Chapter 3

Hankel wavelet multiplier associated with the unitary representation

3.1 Introduction

In the field of partial differential equations, signal analysis, quantum mechanics, and other areas of mathematics, the localization operator played a key role. Many authors have used the theory of localization operators to develop many research works by utilizing the various integral transform tools. Wong et al. [27, 34], Upadhyay [66] examined the boundedness of the localization operator on various functional spaces in terms of the wavelet multiplier. Baccar et al. [2, 3], Ghober [22], and Mejjaoli et al. [36, 37] investigated the properties of wavelet multipliers by using the Hankel transform technique and they discussed the L^p-boundedness and compactness of wavelet multipliers.

Our present intention in this chapter is to study the boundedness of Hankel wavelet multipliers associated with the unitary representation on L^p-space by exploiting the theory of the Hankel transform technique. We also discussed the Hilbert- Schmidt class and the compactness of Hankel wavelet multipliers. The application of Hankel wavelet multipliers in form of the Landau-Pollak Slepian operator is given. An analysis of the properties of the Sobolev-type space associated with the Hankel transform is carried out using the Hankel wavelet multiplier.

By Wong [80], it is shown that the $(\mathbb{R}^+ = (0,\infty), .)$, multiplicative group of a positive real number $\mathbb{R}^+$ (under the subspace topology of the usual topology) is a topological group. We define a new unitary representation, on this multiplicative group. Let $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ be the unitary representation of the multiplicative group $\mathbb{R}^+$ on $L^2(\mathbb{R}^+)$ is defined by

$$(\pi(\zeta)u)(x) = (x\zeta)^{1/2} J_\mu(x\zeta)u(x) \qquad x, \zeta \in (0,\infty), \tag{3.1.1}$$

for all functions, u in $L^2(0,\infty)$ and $U(L^2(0,\infty))$ is the group of all unitary operators on $L^2(0,\infty)$.

Let $\sigma \in L^1(0,\infty) \cap L^\infty(0,\infty)$ and $\phi \in L^2(0,\infty) \cap L^\infty(0,\infty)$. Then for $u, v \in H_\mu$, we give the definition of $R_{\sigma,\phi}$ with the help of the above unitary representation

$$\langle R_{\sigma,\phi}u, v\rangle = \int_0^\infty \sigma(\zeta)\langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta, \quad \zeta \in (0,\infty). \tag{3.1.2}$$

where $\langle , \rangle$ represents the inner product in $L^2(0,\infty)$.

Then, it can be proved easily that $R_{\sigma,\phi}u \in L^2(0,\infty)$ for all $u \in H_\mu$ and $R_{\sigma,\phi}$ initially defined on H_μ, can be extended to a bounded linear operator from $L^2(0,\infty)$ into $L^2(0,\infty)$.

Lemma 3.1.1. *For $u, v \in L^2(0,\infty)$, we obtain*

$$\langle u, \pi(\zeta)\phi\rangle = h_\mu(u\bar{\phi})(\zeta) \tag{3.1.3}$$

and,

$$\langle \pi(\zeta)\phi, v\rangle = h_\mu(\phi\bar{v})(\zeta). \tag{3.1.4}$$

Proof. Since

$$\langle u, \pi(\zeta)\phi\rangle = \int_0^\infty u(x)\overline{(\pi(\zeta)\phi)(x)}dx.$$

By (3.1.1), we get

$$\begin{aligned}\langle u, \pi(\zeta)\phi\rangle &= \int_0^\infty u(x)\overline{(x\zeta)^{1/2}J_\mu(x\zeta)\phi(x)}dx\\ &= h_\mu(u\bar{\phi})(\zeta).\end{aligned}$$

In a similar manner we get,

$$\begin{aligned}\langle \pi(\zeta)\phi, v\rangle &= \int_0^\infty (\pi(\zeta)\phi)(y)\overline{v}(y)dy\\ &= \int_0^\infty (y\zeta)^{1/2}J_\mu(y\zeta)\phi(y)\overline{v}(y)dy\\ &= h_\mu(\phi\overline{v})(\zeta).\end{aligned}$$

□

Lemma 3.1.2. *Let $\phi \in L^2(0,\infty) \cap L^\infty(0,\infty)$, such that $\|\phi\|_2 = 1$. Then for functions $u, v \in H_\mu(I)$, we have*

$$\int_0^\infty \langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta = \langle u\bar{\phi}, \bar{v}\phi\rangle. \tag{3.1.5}$$

Proof. Using Lemma 3.1.1, we have

$$\int_0^\infty \langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta = \int_0^\infty h_\mu(u\bar{\phi})(\zeta)h_\mu(\phi\overline{v})(\zeta)d\zeta.$$

By applying the Parseval relation of the Hankel transform (1.3.13), the above yields

$$\int_0^\infty \langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta = \langle u\bar{\phi}, \phi\overline{v}\rangle.$$

□

Theorem 3.1.3. *Let* $\sigma \in L^\infty(0,\infty)$, $\phi \in L^2(0,\infty)\cap L^\infty(0,\infty)$, *and* $u, v \in H_\mu$, *such that* $\|\phi\|_2 = 1$. *Then we have*

$$\langle R_{\sigma,\phi}u, v\rangle = \langle \phi h_{\mu,\sigma}\bar{\phi}u, v\rangle. \tag{3.1.6}$$

Proof. Using (3.1.2), we have

$$\langle R_{\sigma,\phi}u, v\rangle = \int_0^\infty \sigma(\zeta)\langle u, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, v\rangle d\zeta, \quad \zeta \in I = (0,\infty).$$

By Lemma 3.1.1

$$\langle R_{\sigma,\phi}u, v\rangle = \int_0^\infty \sigma(\zeta)h_\mu(u\bar{\phi})(\zeta)h_\mu(\phi\overline{v})(\zeta)d\zeta.$$

From (1.3.1), we get

$$\begin{aligned}
\langle R_{\sigma,\phi}u, v\rangle &= \int_0^\infty \sigma(\zeta)h_\mu(u\bar{\phi})(\zeta)\left(\int_0^\infty (x\zeta)^{1/2}J_\mu(x\zeta)(\phi\overline{v})(x)dx\right)d\zeta \\
&= \int_0^\infty \left(\int_0^\infty (x\zeta)^{1/2}J_\mu(x\zeta)\sigma(\zeta)h_\mu(u\bar{\phi})(\zeta)d\zeta\right)(\phi\overline{v})(x)dx \\
&= \int_0^\infty \phi(x)h_\mu^{-1}\left(\sigma(\zeta)h_\mu(u\bar{\phi})\right)(x)\overline{v}(x)dx.
\end{aligned}$$

Therefore by definition of inner product, we obtain

$$\begin{aligned}\langle R_{\sigma,\phi}u, v\rangle =&\langle \phi h_{\mu}^{-1}(\sigma(\zeta)h_{\mu}(u\bar{\phi})), v\rangle \\ =&\langle (\phi h_{\mu,\sigma}\bar{\phi})u, v\rangle.\end{aligned}$$

□

Remark: It is proved in Theorem 3.1.3, that the bounded linear operator $R_{\sigma,\phi}$: $L^2(0,\infty) \to L^2(0,\infty)$ and $\phi h_{\mu,\sigma}\bar{\phi} : L^2(0,\infty) \to L^2(0,\infty)$ are unitarily equivalent, hence we denote $\phi h_{\mu,\sigma}\bar{\phi}$ as $R_{\sigma,\phi}$. And here ϕ plays the role of admissible wavelet in the localization operator $R_{\sigma,\phi}$, then the localization operator $R_{\sigma,\phi}$ is the Hankel wavelet multiplier associated with the unitary representation $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ of the multiplicative group $\mathbb{R}^+$.

A function $\phi \in L^2(0,\infty)$ satisfying $\|\phi\|_2 = 1$ and

$$\int_0^\infty |\langle \phi, \pi(\zeta)\phi\rangle|^2 \, d\zeta < \infty \tag{3.1.7}$$

is said to be an admissible wavelet of $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$.

Now we define c_ϕ

$$c_\phi = \int_0^\infty |\langle \phi, \pi(\zeta)\phi\rangle|^2 \, d\zeta. \tag{3.1.8}$$

Using (3.1.1), the above can be written as

$$\begin{aligned}c_\phi &= \int_0^\infty \Big| \int_0^\infty \phi(x)\overline{(\pi(\zeta)\phi)}(x)dx\Big|^2 d\zeta \\ &= \int_0^\infty \Big| \int_0^\infty (x\zeta)^{1/2}J_\mu(x\zeta)\phi^2(x)dx\Big|^2 d\zeta \\ &= \int_0^\infty \big|h_\mu(\phi^2)(\zeta)\big|^2 d\zeta.\end{aligned}$$

By the Parseval formula of the Hankel transform (1.3.13), we have

$$c_\phi = \|\phi\|_2^4. \tag{3.1.9}$$

From the above discussions, our present chapter is organized in the following way: Section 3.1 is introductory, in which the notations, terminologies, definitions and properties are given. In this section, we prove many auxiliary results that will be useful in the subsequent sections. In Section 3.2, it is shown that the Hankel wavelet multiplier $R_{\sigma,\phi}$ are bounded linear operator on L^p space, $1 \leq p \leq \infty$ for a suitable choice of the admissible wavelet ϕ in $L^1(0,\infty) \cap L^\infty(0,\infty)$ and symbol $\sigma \in L^1(0,\infty)$. Furthermore, the Hankel wavelet multiplier $R_{\sigma,\phi}$ is also shown to be bounded linear operator on $L^p(0,\infty)$, $r \leq p \leq r'$ for the symbols $\sigma \in L^r(0,\infty)$ for $1 \leq r \leq 2$ and ϕ in $L^1(0,\infty) \cap L^2(0,\infty) \cap L^\infty(0,\infty)$. In Section 3.3, we have shown that the Hankel wavelet multiplier $R_{\sigma,\phi} : L^2(0,\infty) \to L^2(0,\infty)$ is a Hilbert-Schmidt operator and also studied its compactness property for the suitable choice of admissible wavelet associated with the unitary representation. In Section 3.4, the Landau-Pollak-Slepian operator associated with the unitary representation is introduced. Section 3.5 investigated the properties of the Hankel wavelet multiplier in Sobolev-type space.

3.2 Boundedness of the Hankel wavelet multiplier on $L^p(0,\infty)$

In this section, the boundedness of the Hankel wavelet multiplier $R_{\sigma,\phi}$ on $L^1(0,\infty)$, $L^\infty(0,\infty)$ and $L^p(0,\infty)$ for $1 < p < \infty$ are investigated, for a suitable choice of the admissible wavelet associated with the unitary representation.

The Hankel wavelet multiplier $R_{\sigma,\phi}$ for $\sigma \in L^1(0,\infty)$ and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$ is defined by

$$(R_{\sigma,\phi}f)(x) = \int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta, \quad x, \zeta \in (0,\infty), \tag{3.2.1}$$

for all $f \in L^1(0,\infty)$, and $\langle\ ,\ \rangle$ is defined by

$$\langle u, v\rangle = \int_0^\infty u(x)\bar{v}(x)dx,$$

for all measurable function u and v on $I = (0,\infty)$, whenever the integral exists.

Theorem 3.2.1. *Let $\sigma \in L^1(0,\infty)$, and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then the Hankel wavelet multiplier $R_{\sigma,\phi} : L^1(0,\infty) \to L^1(0,\infty)$ is a bounded linear operator such that*

$$\|R_{\sigma,\phi}\|_{B(L^1(0,\infty))} \le A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty, \tag{3.2.2}$$

where $\|\ \|_{B(L^1(0,\infty))}$ is the norm in the Banach space $B(L^1(0,\infty))$ of all bounded linear operators from $L^1(0,\infty)$ into $L^1(0,\infty)$.

Proof. Using (3.2.1), we have

$$(R_{\sigma,\phi}f)(x) = \int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta.$$

Now taking L^1 norm, we get

$$\begin{aligned}\|R_{\sigma,\phi}f\|_1 &= \int_0^\infty |\int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta|dx \\ &\le \int_0^\infty \left(\int_0^\infty |\sigma(\zeta)| \cdot |\langle f, \pi(\zeta)\phi\rangle| \cdot |(\pi(\zeta)\phi)(x)|d\zeta\right) dx.\end{aligned}$$

Using (3.1.1), we have

$$\|R_{\sigma,\phi}f\|_1 \leq \int_0^\infty \left(\int_0^\infty |\sigma(\zeta)| \cdot |\langle f, \pi(\zeta)\phi\rangle| \cdot |(x\zeta)^{1/2} J_\mu(x\zeta)\phi(x)| d\zeta \right) dx.$$

By Lemma 3.1.1, we obtain

$$\begin{aligned}
\|R_{\sigma,\phi}f\|_1 &\leq A_\mu \int_0^\infty (\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\phi)(\zeta)| \cdot |\phi(x)| d\zeta) dx \\
&\leq A_\mu \sup_{\zeta \in I} |h_\mu(f\phi)(\zeta)| \int_0^\infty \left(\int_0^\infty |\sigma(\zeta)| d\zeta \right) |\phi(x)| dx \\
&\leq A_\mu \|h_\mu(f\phi)\|_\infty \|\sigma\|_1 \|\phi\|_1 \\
&\leq A_\mu \|f\|_1 \|\phi\|_\infty \|\sigma\|_1 \|\phi\|_1.
\end{aligned}$$

Thus the above expression yields the required results,

$$\|R_{\sigma,\phi}\|_{B(L^1(0,\infty))} \leq A_\mu \|\phi\|_\infty \|\sigma\|_1 \|\phi\|_1.$$

□

Theorem 3.2.2. *Let $\sigma \in L^1(0,\infty)$, and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$ then $R_{\sigma,\phi} : L^\infty(0,\infty) \to L^\infty(0,\infty)$ is a bounded linear operator and*

$$\|R_{\sigma,\phi}\|_{B(L^\infty(0,\infty))} \leq A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty. \tag{3.2.3}$$

Proof. From (3.2.1), we have

$$(R_{\sigma,\phi}f)(x) = \int_0^\infty \sigma(\zeta) \langle f, \pi(\zeta)\phi\rangle (\pi(\zeta)\phi)(x) d\zeta.$$

$$\|R_{\sigma,\phi}f\|_\infty = \sup_{x \in I} |(R_{\sigma,\phi}f)(x)|.$$

This implies

$$|R_{\sigma,\phi}f| = |\int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta|.$$

Using (3.1.1), we get

$$\begin{aligned}|R_{\sigma,\phi}f| &\leq \left(\int_0^\infty |\sigma(\zeta)| \cdot |\langle f, \pi(\zeta)\phi\rangle| \cdot |(x\zeta)^{1/2}J_\mu(x\zeta)\phi(x)|d\zeta\right)\\ &\leq A_\mu \left(\int_0^\infty |\sigma(\zeta)| \cdot |\langle f, \pi(\zeta)\phi\rangle| \cdot |\phi(x)|d\zeta\right).\end{aligned}$$

By Lemma 3.1.1

$$|R_{\sigma,\phi}f| \leq A_\mu \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\phi)(\zeta)| \cdot |\phi(x)|d\zeta\right).$$

Now

$$\begin{aligned}\|R_{\sigma,\phi}f\|_\infty &\leq A_\mu \sup_{x\in I} |\phi(x)| \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\phi)(\zeta)|d\zeta\right)\\ &\leq A_\mu\|\phi\|_\infty\|h_\mu(f\phi)\|_\infty\|\sigma\|_1\\ &\leq A_\mu\|\phi\|_\infty\|f\|_\infty\|\phi\|_1\|\sigma\|_1.\end{aligned}$$

Therefore, we get the required result

$$\|R_{\sigma,\phi}\|_{B(L^\infty(0,\infty))} \leq A_\mu\|\phi\|_1\|\sigma\|_1\|\phi\|_\infty.$$

□

Now from Wong [78, p. 2916], we recall the interpolation theorem which is useful for our following results.

Theorem 3.2.3. *Let (X, μ) be a measure space and (Y, ν) a σ-finite measure space. Let T be a linear transformation with domain D consisting of all μ-simple functions*

f on X such that $\mu\{x \in X : f(x) \neq 0\} < \infty$ and such that range of T is contained in the set of all ν-measurable functions on Y. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers in [0 1] and there exist positive constant M_1 and M_2 such that

$$\|Tf\|_{L^{1/\beta_j}(Y)} \leq M_j \|f\|_{L^{1/\alpha_j}(X)} \qquad f \in D \quad j = 1, 2. \tag{3.2.4}$$

Then for $\;0 < \theta < 1$

$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2 \tag{3.2.5}$$

$$\beta = (1-\theta)\beta_1 + \theta\beta_2. \tag{3.2.6}$$

We have

$$\|Tf\|_{L^{1/\beta_j}(Y)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^{1/\alpha_j}(X)}. \tag{3.2.7}$$

With the help of the above interpolation results, we prove a theorem on the L^p-boundedness of the Hankel wavelet multiplier associated with the unitary representation.

Theorem 3.2.4. *For $1 \leq p \leq \infty$, there exists a bounded linear operator, $R_{\sigma,\phi}$: $L^p(0,\infty) \to L^p(0,\infty)$ such that*

$$\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} \leq A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty. \tag{3.2.8}$$

Proof. By Theorem 3.2.1 and Theorem 3.2.2 we have $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = 1, \beta_2 = 0$. Then from (3.2.5) and (3.2.6) we find that $\alpha = 1-\theta$, $\beta = 1-\theta$, and applying interpolation Theorem 3.2.3, we get

$$\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} \leq A_\mu (\|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty)^{1-\theta} (\|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty)^{\theta}.$$

$$\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} \leq A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty.$$

for $0 < \theta < 1$ and $1/\alpha \in (1, \infty)$. $\square$

Theorem 3.2.5. *Let $\phi \in L^2(0,\infty) \cap L^\infty(0,\infty)$ be such that $\|\phi\|_2 = 1$ and $\sigma \in L^2(0,\infty)$. Then $R_{\sigma,\phi} : L^2(0,\infty) \to L^2(0,\infty)$ is a bounded linear operator and*

$$\|R_{\sigma,\phi}\|_{B(L^2(0,\infty))} \leq A_\mu \|\sigma\|_2 \|\phi\|_\infty. \tag{3.2.9}$$

Proof. Let

$$(R_{\sigma,\phi} f)(x) = \int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle (\pi(\zeta)\phi)(x) d\zeta.$$

$$\begin{aligned}
\|R_{\sigma,\phi} f\|_2^2 &= \int_0^\infty |(R_{\sigma,\phi} f)(x)|^2 dx \\
&= \int_0^\infty |\int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle (\pi(\zeta)\phi)(x) d\zeta|^2 dx.
\end{aligned}$$

By using Lemma 3.1.1, we get

$$\begin{aligned}
\|R_{\sigma,\phi} f\|_2^2 &= \int_0^\infty |\int_0^\infty \sigma(\zeta) h_\mu(f\bar{\phi})(\zeta) \left((x\zeta)^{1/2} J_\mu(x\zeta)\phi(x)\right) d\zeta|^2 dx \\
&\leq \int_0^\infty \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\bar{\phi})(\zeta)| \cdot |(x\zeta)^{1/2} J_\mu(x\zeta)| \cdot |\phi(x)| d\zeta\right)^2 dx \\
&\leq A_\mu \int_0^\infty \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\bar{\phi})(\zeta)| \cdot |\phi(x)| d\zeta\right)^2 dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|R_{\sigma,\phi} f\|_2^2 &\leq A_\mu \int_0^\infty |\phi(x)|^2 \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\bar{\phi})(\zeta)| d\zeta\right)^2 dx \\
&\leq A_\mu \int_0^\infty |\phi(x)|^2 dx \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\bar{\phi})(\zeta)| d\zeta\right)^2 \\
&= A_\mu \|\phi\|_2^2 \left(\int_0^\infty |\sigma(\zeta)| \cdot |h_\mu(f\bar{\phi})(\zeta)| d\zeta\right)^2.
\end{aligned}$$

By Holder inequality and (1.3.13), we get

$$\begin{aligned}\|R_{\sigma,\phi}f\|_2 &\leq A_\mu\|\phi\|_2\|\sigma\|_2\|h_\mu(f\bar{\phi})\|_2\\ &= A_\mu\|\phi\|_2\|\sigma\|_2\|(f\bar{\phi})\|_2\\ &\leq A_\mu\|\phi\|_2\|\sigma\|_2\|f\|_2\|\phi\|_\infty.\end{aligned}$$

Therefore, we obtain the required result

$$\|R_{\sigma,\phi}\|_{B(L^2(0,\infty))} \leq A_\mu\|\sigma\|_2\|\phi\|_\infty.$$

□

Theorem 3.2.6. *Let $\phi \in L^1(0,\infty) \cap L^2(0,\infty) \cap L^\infty(0,\infty)$ such that $\|\phi\|_2 = 1$. Let $\sigma \in L^r(0,\infty)$ for $1 \leq r \leq 2$ then there exists a bounded linear operator $R_{\sigma,\phi} : L^p(0,\infty) \to L^p(0,\infty)$ for all $p \in [r,\ r']$ where r is conjugate index of r'such that*

$$\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} \leq A_\mu\|\sigma\|_1^{1-2/r'}\|\sigma\|_2^{2/r'}\|\phi\|_1^{1-2/r'}\|\phi\|_2^{2/r'}\|\phi\|_\infty. \tag{3.2.10}$$

Proof. By interpolation of the Theorem 3.2.1 and 3.2.5, we get a bounded linear operator $R_{\sigma,\phi} : L^p(0,\infty) \to L^p(0,\infty)$.
Taking the suitable choice of $\alpha_1 = 1$, $\alpha_2 = 1/2$, $\dfrac{1}{\beta_1} = p$ and $\dfrac{1}{\beta_2} = p$. By (3.2.5) and (3.2.6) we find that $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2 = 1 - \dfrac{\theta}{2}$ and $\beta = (1-\theta)(1/p) + \theta(1/p)$. Now suppose $\alpha = 1/r$ this implies $1/r = 1-\theta/2$ and $1 - 1/r' = 1 - \theta/2$ or $\theta = 2/r'$. This implies

$$\begin{aligned}\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} &\leq A_\mu(\|\sigma\|_1\|\phi\|_1\|\phi\|_\infty)^{1-\theta}(\|\phi\|_2\|\sigma\|_2\|\phi\|_\infty)^\theta\\ &= A_\mu(\|\sigma\|_1\|\phi\|_1\|\phi\|_\infty)^{1-2/r'}(\|\phi\|_2\|\sigma\|_2\|\phi\|_\infty)^{2/r'}.\end{aligned}$$

Therefore, we get the required result

$$\|R_{\sigma,\phi}\|_{B(L^p(0,\infty))} \leq A_\mu \|\sigma\|_1^{1-2/r'} \|\sigma\|_2^{2/r'} \|\phi\|_1^{1-2/r'} \|\phi\|_2^{2/r'} \|\phi\|_\infty.$$

□

3.3 Hilbert-Schmidt operator and compactness

In this section, we demonstrates that the Hankel wavelet multiplier $R_{\sigma,\phi}$ associated with the unitary representation is a Hilbert-Schmidt operator. We examine the compactness property of the Hankel wavelet multiplier $R_{\sigma,\phi}$.

Definition 3.3.1. Hilbert-schmidt operator:

From Wong [80, p. 17], let $T : H \to H$ be a bounded linear operator in a Hilbert space H such that,

$$\sum_k \|T\phi_k\|^2 < \infty, \tag{3.3.1}$$

for all orthonormal bases $\{\phi_k : k \in \mathbb{N}\}$ in H. Then $T : H \to H$ is in the Hilbert-Schmidt class S_2 and satisfies the following norm

$$\|T\|_{S_2}^2 = \sum_k \|T\phi_k\|^2. \tag{3.3.2}$$

Theorem 3.3.2. *If $\phi \in L^2(0,\infty) \cap L^\infty(0,\infty)$, Then for all $\sigma \in L^1(0,\infty) \cap L^2(0,\infty)$, $R_{\sigma,\phi} : L^2(0,\infty) \to L^2(0,\infty)$ is a Hilbert-schmidt operator such that*

$$\|R_{\sigma,\phi}\|_{S_2}^2 \leq \|\sigma\|_2 \|\sigma\|_1 \|x^{\mu+1/2}\phi(x)\|_2^2 \|\bar{\phi}(x)\|_\infty^2.$$

Proof. We take $\sigma \in L^1(0,\infty) \cap L^2(0,\infty)$ and $\{\phi_k : k \in \mathbb{N}\}$ be an orthonormal basis in $L^2(0,\infty)$. Then by definition of the inner-product, we have

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \sum_k \langle R_{\sigma,\phi}\phi_k, R_{\sigma,\phi}\phi_k\rangle. \tag{3.3.3}$$

Using (3.1.2), we have

$$\langle R_{\sigma,\phi}f, g\rangle = \int_0^\infty \sigma(\zeta)\langle f, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, g\rangle d\zeta. \tag{3.3.4}$$

Therefore from (3.3.3), we obtain

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &= \sum_k \int_0^\infty \sigma(\zeta)\langle \phi_k, \pi(\zeta)\phi\rangle\langle \pi(\zeta)\phi, R_{\sigma,\phi}\phi_k\rangle d\zeta \\ &= \sum_k \int_0^\infty \sigma(\zeta)\langle \phi_k, \pi(\zeta)\phi\rangle\langle R_{\bar{\sigma},\phi}\pi(\zeta)\phi, \phi_k\rangle d\zeta.\end{aligned}$$

With the help of Wong [34, p. 1014] and Plancheral relation from Ponnusamy [54, p. 409], we get

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \int_0^\infty \sigma(\zeta)\langle R_{\bar{\sigma},\phi}\pi(\zeta)\phi, \pi(\zeta)\phi\rangle d\zeta.$$

Using (3.3.4), we have

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\langle \pi(\zeta)\phi, \pi(\eta)\phi\rangle\langle \pi(\eta)\phi, \pi(\zeta)\phi\rangle d\eta\Big)d\zeta. \tag{3.3.5}$$

From (3.3.5) and Fubini theorem, we get

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)|\langle \pi(\zeta)\phi, \pi(\eta)\phi\rangle|^2 d\eta\Big)d\zeta.$$

By the definition of the inner product and take $(\pi(\zeta)\phi)(x) = (x\zeta)^{1/2}J_\mu(x\zeta)\phi(x)$ and $(\pi(\eta)\phi)(x) = (x\eta)^{1/2}J_\mu(x\eta)\phi(x)$, we have

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(\Big|\int_0^\infty (x\zeta)^{\frac{1}{2}}J_\mu(x\zeta)\phi(x)(x\eta)^{\frac{1}{2}}J_\mu(x\eta)\bar{\phi}(x)dx\Big|^2\big) \\ &\qquad \times d\eta\Big)d\zeta \\ &= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(\Big|\int_0^\infty (x\zeta)^{\frac{1}{2}}J_\mu(x\zeta)(x\eta)^{\frac{1}{2}}J_\mu(x\eta)\phi(x)\bar{\phi}(x)dx\Big|^2\big) \\ &\qquad \times d\eta\Big)d\zeta.\end{aligned}$$

Using (1.3.6), we obtain

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(\Big|\int_0^\infty \Big(\int_0^\infty x^{\mu+1/2}(xz)^{\frac{1}{2}}J_\mu(xz)D_\mu(\zeta,\eta,z)dz\Big) \\ &\qquad \times \phi(x)\bar{\phi}(x)dx\Big|^2\big)d\eta\Big)d\zeta \\ &= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(\Big|\int_0^\infty \big(\int_0^\infty x^{\mu+1/2}(xz)^{\frac{1}{2}}J_\mu(xz)\phi(x)\bar{\phi}(x)dx\big) \\ &\qquad \times D_\mu(\zeta,\eta,z)dz\Big|^2\big)d\eta\Big)d\zeta.\end{aligned}$$

Thus by definition of Hankel transform (1.3.1), we get

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &= \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(\Big|\int_0^\infty h_\mu(x^{\mu+1/2}\phi(x)\bar{\phi}(x))(z)D_\mu(\zeta,\eta,z)dz\Big|^2\big) \\ &\qquad \times d\eta\Big)d\zeta.\end{aligned}$$

Thus by (1.3.5), we find

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \int_0^\infty \sigma(\zeta)\Big(\int_0^\infty \bar{\sigma}(\eta)\big(|h_\mu(x^{\mu+1/2}\phi(x)\bar{\phi}(x))(\zeta,\eta)|^2\big)d\eta\Big)d\zeta.$$

If we take $\left(h_\mu(x^{\mu+1/2}\phi(x)\bar{\phi}(x))(\zeta,\eta)\right)^2 = G(\zeta,\eta)$, then above expression yields

$$\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 = \int_0^\infty \sigma(\zeta)(\int_0^\infty \bar{\sigma}(\eta)|G(\zeta,\eta)|d\eta)d\zeta.$$

Using (1.3.4), we have

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &= \int_0^\infty \sigma(\zeta)(\sigma\#|G|)(\zeta)d\zeta \\ &\leq \int_0^\infty |\sigma(\zeta)(\sigma\#|G|)(\zeta)|d\zeta \\ &\leq \|\sigma\|_2\|(\sigma\#|G|)\|_2.\end{aligned}$$

From [5, p. 111], we get

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &\leq \|\sigma\|_2\|\sigma\|_1\||G|\|_2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|\left|h_\mu(x^{\mu+1/2}\phi(x)\bar{\phi}(x))\right|^2\|_2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|h_\mu(x^{\mu+1/2}\phi(x)\bar{\phi}(x))\|_2^2.\end{aligned}$$

Using Parseval formula of Hankel transform (1.3.13), we have

$$\begin{aligned}\sum_k \|R_{\sigma,\phi}\phi_k\|_2^2 &\leq \|\sigma\|_2\|\sigma\|_1\|x^{\mu+1/2}\phi(x)\bar{\phi}(x)\|_2^2 \\ &\leq \|\sigma\|_2\|\sigma\|_1\|x^{\mu+1/2}\phi(x)\|_2^2\|\bar{\phi}(x)\|_\infty^2.\end{aligned}$$

This implies $R_{\sigma,\phi} : L^2(0,\infty) \to L^2(0,\infty)$ is a Hilbert-Schmidt operator. □

Theorem 3.3.3. *Let $\sigma \in L^1(0,\infty)$, and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then $R_{\sigma,\phi}$: $L^1(0,\infty) \to L^1(0,\infty)$ is a compact operator.*

Proof. From Theorem 3.2.1, the bounded linear operator $R_{\sigma,\phi} : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ satisfies

$$\|R_{\sigma,\phi}\|_{B(L^1(0,\infty))} \leq A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty. \tag{3.3.6}$$

Consider $\{u_j\}$ be a sequence of functions in $L^1(0,\infty)$ such that $\|u_j\| \leq 1$, then (3.3.6) gives

$$\|R_{\sigma,\phi} u_j\|_\infty \leq A_\mu \|\sigma\|_1 \|\phi\|_1 \|\phi\|_\infty \quad \text{for} \quad j = 1, 2, 3.... \tag{3.3.7}$$

the above expression shows that $\{R_{\sigma,\phi} u_j\}$ is uniformly bounded in $L^1(0,\infty)$.

Now, consider the function $\phi \in C_0^\infty(0,\infty)$, which is compactly supported then the function $(\pi(\zeta)\phi)(x) = (x\zeta)^{1/2} J_\mu(x\zeta)\phi(x)$ is uniformly continuous.

Then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that for all x and y in supp(ϕ) with $|x - y| < \delta$, we get

$$|(\pi(\zeta)\phi)(x) - (\pi(\zeta)\phi)(y)| < \epsilon. \tag{3.3.8}$$

Hence for all $j = 1, 2, 3\ldots$, we have

$$\begin{aligned}
&|(R_{\sigma,\phi} u_j)(x) - (R_{\sigma,\phi} u_j)(y)| \\
&\qquad = |\int_0^\infty \sigma(\zeta)\langle u_j, \pi(\zeta)\phi\rangle(\phi(\zeta)\phi)(x) d\zeta \\
&\qquad - \int_0^\infty \sigma(\zeta)\langle u_j, \pi(\zeta)\phi\rangle(\phi(\zeta)\phi)(y) d\zeta|, \quad x, y \in (0,\infty) \\
&\qquad = |\int_0^\infty \sigma(\zeta)\langle u_j, \pi(\zeta)\phi\rangle \left((\pi(\zeta)\phi)(x) - (\pi(\zeta)\phi)(y)\right) d\zeta| \\
&\qquad \leq \int_0^\infty |\sigma(\zeta)| \cdot |\langle u_j, \pi(\zeta)\phi\rangle| \cdot |(\pi(\zeta)\phi)(x) - (\pi(\zeta)\phi)(y)| d\zeta.
\end{aligned}$$

Using (3.3.8), we find that

$$\begin{aligned}
|(R_{\sigma,\phi} u_j)(x) - (R_{\sigma,\phi} u_j)(y)| &\leq \epsilon \int_0^\infty |\sigma(\zeta)| \cdot |\langle u_j, \pi(\zeta)\phi\rangle| d\zeta \\
&\leq \epsilon \cdot \sup |\langle u_j, \pi(\zeta)\phi\rangle| \int_0^\infty |\sigma(\zeta)| d\zeta
\end{aligned}$$

Using (3.1.1), we get

$$\begin{aligned}|(R_{\sigma,\phi}u_j)(x)-(R_{\sigma,\phi}u_j)(y)| &\leq \epsilon\cdot A_\mu\|u_j\|_1\|\phi\|_\infty\int_0^\infty|\sigma(\zeta)|d\zeta\\ &\leq \epsilon\cdot A_\mu\|u_j\|_1\|\phi\|_\infty\|\sigma\|_1\\ &\leq \epsilon\cdot A_\mu\|\phi\|_\infty\|\sigma\|_1.\end{aligned}$$

So $\{R_{\sigma,\phi}u_j\}_{j=1}^\infty$ is equicontinuous on $(0,\infty)$.

Therefore for every compact subset K of $(0,\infty)$, the Ascoli-Arzela theorem implies that the Hankel wavelet multiplier $\{R_{\sigma,\phi}u_j\}_{j=1}^\infty$ has a subsequence that converges uniformly on K.

Thus by Cantor diagonal procedure we can find a subsequence $\{u_{j_k}\}$ of $\{u_j\}_{j=1}^\infty$ such that $\{R_{\sigma,\phi}u_{j_k}\}_{k=1}^\infty$ converges to pointwise to a function on $(0,\infty)$. Using (3.3.7) and by applying Lebesgue dominated convergence theorem, the sequence $\{R_{\sigma,\phi}u_{j_k}\}_{k=1}^\infty$ is converges in $L^1(0,\infty)$.

Therefore $R_{\sigma,\phi}:L^1(\mathbb{R}_+)\to L^1(\mathbb{R}_+)$ is compact.

Now let $\psi\in L^1(0,\infty)\cap L^\infty(0,\infty)$ and $\{\psi_j\}_{j=1}^\infty$ be a sequence of functions in $C_0^\infty(0,\infty)$ such that $\psi_j\to\psi$ in $L^1(0,\infty)$ as $j\to\infty$. So by (3.3.6)

$$\|P_{\sigma,\phi,\psi_j}-P_{\sigma,\phi,\psi}\|_{B(L^1(0,\infty))}\leq\|\phi\|_\infty\|\psi_j-\psi\|_1\|\sigma\|_1\to 0$$

as $j\to\infty$. Therefore $R_{\sigma,\phi}:L^1(0,\infty)\to L^1(0,\infty)$ is compact provided that the support of σ is compact. □

Theorem 3.3.4. *For $1\leq p\leq\infty$ and for any $\sigma\in L^1(0,\infty)$ and $\phi\in L^1(0,\infty)\cap L^\infty(0,\infty)$. Then $R_{\sigma,\phi}:L^p(0,\infty)\to L^p(0,\infty)$ is a compact operator.*

Proof. Using the interpolation technique on Theorem 3.2.3, we get the required result. □

3.4 Applications of the Hankel wavelet multiplier and construction of Sobolev-type space

In this section, it is shown that the Landau-Pollak Slepian operator associated with the unitary representation $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ arising in signal analysis, is a Hankel wavelet multiplier.

Definition 3.4.1. Let $C_1 > 0$ and $C_2 > 0$. Then the linear operators $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ is defined by

$$h_\mu(P_{C_1}f)(\zeta) = \begin{cases} (h_\mu f)(\zeta) & 0 \leq \zeta \leq C_1 \\ 0 & \zeta > C_1 \end{cases} \tag{3.4.1}$$

and

$$(Q_{C_2}f)(x) = \begin{cases} f(x) & 0 \leq x \leq C_2 \\ 0 & x > C_2 \end{cases} \tag{3.4.2}$$

for all functions f in $L^2(0,\infty)$.

Theorem 3.4.2. *$P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ are self-adjoint.*

Proof. By Parseval relation of the Hankel transform, we have

$$\begin{aligned} \langle P_{c_1}f, g\rangle_{L^2(0,\infty)} &= \langle h_\mu(P_{C_1}f), h_\mu g\rangle_{L^2(0,\infty)} \\ &= \int_0^\infty h_\mu(P_{C_1}f)(\zeta)\overline{(h_\mu g)}(\zeta)d\zeta. \end{aligned}$$

Using (3.4.1), we get

$$\begin{aligned}\langle P_{c_1} f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_1}} (h_\mu f)(\zeta)\overline{(h_\mu g)}(\zeta) d\zeta \\ &= \int_{B_{C_1}} (h_\mu f)(\zeta)\overline{h_\mu(P_{c_1} g)}(\zeta) d\zeta \\ &= \int_0^\infty (h_\mu f)(\zeta)\overline{h_\mu(P_{C_1} g)}(\zeta) d\zeta \\ &= \langle (h_\mu f), h_\mu(P_{C_1} g)\rangle.\end{aligned}$$

By Parseval relation (1.3.13) , we have

$$\langle P_{c_1} f, g\rangle_{L^2(0,\infty)} = \langle f, P_{C_1} g\rangle_{L^2(0,\infty)}. \tag{3.4.3}$$

This implies that $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint operator.

Similarily

$$\langle Q_{c_2} f, g\rangle_{L^2(0,\infty)} = \int_0^\infty (Q_{C_2} f)(x)\bar{g}(x) dx.$$

From (3.4.2), we have

$$\begin{aligned}\langle Q_{c_2} f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_2}} f(x)\bar{g}(x) dx \\ &= \int_{B_{C_2}} f(x)\overline{Q_{C_2} g(x)} dx \\ &= \int_0^\infty f(x)\overline{Q_{C_2} g(x)} dx \\ &= \langle f, Q_{C_2} g\rangle.\end{aligned} \tag{3.4.4}$$

This implies that $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint operator. $\square$

Theorem 3.4.3. *$P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ and $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ are projection.*

Proof. Since $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint, so we have

$$\langle P_{C_1}^2 f, g\rangle_{L^2(0,\infty)} = \langle P_{C_1} f, P_{C_1} g\rangle_{L^2(0,\infty)}.$$

and by Parseval relation, we get

$$\begin{aligned}\langle P_{C_1}^2 f, g\rangle_{L^2(0,\infty)} &= \langle h_\mu(P_{C_1} f), h_\mu(P_{C_1} g)\rangle_{L^2(0,\infty)}\\ &= \int_0^\infty h_\mu(P_{C_1} f)(\zeta)\overline{h_\mu(P_{C_1} g)}(\zeta)d\zeta.\end{aligned}$$

By (3.4.1), we have

$$\begin{aligned}\langle P_{C_1}^2 f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_1}} (h_\mu f)(\zeta)\overline{(h_\mu g)}(\zeta)d\zeta\\ &= \int_0^\infty h_\mu(P_{C_1} f)(\zeta)\overline{(h_\mu g)}(\zeta)d\zeta\\ &= \langle h_\mu(P_{C_1} f), h_\mu g\rangle_{L^2(0,\infty)}\\ &= \langle P_{C_1} f, g\rangle_{L^2(0,\infty)}\end{aligned}$$

for all $f, g \in L^2(0,\infty)$.

This implies that $P_{C_1}^2 = P_{C_1}$, hence we say that $P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a projection.

Since $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint, we get

$$\begin{aligned}\langle Q_{C_2}^2 f, g\rangle_{L^2(0,\infty)} &= \langle Q_{C_2} f, Q_{C_2} g\rangle_{L^2(0,\infty)}\\ &= \int_0^\infty (Q_{C_2} f)(x)\overline{(Q_{C_2} g)}(x)dx\end{aligned}$$

By (3.4.2), we get

$$\begin{aligned}
\langle Q_{C_2}^2 f, g\rangle_{L^2(0,\infty)} &= \int_{B_{C_2}} f(x)\overline{g}(x)dx \\
&= \int_0^\infty (Q_{C_2}f)(x)\overline{g}(x)dx \\
&= \langle Q_{C_2} f, g\rangle_{L^2(0,\infty)}
\end{aligned}$$

for all $f, g \in L^2(0,\infty)$.

This implies that $Q_{C_2}^2 = Q_{C_2}$, hence we say that $Q_{C_2} : L^2(0,\infty) \to L^2(0,\infty)$ is a projection. □

Remark: Thus $P_{C_1}Q_{C_2}P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is a bounded linear operator and known as the generalized Landau-Pollak Slepian operator. We have to show that this generalized Landau-Pollak Slepian operator is a Hankel wavelet multiplier.

Theorem 3.4.4. *Let ϕ be any function in $L^2(0,\infty)$ is defined by*

$$\phi(x) = \begin{cases} \dfrac{1}{\sqrt{\mu(B_{C_1})}} & 0 \le x \le C_1 \\ 0 & x > C_1 \end{cases} \tag{3.4.5}$$

where $\mu(B_{C_1})$ is the volume of B_{C_1}. Let σ be the characteristic function on B_{C_2},

$$\sigma(\zeta) = \begin{cases} 1 & 0 \le \zeta \le C_2 \\ 0 & \zeta > C_2 \end{cases} \tag{3.4.6}$$

then the Landau-Pollak Slepian operator $P_{C_1}Q_{C_2}P_{C_1} : L^2(0,\infty) \to L^2(0,\infty)$ is unitarily equivalent to a scalar multiple of the Hankel wavelet multiplier $\phi h_{\mu,\sigma}\bar{\phi}$: $L^2(0,\infty) \to L^2(0,\infty)$.

Proof. By (3.4.5), we see that ϕ is a function in $L^2(0,\infty)\cap L^\infty(0,\infty)$ such that

$$\begin{aligned}\|\phi\|^2_{L^2(0,\infty)} &= \int_0^\infty |\phi(x)|^2 dx\\ &= \frac{1}{\mu(B_{C_1})}\int_{B_{C_1}} dx\\ &= 1.\end{aligned}$$

Thus, by Theorem 3.1.3, we have

$$\langle(\phi h_{\mu,\sigma}\bar{\phi})u,v\rangle = \int_0^\infty \sigma(\zeta)\langle u,\pi(\zeta)\phi\rangle\langle\pi(\zeta)\phi,v\rangle d\zeta. \tag{3.4.7}$$

From (3.4.5), we get

$$\begin{aligned}\langle u,\pi(\zeta)\phi\rangle &= \int_0^\infty (x\zeta)^{1/2}J_\mu(x\zeta)u(x)\bar{\phi}(x)dx \qquad (3.4.8)\\ &= \frac{1}{\sqrt{\mu(B_{C_1})}}\int_{B_{C_1}}(x\zeta)^{1/2}J_\mu(x\zeta)u(x)dx.\end{aligned}$$

By definition of P_{C_1}, we have

$$h_\mu(P_{C_1}h_\mu^{-1}(u))(x) = \begin{cases} u(x) & 0\le x\le C_1\\ 0 & x > C_1\end{cases} \tag{3.4.9}$$

function u in H_μ, where h_μ^{-1} is the inverse Hankel transform of u.

By (3.4.8) and (3.4.9), we obtain

$$\begin{aligned}\langle u,\pi(\zeta)\phi\rangle &= \frac{1}{\sqrt{\mu(B_{C_1})}}\int_{B_{C_1}}(x\zeta)^{1/2}J_\mu(x\zeta)h_\mu(P_{C_1}h_\mu^{-1}(u))(x)dx.\\ &= \frac{1}{\sqrt{\mu(B_{C_1})}}(P_{C_1}h_\mu^{-1}(u))(\zeta) \quad \text{for all} \quad u\in H_\mu. \qquad (3.4.10)\end{aligned}$$

Since P_{C_1} is a self-adjoint and using (3.4.6), (3.4.7) and (3.4.10) we get

$$\begin{aligned}
\langle(\phi h_{\mu,\sigma}\bar{\phi})u, v\rangle &= \frac{1}{\mu(B_{C_1})}\int_0^\infty \sigma(\zeta)(P_{C_1}h_\mu^{-1}(u))(\zeta)\overline{(P_{C_1}h_\mu^{-1}(v))(\zeta)}d\zeta \\
&= \frac{1}{\mu(B_{C_1})}\int_{B_{C_2}} (P_{C_1}h_\mu^{-1}(u))(\zeta)\overline{(P_{C_1}h_\mu^{-1}(v))(\zeta)}d\zeta \\
&= \frac{1}{\mu(B_{C_1})}\int_0^\infty (Q_{C_2}P_{C_1}(h_\mu^{-1}u))(\zeta)\overline{(P_{C_1}(h_\mu^{-1}v))(\zeta)}d\zeta \\
&= \frac{1}{\mu(B_{C_1})}\langle(Q_{C_2}P_{C_1}(h_\mu^{-1}u)), (P_{C_1}(h_\mu^{-1}v))\rangle \\
&= \frac{1}{\mu(B_{C_1})}\langle(P_{C_1}Q_{C_2}P_{C_1}(h_\mu^{-1}u)), (h_\mu^{-1}v)\rangle \\
&= \frac{1}{\mu(B_{C_1})}\langle h_\mu P_{C_1}Q_{C_2}P_{C_1}h_\mu^{-1}u, v\rangle
\end{aligned}$$

for all $u, v \in H_\mu$. Thus the Landau-Pollak Slepian operator $P_{C_1}Q_{C_2}P_{C_1}$ is unitary equivalent to $(\phi h_{\mu,\sigma}\bar{\phi})$ Hankel wavelet multiplier. $\square$

3.5 Hankel wavelet multiplier in Sobolev-type space

In this section, with the help of the Hankel wavelet multiplier, the Sobolev-type space is constructed involving the Hankel transform technique.

Theorem 3.5.1. *If $\sigma_s(\zeta) = (1+\zeta^2)^{-s/2} \in L^1(0,\infty)$, for $s > 0$ and $\phi \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Then the Hankel wavelet multiplier*

$$(R_{\sigma_s,\phi}f)(x) = \int_0^\infty \sigma_s(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta, \quad \zeta \in (0,\infty) \tag{3.5.1}$$

can be expressed in the following form:

$$(R_{\sigma_s,\phi}f)(x) = \phi(x)h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\bar{\phi})(\zeta)](x), \quad f \in L^1(0,\infty). \tag{3.5.2}$$

Proof. Using (3.5.1), we have

$$(R_{\sigma_s,\phi}f)(x) = \int_0^\infty \sigma_s(\zeta)\langle f, \pi(\zeta)\phi\rangle(\pi(\zeta)\phi)(x)d\zeta, \quad \zeta \in (0,\infty).$$

From (3.1.1), we get

$$\begin{aligned}
(R_{\sigma_s,\phi}f)(x) &= \int_0^\infty \sigma_s(\zeta)\left(\int_0^\infty f(\eta)\overline{(\pi(\zeta)\phi)}(\eta)d\eta\right)(\pi(\zeta)\phi)(x)d\zeta \\
&= \int_0^\infty \sigma_s(\zeta)\left(\int_0^\infty (\eta\zeta)^{1/2}J_\mu(\eta\zeta)f(\eta)\bar{\phi}(\eta)d\eta\right)(x\zeta)^{1/2}J_\mu(x\zeta)\phi(x)d\zeta \\
&= \int_0^\infty \sigma_s(\zeta)h_\mu(f\bar{\phi})(\zeta)(x\zeta)^{1/2}J_\mu(x\zeta)\phi(x)d\zeta \\
&= \phi(x)\int_0^\infty (x\zeta)^{1/2}J_\mu(x\zeta)\sigma_s(\zeta)h_\mu(f\bar{\phi})(\zeta)d\zeta \\
&= \phi(x)h_\mu^{-1}\left[\sigma_s(\zeta)h_\mu(f\bar{\phi})(\zeta)\right](x).
\end{aligned}$$

□

Definition 3.5.2. If O_M denotes the linear space of all smooth functions $\theta(x)$ defined on I such that for each non-negative integer ν, there is a non-negative integer n_ν for which

$$|(1+x^2)^{-n_\nu}(x^{-1}D_x)^\nu\theta(x)| < \infty \tag{3.5.3}$$

for all $x \in I = (0,\infty)$, then $\theta(x)$ is called multiplier in $H_\mu(I)$.

Theorem 3.5.3. *Let $\phi \in O_M$ and $\psi \in H_\mu(I)$. Then $\phi\psi \in H_\mu(I)$.*

Proof. For $\phi \in O_M$ and $\psi \in H_\mu(I)$, then we have

$$
\begin{aligned}
&|(1+x^2)^m (x^{-1}D_x)^k [x^{-\mu-1/2}(\phi\psi)(x)]| \\
&\qquad = |(1+x^2)^m \sum_{r=0}^{k} \binom{k}{r} (x^{-1}D_x)^r \phi(x) (x^{-1}D_x)^{k-r}[x^{-\mu-1/2}\psi(x)]| \\
&\qquad = |\sum_{r=0}^{k} \binom{k}{r} (1+x^2)^m (x^{-1}D_x)^r \phi(x) (x^{-1}D_x)^{k-r}[x^{-\mu-1/2}\psi(x)]| \\
&\qquad = |\sum_{r=0}^{k} \binom{k}{r} (1+x^2)^{-l} (x^{-1}D_x)^r \phi(x) (1+x^2)^{m+l} \\
&\qquad\qquad \times (x^{-1}D_x)^{k-r}[x^{-\mu-1/2}\psi(x)]|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|(1+x^2)^m (x^{-1}D_x)^k [x^{-\mu-1/2}(\phi\psi)(x)]| \\
&\qquad = |\sum_{r=0}^{k} \binom{k}{r} (1+x^2)^{-l} (x^{-1}D_x)^r \phi(x) (1+x^2)^{m+l} \\
&\qquad\qquad \times (x^{-1}D_x)^{k-r}[x^{-\mu-1/2}\psi(x)]| \\
&\qquad \leq \sum_{r=0}^{k} \binom{k}{r} |(1+x^2)^{-l} (x^{-1}D_x)^r \phi(x)| |(1+x^2)^{m+l} \\
&\qquad\qquad \times (x^{-1}D_x)^{k-r}[x^{-\mu-1/2}\psi(x)]|.
\end{aligned}
$$

Since $\phi \in O_M$ and using (1.3.3), we get

$$\gamma^{\mu}_{m,k}(\phi\psi) \leq C_{k,l}\gamma^{\mu}_{m+l,k-r}(\psi) < \infty.$$

This implies

$$\phi\psi \in H_\mu(I).$$

□

Theorem 3.5.4. *Let $\phi \in O_M$ and $\psi \in H_\mu(I)$. Then $h_\mu(\phi\psi) \in H_\mu(I)$.*

Proof. From Theorem 3.5.3, we get $\phi\psi \in H_\mu(I)$. Then by using [83, p. 141], we find that

$$h_\mu(\phi\psi) \in H_\mu(I).$$

□

Theorem 3.5.5. *Let $\phi \in O_M$ and $\psi \in H_\mu(I)$. Then*

$$H^s_\mu(\phi\psi) = h^{-1}_\mu[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x) \in H_\mu(I). \tag{3.5.4}$$

Proof. In view of the Zemanian [83, p. 141], we find that

$$\begin{aligned}&(-1)^{m+k}x^m(x^{-1}D_x)^k x^{-\mu-1/2}h^{-1}_\mu[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)\\ &\quad= \int_0^\infty \zeta^{2\mu+2k+m+1}(\zeta^{-1}D_\zeta)^m\zeta^{-\mu-1/2}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)]J_{\mu+k+m}(x\zeta)(x\zeta)^{-\mu-k}d\zeta.\end{aligned}$$

Suppose that $n \geq 2\mu+2k+m+1$. Then $\zeta^{2\mu+2k+m+1} < (1+\zeta^2)^n$ for $\zeta > 0$. Therefore

$$\begin{aligned}&|(-1)^{m+k}x^m(x^{-1}D_x)^k x^{-\mu-1/2}h^{-1}_\mu[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)|\\ &\quad\leq \int_0^\infty |(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^m\zeta^{-\mu-1/2}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x\zeta)^{-\mu-k}J_{\mu+k+m}(x\zeta)|d\zeta\\ &\quad= \int_0^\infty |(1+\zeta^2)^n\sum_{r=0}^m\binom{m}{r}(\zeta^{-1}D_\zeta)^r\sigma_s(\zeta)(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)\\ &\qquad\qquad \times (x\zeta)^{-\mu-k}J_{\mu+k+m}(x\zeta)|d\zeta\\ &\quad\leq \sum_{r=0}^m\binom{m}{r}\int_0^\infty |(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^r\sigma_s(\zeta)(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)\\ &\qquad\qquad \times (x\zeta)^{-\mu-k}J_{\mu+k+m}(x\zeta)|d\zeta.\end{aligned}$$

Since $|(x\zeta)^{-\mu-k}J_{\mu+k+m}(x\zeta)|$ is bounded on $I=(0,\infty)$ by the constant $A_{k,m}$. Thus

$$
\begin{aligned}
&|x^m(x^{-1}D_x)^k x^{-\mu-1/2}h_\mu^{-1}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)| \\
&\quad\leq \sum_{r=0}^{m}\binom{m}{r}A_{k,m}\int_0^\infty |(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^r\sigma_s(\zeta)(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)|d\zeta.
\end{aligned}
$$

Since $\sigma_s(\zeta)=(1+\zeta^2)^{-s/2}$, we have

$$
\begin{aligned}
&|x^m(x^{-1}D_x)^k x^{-\mu-1/2}h_\mu^{-1}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)| \\
&\leq \sum_{r=0}^{m}\binom{m}{r}A_{k,m}\int_0^\infty |(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^r(1+\zeta^2)^{-s/2}(\zeta^{-1}D_\zeta)^{m-r}(\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta))|d\zeta \\
&= \sum_{r=0}^{m}\binom{m}{r}A_{k,m}\int_0^\infty |(1+\zeta^2)^n\left\{(-1)^r s(-s-2)(-s-4)...(-s-2r)(1+\zeta^2)^{-s/2-r}\right\} \\
&\qquad\quad \times(\zeta^{-1}D_\zeta)^{m-r}(\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta))|d\zeta \\
&\quad\leq \sum_{r=0}^{m}\binom{m}{r}A_{k,m}s(s+2)(s+4)...(s+2r)\int_0^\infty (1+\zeta^2)^n(1+\zeta^2)^{-s/2-r} \\
&\qquad\qquad \times|(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)|d\zeta \\
&\quad= \sum_{r=0}^{m}\binom{m}{r}A_{k,m}s(s+2)(s+4)...(s+2r)\int_0^\infty |(1+\zeta^2)^{-s/2-r} \\
&\qquad\qquad \times(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)|d\zeta.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|x^m(x^{-1}D_x)^k x^{-\mu-1/2}h_\mu^{-1}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)| \\
&\quad\leq A_{k,m}\sum_{r=0}^{m} sup_{\zeta\in I}|(1+\zeta^2)^n(\zeta^{-1}D_\zeta)^{m-r}\zeta^{-\mu-1/2}h_\mu(\phi\psi)(\zeta)| \\
&\qquad\quad \times\binom{m}{r}s(s+2)(s+4)...(s+2r)\int_0^\infty |(1+\zeta^2)^{-s/2-r}|d\zeta \\
&\quad\leq A_{k,m}\gamma^\mu_{n,m-r}(h_\mu(\phi\psi))\sum_{r=0}^{m}\binom{m}{r}s(s+2)(s+4)...(s+2r)\int_0^\infty |(1+\zeta^2)^{-s/2-r}|d\zeta.
\end{aligned}
$$

Therefore using (1.3.3), we have

$$\begin{aligned}
&\gamma^{\mu}_{m,k}(h_{\mu}^{-1}[\sigma_s(\zeta)h_{\mu}(\phi\psi)(\zeta)])(x)\\
&\quad\leq A_{k,m}\gamma^{\mu}_{n,m-r}(h_{\mu}(\phi\psi))\sum_{r=0}^{m}\binom{m}{r}s(s+2)(s+4)...(s+2r)\int_0^{\infty}|(1+\zeta^2)^{-s/2-r}|d\zeta\\
&\quad\leq A_{k,m,}B_{s,r}\gamma^{\mu}_{n,m-r}(h_{\mu}(\phi\psi))<\infty
\end{aligned}$$

for taking $s>0$ and sufficiently large.

This implies $h_{\mu}^{-1}[\sigma_s(\zeta)h_{\mu}(\phi\psi)(\zeta)]\in H_{\mu}(I)$. □

Theorem 3.5.6. *Let $\phi(x)\in O_M$ and $H^s_{\mu}(\phi\psi)(x)=h_{\mu}^{-1}[\sigma_s(\zeta)h_{\mu}(\phi\psi)(\zeta)](x)\in H_{\mu}(I)$. Then $\phi(x)H^s_{\mu}(\phi\psi)(x)\in H_{\mu}(I)$.*

Proof. Let

$$\begin{aligned}
&|(1+x^2)^m x^{-\mu-1/2}\phi(x)H^s_{\mu}(\phi\psi)(x)|\\
&\qquad\leq|(1+x^2)^{-1}\phi(x)(1+x^2)^{m+1}x^{-\mu-1/2}H^s_{\mu}(\phi\psi)(x)|\\
&\qquad\leq|(1+x^2)^{-1}\phi(x)||(1+x^2)^{m+1}x^{-\mu-1/2}H^s_{\mu}(\phi\psi)(x)|.
\end{aligned}$$

Since $\phi\in O_M$ and $H^s_{\mu}(\phi\psi)\in H_{\mu}(I)$

$$\begin{aligned}
|(1+x^2)^m x^{-\mu-1/2}\phi(x)H^s_{\mu}(\phi\psi)(x)|&\leq C_m sup_{x\in I}|(1+x^2)^{m+1}x^{-\mu-1/2}H^s_{\mu}(\phi\psi)(x)|\\
&\leq C_k\gamma^0_{\mu,m+1}(H^s_{\mu}(\phi\psi)(x))|<\infty.
\end{aligned}$$

Therefore $\phi(x)H^s_{\mu}(\phi\psi)(x)\in H_{\mu}(I)$. □

Theorem 3.5.7. *If $\phi\in O_M$ and $\psi\in H_{\mu}(I)$, then $R_{\sigma_s,\phi}$ and $f\in H'_{\mu}(I)$ also we have the following relation:*

$$\langle(R_{\sigma_s,\phi}f)(x),\psi(x)\rangle=\langle f(x),\phi(x)h_{\mu}^{-1}[\sigma_s(\zeta)h_{\mu}(\phi\psi)(\zeta)](x)\rangle. \tag{3.5.5}$$

Proof. Using Theorem 3.5.1, we have

$$\begin{aligned}\langle (R_{\sigma_s,\phi}f)(x), \psi(x)\rangle &= \langle \phi(x)h_\mu^{-1}[\sigma(\zeta)h_\mu(f\phi)(\zeta)](x), \psi(x)\rangle \\ &= \langle h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\phi)(\zeta)](x), (\phi\psi)(x)\rangle.\end{aligned}$$

By duality of Hankel transform technique, the above yields

$$\begin{aligned}\langle (R_{\sigma_s,\phi}f)(x), \psi(x)\rangle &= \langle \sigma_s(\zeta)h_\mu(f\phi)(\zeta), h_\mu(\phi\psi)(\zeta)\rangle \\ &= \langle h_\mu(f\phi)(\zeta), \sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)\rangle \\ &= \langle f\phi, h_\mu^{-1}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)]\rangle \\ &= \langle f(x), \phi(x)h_\mu^{-1}[\sigma_s(\zeta)h_\mu(\phi\psi)(\zeta)](x)\rangle \\ &= \langle f(x), \phi(x)H_\mu^s(\phi\psi)(x)\rangle.\end{aligned}$$

this implies that $f \in H_\mu^{'}(I)$, and also we can say that the Hankel wavelet multiplier $R_{\sigma_s,\phi}$ is also an element of $H_\mu^{'}(I)$. □

With the help of all above Theorems given in this section and from [40], we find a Hankel wavelet multiplier on the Sobolev space $G_\mu^{s,2}$.

Theorem 3.5.8. *Let $R_{\sigma_s,\phi}$ is a Hankel wavelet multiplier on $L^p(I)$ and $\phi \in O_M$, which satisfies the norm inequality*

$$\|(1+x^2)^{-s/2}\phi(x)\|_2 \le C_s \quad for \quad s > 0. \tag{3.5.6}$$

Then for $f \in H_\mu^{'}(I)$, $R_{\sigma_s,\phi}$ can be estimated by the

$$\|R_{\sigma_s,\phi}f\|_1 \le C_s\|\sigma_s(\zeta)h_\mu(f\phi)(\zeta)\|_{s,2}. \tag{3.5.7}$$

Proof. From Theorem 3.5.1, we have

$$\begin{aligned}
\|R_{\sigma_s,\phi}f\|_1 &= \left(\int_0^\infty |\phi(x)h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\phi)(\zeta)](x)|dx\right) \\
&= \left(\int_0^\infty |(1+x^2)^{-s}\phi(x)(1+x^2)^s h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\phi)(\zeta)](x)|dx\right) \\
&\leq \|(1+x^2)^{-s}\phi(x)\|_2 \left(\int_0^\infty |(1+x^2)^s h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\phi)(\zeta)](x)|^2 dx\right)^{1/2} \\
&\leq C_s \left(\int_0^\infty |(1+x^2)^s h_\mu^{-1}[\sigma_s(\zeta)h_\mu(f\phi)(\zeta)](x)|^2 dx\right)^{1/2} \\
&\leq C_s\|\sigma_s(\zeta)h_\mu(f\phi)(\zeta)\|_{s,2}.
\end{aligned}$$

From [40], we can conclude that the above norm of the R.H.S is an element of Sobolev space, for $f\phi \in H_\mu'(I)$. □

3.6 Conclusions

This chapters introduced the Hankel wavelet multipliers associated with the unitary representation. Motivated from the work of Ghober [22], Wong [34] and Mejjaoli [36], we observed that the Hankel wavelet multiplier associated with the unitary representation on L^p-space for $1 \leq p \leq \infty$ are bounded linear operator. Furthermore the applications of Hankel wavelet multipliers can be found in the Landau-Pollak Slepian operators and in the construction of Sobolev-type spaces related to Hankel transforms. we also explored the various properties of Hankel wavelet multiplier associated with the Hankel transform technique.

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