{5.45}
$$

If $\rho_- \neq \rho_+$ and $(\rho_+, \nu_+) \in R_1R_2(\rho_-, \nu_-)$, then from (5.45), we have

$$
\begin{aligned}
\nu_+ &> \nu_- + 2\left(\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right), \rho_+ < \rho_-, \\
\nu_+ &> \nu_- + 2\left(\sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}} - \sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}}\right), \rho_+ > \rho_-,
\end{aligned}
\tag{5.46}
$$

from (5.46), we obtain

$$\left|2\left(\sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}}-\sqrt{\frac{c_1^2\rho_+}{\mu}+\frac{c_2}{\rho_+}}\right)\right|<(\nu_+-\nu_-),$$

which implies that

$$c_1,c_2<\frac{(\nu_+-\nu_-)^2}{4\left(\sqrt{\frac{\rho_-}{\mu}+\frac{1}{\rho_-}}-\sqrt{\frac{\rho_+}{\mu}+\frac{1}{\rho_+}}\right)^2}.$$

Let $C=\frac{(\nu_+-\nu_-)^2}{4\left(\sqrt{\frac{\rho_-}{\mu}+\frac{1}{\rho_-}}-\sqrt{\frac{\rho_+}{\mu}+\frac{1}{\rho_+}}\right)^2}$, then $(\rho_+,\nu_+)\in R_1R_2(\rho_-,\nu_-)$, when $0<c_1,c_2<C$. When $\nu_-<\nu_+$, by Lemma 5.8 for any given $c_1,c_2\in(0,C)$ the Riemann solution of (5.1)-(5.2) with (5.3) is given below

$$(\rho_-,\nu_-+\kappa t)+R_1+(\rho_*,u_*+\kappa t)+R_2+(\rho_+,\nu_++\kappa t),$$

where

$$R_1(\rho_-,\nu_-):\begin{cases}\frac{dx}{dt}=\lambda_1=u+\kappa t-\sqrt{\frac{c_1^2\rho}{\mu}+\frac{c_2}{\rho}},\\ u_*=\nu_-+2\left(\sqrt{\frac{c_1^2\rho_*}{\mu}+\frac{c_2}{\rho_*}}-\sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}}\right),\\ \rho_*\le\rho\le\rho_-,\end{cases}\tag{5.47}$$

and

$$R_2(\rho_-,\nu_-):\begin{cases}\frac{dx}{dt}=\lambda_1=u+\kappa t+\sqrt{\frac{c_1^2\rho}{\mu}+\frac{c_2}{\rho}},\\ u_*=\nu_++2\left(\sqrt{\frac{c_1^2\rho_+}{\mu}+\frac{c_2}{\rho_+}}-\sqrt{\frac{c_1^2\rho_*}{\mu}+\frac{c_2}{\rho_*}}\right),\\ \rho_*\le\rho\le\rho_+.\end{cases}\tag{5.48}$$

From (5.47) and (5.48), it can be derived that

$$u_*=\frac{\nu_-+\nu_+}{2}+\left(\sqrt{\frac{c_1^2\rho_+}{\mu}+\frac{c_2}{\rho_+}}-\sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}}\right),\tag{5.49}$$

$$\nu_+ - \nu_- + 2\left(\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} - \sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} + \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right) = 0,$$

$$2\sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} = \frac{\nu_+ - \nu_-}{2} + \sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} + \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}. \tag{5.50}$$

Theorem 5.9. *In the case $\nu_+ > \nu_-$, the Riemann solution of (5.1) - (5.2) with initial condition (5.3), which contains two rarefaction waves, converges to two contact discontinuities joining the states $(\rho_-, \nu_- + \kappa t)$ and $(\rho_+, \nu_+ + \kappa t)$ accompanied by a vacuum state $(\rho = 0)$ between them as $c_1, c_2 \to 0$, neglecting the virtual velocity in the vacuum state, that consists of just the Riemann solution of (5.3) and (5.5).*

Proof We obtained from (5.50) that

$$\nu_+ - \nu_- = 4\sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} - 2\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - 2\sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}. \tag{5.51}$$

Let us assume that

$$\lim_{c_1,c_2\to 0} \rho_* = M \in (0, min(\rho_-, \rho_+)).$$

On taking the limit $(c_1, c_2 \to 0)$ in (5.51), we get $\nu_- = \nu_+$, which is in contradiction to the fact that $\nu_- < \nu_+$. Hence, $\lim_{c_1,c_2\to 0} \rho_* = 0$, indicating the presence of vacuum state as $c_1, c_2 \to 0$.

The Riemann solution of (5.1), (5.2) with initial condition (5.3) can be expressed directly in the following way:

$$(\rho,\nu)(x,t) = \begin{cases} (\rho_-, \nu_- + \kappa t), & -\infty < \xi < \lambda_1(\rho_-, \nu_-), \\ (\rho_{R_1}, \nu_{R_1}), & \lambda_1(\rho_-, \nu_-) \le \xi \le \lambda_1(\rho_*, u_*), \\ (\rho_*, u_* + \kappa t), & \lambda_1(\rho_*, u_*) < \xi < \lambda_2(\rho_*, u_*), \\ (\rho_{R_2}, \nu_{R_2}), & \lambda_2(\rho_*, u_*) \le \xi \le \lambda_2(\rho_+, \nu_+), \\ (\rho_+, \nu_+ + \kappa t), & \lambda_2(\rho_+, \nu_+) < \xi < +\infty. \end{cases} \tag{5.52}$$

The state

$$(\rho_{R_1}, \nu_{R_1}) = \left(2\xi + 2\sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}} - \nu_- - \kappa t, c\left(\xi + 2\sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}} - \nu_- - \kappa t\right)^{-2}\right),$$

is involved in the rarefaction wave fan R_1 which can be uniquelly expressed by

$$\xi = u + \kappa t - \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, u = \nu_- + 2\left(\sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right),$$

and the state

$$(\rho_{R_2}, \nu_{R_2}) = \left(2\xi - 2\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \nu_+ - \kappa t, c\left(\nu_+ + \kappa t + 2\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \xi\right)^{-2}\right),$$

is involved in the fan region of rarefaction wave R_2 which are provided by

$$\xi = u + \kappa t + \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}, u = \nu_+ + 2\left(\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \sqrt{\frac{c_1^2\rho}{\mu} + \frac{c_2}{\rho}}\right).$$

From (5.49) and (5.50), it can be observed that

$$\lim_{c_1,c_2\to 0}(\rho_*, u_*) = \left(\frac{\nu_- + \nu_+}{2}, 0\right).$$

Consider the limit $(c_1, c_2 \to 0)$ in (5.50), which gives

$$\lim_{c_1,c_2\to 0}\sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} = \frac{\nu_+ - \nu_-}{4}.$$

From the other side, a straight forward calculation leads to

$$\begin{aligned}\lim_{c_1,c_2\to 0}\lambda_1(\rho_*,u_*) &= \lim_{c_1,c_2\to 0}\left(u_* + \kappa t - \sqrt{\frac{c_1^2\rho_*}{\mu}+\frac{c_2}{\rho_*}}\right),\\ &= \frac{\nu_- + \nu_+}{2} - \frac{\nu_+ - \nu_-}{4} + \kappa t,\\ &= \frac{3\nu_- + \nu_+}{4} + \kappa t,\end{aligned}$$

$$\begin{aligned}\lim_{c_1,c_2\to 0}\lambda_1(\rho_-,\nu_-) &= \lim_{c_1,c_2\to 0}\left(\nu_- + \kappa t - \sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}}\right),\\ &= \nu_- + \kappa t.\end{aligned}$$

In the limit $c_1, c_2 \to 0$, the tail of 1-rarefaction wave R_1 matches with the contact discontinuity with speed $\nu_- + \kappa t$, while the head of 1-rarefaction wave R_1 matches with the contact discontinuity with speed $\frac{3\nu_- + \nu_+}{4} + \kappa t$. Furthermore, it can be seen that

$$\lim_{c_2,c_2\to 0}\left(2\xi + 2\sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}} - \nu_- - \kappa t, c\left(\xi + 2\sqrt{\frac{c_1^2\rho_-}{\mu}+\frac{c_2}{\rho_-}} - \nu_- - \kappa t\right)^{-2}\right) = (2\xi - \nu_- - \kappa t, 0),$$

for $\xi > \nu_- + \kappa t$. It shows that as $c \to 0$, the state in R_1 represents the vacuum state in which the virtual velocity is $(2\xi - \nu_- - \kappa t)$. From the other side, a straight forward calculation leads to

$$\begin{aligned}\lim_{c_1,c_2\to 0}\lambda_2(\rho_*,u_*) &= \lim_{c_1,c_2\to 0}\left(u_* + \kappa t + \sqrt{\frac{c_1^2\rho_*}{\mu}+\frac{c_2}{\rho_*}}\right),\\ &= \frac{\nu_- + \nu_+}{2} + \frac{\nu_+ - \nu_-}{4} + \kappa t,\\ &= \frac{\nu_- + 3\nu_+}{4} + \kappa t,\end{aligned}$$

$$\lim_{c_1,c_2\to 0}\lambda_2(\rho_+,\nu_+) = \lim_{c_1,c_2\to 0}\left(\nu_+ + \kappa t + \sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}}\right),$$
$$= \nu_+ + \kappa t.$$

In the limit $c_1, c_2 \to 0$, the tail of 2-rarefaction wave R_2 matches with the contact discontinuity with speed $\frac{\nu_- + 3\nu_+}{4} + \kappa t$, while the head of 2-rarefaction wave R_2 matches with the contact discontinuity with speed $\nu_+ + \kappa t$. Furthermore, it can be seen that

$$\lim_{c_1,c_2\to 0}\left(2\xi - 2\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \nu_+ - \kappa t, c\left(\nu_+ + \kappa t + 2\sqrt{\frac{c_1^2\rho_+}{\mu} + \frac{c_2}{\rho_+}} - \xi\right)^{-2}\right) = (2\xi - \nu_+ - \kappa t, 0),$$

for $\xi < \nu_+ + \kappa t$. It shows that as $c \to 0$, the state in R_2 represents the vacuum state in which virtual velocity is $(2\xi - \nu_+ - \kappa t)$. On taking $(c_1, c_2 \to 0)$ in (5.52), we obtain

$$\lim_{c_1,c_2\to 0}(\rho,\nu)(\xi) = \begin{cases} (\rho_-, \nu_- + \kappa t), & -\infty < \xi < \nu_- + \kappa t, \\ (2\xi - \nu_- - \kappa t, 0), & \nu_- + \kappa t \le \xi \le \frac{3\nu_- + \nu_+}{4} + \kappa t, \\ (\frac{\nu_- + \nu_+}{2} + \kappa t, 0), & \frac{3\nu_- + \nu_+}{4} + \kappa t < \xi < \frac{\nu_- + 3\nu_+}{4} + \kappa t, \\ (2\xi - \nu_+ - \kappa t, 0), & \frac{\nu_- + 3\nu_+}{2} + \kappa t \le \xi \le \nu_+ + \kappa t, \\ (\rho_+, \nu_+ + \kappa t), & \nu_+ + \kappa t < \xi < +\infty. \end{cases} \tag{5.53}$$

Which is made up of three different vacuum states; these separate four contact discontinuities are different from the solution obtained in (5.34). When $\nu_- < \nu_+$, if we ignore the virtual velocity in the vacuum state, we can see that in the limit $c_1, c_2 \to 0$, the Riemann solutions (5.1) and (5.2) with initial condition (5.3) is a vacuum region separating two contact discontinuities on either side, which corresponds to the Riemann solutions (5.5) and (5.3) (see Fig.5.5).

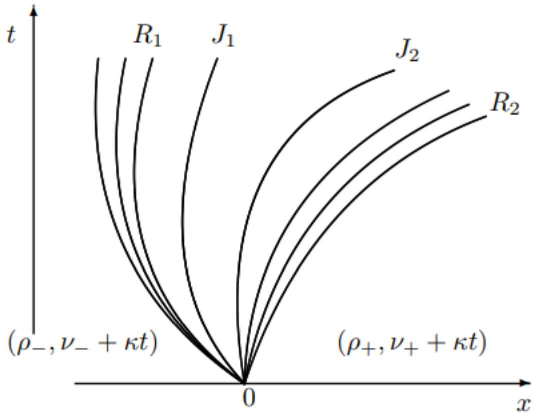

FIGURE 5.5: Riemann solution when $(\rho_+, \nu_+) \in \mathrm{R}_1\mathrm{R}_2(\rho_-, \nu_-)$ in (x, t)-plane, where $J_1 : x = \nu_- t + \frac{1}{2}\kappa t^2$ and $J_2 : x = \nu_+ t + \frac{1}{2}\kappa t^2$

Remark 5.10. [166] established that the velocity in the vacuum region is virtual in the study of isentropic gas dynamics equations. They illustrated the contrast between different vacuum regions with varying velocities, then dealt with a vacuum region that was similar to the rarefaction wave. In this section, we are not considering virtual velocity in the vacuum region.

5.4.3 Formation of contact discontinuity

This segment is consisting of the behavior of the Riemann solution for system (5.1) when $\nu_- = \nu_+$ as under the limit $(c_1, c_2 \to 0)$. Now we consider the case $(\rho_+, \nu_+) \in S_1R_2 \cup R_1S_2(\rho_-, \nu_-)$, the Riemann solution consists of either a S_1 (1-shock) followed by a R_2 (2-Rarefaction) or a R_1 (1-Rarefaction) followed by S_2 (2-Shock wave). This discussion has two cases that depend on the right-side (ρ_+, ν_+) in a different state. If $(\rho_+, \nu_+) \in R_1S_2(\rho_-, \nu_-)$, then the Riemann solution can be expressed as

$$(\rho_-, \nu_- + \kappa t) + R_1 + (\rho_*, \nu_* + \kappa t) + S_2 + (\rho_+, \nu_+ + \kappa t),$$

where R_1 and S_2 are given by (5.47) and (5.37). Intermediate state $(\rho_*, u_* + \kappa t)$ can be calculated from the following relation

$$\begin{cases} u_* = \nu_- + 2\left(\sqrt{\frac{c_1^2\rho_*}{\mu} + \frac{c_2}{\rho_*}} - \sqrt{\frac{c_1^2\rho_-}{\mu} + \frac{c_2}{\rho_-}}\right), & \rho_- > \rho_*, \\ \nu_+ = u_* + \sqrt{\frac{1}{\rho_*\rho_+}\left(\frac{c_1^2(\rho_+ + \rho_*)}{2\mu} + \frac{c_2(ln\rho_+ - ln\rho_*)}{(\rho_+ - \rho_*)}\right)(\rho_+ - \rho_*)}, & \rho_* > \rho_+. \end{cases}$$

This demonstrates that ρ_* is satisfies for

$$\rho_+ < \rho_* < \rho_-.$$

Taking the limit $(c_1, c_2 \to 0)$ in (5.47), we have

$$\lim_{c_1,c_2\to 0} u = \nu_- on \ \ R_1, \lim_{c_1,c_2\to 0} \lambda_1 = \nu_- + \kappa t. \tag{5.54}$$

Similarly, if we consider the limit $(c_1, c_2 \to 0)$ in (5.37), we get

$$\lim_{c_1,c_2\to 0} Q_2(t) = \lim_{c_1,c_2\to 0}(u_* + \kappa t) = \nu_+ + \kappa t = \nu_- + \kappa t. \tag{5.55}$$

From (5.54) and (5.55), it is clear that as $c_1, c_2 \to 0$ the rarefaction wave R_1 and the shock wave S_2 converges to a single contact discontinuity that connects the states $(\rho_-, \nu_- + \kappa t)$ and $(\rho_+, \nu_+ + \kappa t)$ with the speed $\nu_- + \kappa t$ (see Fig.5.6).

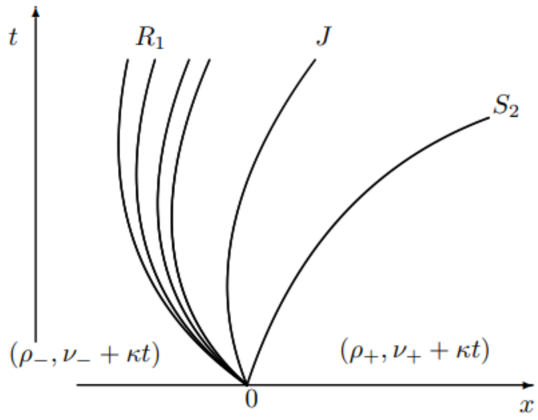

FIGURE 5.6: Riemann solution when $(\rho_+, \nu_+) \in \mathrm{R_1S_2}(\rho_-, \nu_-)$ in (x,t)-plane, where $J : x = \nu_- t + \frac{1}{2}\kappa t^2$

Similarly, when $(\rho_+, \nu_+) \in S_1R_2(\rho_-, \nu_-)$, then the Riemann solution of (5.1) can be expressed as

$$(\rho_-, \nu_- + \kappa t) + S_1 + (\rho_*, u_* + \kappa t) + R_2 + (\rho_+, \nu_+ + \kappa t),$$

where S_1 and R_2 are given by (5.36) and (5.48). In this case also ρ_* satisfies

$$\rho_- < \rho_* < \rho_+.$$

Taking the limit $c_1, c_2 \to 0$ in (5.36), we have

$$\lim_{c_1,c_2\to 0} \sigma_1(t) = \lim_{c_1,c_2\to 0} (u_* + \kappa t) = \nu_- + \kappa t = \nu_+ + \kappa t. \tag{5.56}$$

Similarly, taking the limit $c_1, c_2 \to 0$ in (5.48), we have

$$\lim_{c_1,c_2\to 0} u = \nu_+ \quad on \quad R_2, \lim_{c_1,c_2\to 0} \lambda_2 = \nu_+ + \kappa t. \tag{5.57}$$

We can obtain the following conclusion based on the above discussion.

Theorem 5.11. *The contact discontinuity which is a convergent solution of the Riemann problem governed by the system (5.1) and (5.2), given as*

$$\lim_{c\to 0}(\rho_c,\nu_c)(x,t) = \begin{cases} (\rho_-,\nu_- + \kappa t), & -\infty < x < \nu_- t + \frac{1}{2}\kappa t^2, \\ (\rho_+,\nu_+ + \kappa t), & \nu_- t + \frac{1}{2}\kappa t^2 < x < +\infty, \end{cases} \tag{5.58}$$

when $\nu_- = \nu_+$, as $c_1, c_2 \to 0$. Which is followed by the Riemann solution of (5.3) and (5.5).

5.5 Conclusions

In the present chapter, we have studied the phenomena of concentration and cavitation in the Riemann solution for the non-homogeneous hyperbolic system with LEoS and magnetic field. We established that the Riemann solution of (5.1), (5.2) with initial condition (5.3) are no longer self-similar due to presence of source term. We investigated the Riemann solution to the systems (5.1) and (5.2) with discontinuous initial condition (5.3) which forms a delta shock wave and a vacuum state when pressure and magnetic field vanishes. This is proved that when $\nu_- > \nu_+$, the solution of RP (5.1)-(5.3) converges to the corresponding solution of RP (5.5) and (5.3) as $c_1, c_2 \to 0$, and for the case $\nu_- < \nu_+$ as $c_1, c_2 \to 0$, the solution of RP (5.1)-(5.3) consisting of two rarefaction waves become two contact discontinuities and a vacuum region in between them. Hence, we conclude that for the case $\nu_- > \nu_+$, the solutions converge to the corresponding solutions of (5.5) and (5.3) as $c_1, c_2 \to 0$, and for the case $\nu_- < \nu_+$, the solutions of (5.1) and (5.2) with initial condition (5.3) converge to the solution of the transport equation (5.5). The results obtained in this study is in close agreement to the result reported by [131, 42].