Chapter 6

Invariant Measures and Fractal Transformations on Graphs of FIF

In this chapter, we examine certain characteristics of measures. We focus on the measures that arise from the iterated function systems generated by Barnsley [9], whose attractor is the graph of a fractal interpolation function. We also investigate a fractal transformation between two fractal interpolation functions that satisfy the same interpolation data. Additionally, we introduce some function spaces and provide several results that establish conditions for the fractal interpolation function to be an element of these spaces. By utilizing these spaces, we are able to estimate the fractal dimension of the fractal interpolation functions.

6.1 Introduction

The measure-theoretic aspects of self-similar IFSs have been the subject of study, as seen in references such as [31, 63]. However, the IFS system constructed by Barnsley [9], whose attractor is the graph of FIF, is not self-similar. In this chapter, we have focused on the measure-theoretic aspects of Barnsley's IFS system. We have examined certain properties of an invariant measure supported on the graph of FIF and compared our findings with previous results. Additionally, we have demonstrated the existence of a fractal transformation between IFS systems whose attractors are the graph of FIF. Furthermore, researchers have investigated the existence of FIF in various spaces to analyze its properties. For example, Massopust [59] has studied FIF in the L^p-space with one-dimensional Hausdorff measure, while this chapter considers a more refined space, specifically the L^p-spaces with any invariant measure. We have extended the results presented in [59]. In [38], a dimensional space has been introduced, and in [47], the existence of FIF in that dimensional space has been explored. In this work, we have defined a more general space and provided some results regarding the existence of FIF in that space, including an upper bound on the box dimension of the graph of the m-th differentiation of FIF. Finally, we have generalized the results given in [29] by introducing a smooth convex-Lipschitz map.

6.1.1 Delineation

The current chapter is organized as follows. In the following section, we have examined the measure-theoretic viewpoint of FIF and various outcomes concerning the invariant measures that are supported on the graph of FIF. In Section 6.3, we have investigated the fractal transformation that occurs between the graphs of FIFs. Then, in Section 6.4, we have defined certain function spaces and established their completeness by assigning norms to them. Subsequently, we have presented certain conditions under which the FIF can be classified as a member of these function spaces. We have concluded the chapter in Section 6.5.

6.2 Measure-Theoretic Aspects of FIFs

In this section, we shall see some measure-theoretic aspects of FIF. The following theorem establishes a relation between the invariant measure generated by IFS $\{I:\ u_1,\dots,u_N\}$ and the invariant measure generated by IFS $\{I\times\mathbb{R}:\ \Omega_1,\dots,\Omega_N\}$ (refer Subsection 1.3.1).

Theorem 6.1. [9, Theorem 2] *For a given probability vector $\boldsymbol{r}=(r_1,\dots,r_N)$, let μ_r and μ_r^* are the invariant measures generated by the IFSs $\{I:\ u_1,\dots,u_N\}$ and $\{I\times\mathbb{R}:\ \Omega_1,\dots,\Omega_N\}$, respectively. Let $\Phi_{h^*}: I\to\mathcal{G}(h^*)$ be the homeomorphism defined as $\Phi_{h^*}(x)=(x,h^*(x))$ for $x\in I$. Then,*

$$\mu_r(\mathcal{B})=\mu_r^*(\Phi_{h^*}(\mathcal{B}))\ \textit{for all Borel subsets } \mathcal{B} \textit{ of } I.$$

Proposition 6.2. *Let $\{X;\upsilon_1,\dots,\upsilon_m\}$ be a system of self-similar IFS and satisfy the OSC. Let $\boldsymbol{r}=(r_1,\dots,r_m)$ be a probability vector. Then, for any countable subset $\boldsymbol{C}$ of X, we have $\mu_r(\boldsymbol{C})=0$, where μ_r is the invariant measure corresponding to $\boldsymbol{r}$.*

Proof. To prove that $\mu_r(\boldsymbol{C})=0$, it is sufficient to prove that $\mu_r(\{x\})=0$ for singleton set $\{x\}$. For this, we take μ_r as an invariant measure generated by a self-similar IFS satisfying OSC. Then, using (1.19), we get $\dim_H(\mu_r)>0$. Now, if possible, $\mu_r(\{x\})>0$, then by the definition of the Hausdorff dimension of a measure, we have $\dim_H(\mu_r)=0$, which is a contradiction. This completes the proof. $\square$

Example 6.3. In the above proposition, the OSC is necessary. To see this, consider $\upsilon_1: I\to\mathbb{R}$ and $\upsilon_2: I\to\mathbb{R}$ be two maps defined as $\upsilon_1(x)=\frac{x}{2}$ and $\upsilon_2(x)=\frac{x}{3}$, respectively. Then, $\{I: \upsilon_1, \upsilon_2\}$ is an IFS whose attractor is $\{0\}$. This IFS is self-similar and does not satisfy the OSC. Then, for any invariant measure μ supported on the attractor of this IFS, we have $\mu(\{0\}) = 1$.

Lemma 6.4. *Let Θ_Ω be the overlapping set of IFS $\{I \times \mathbb{R}: \Omega_1, \ldots, \Omega_N\}$, then $\mu_r^*(\Theta_\Omega) = 0$ for all invariant measures μ_r^* generated by the IFS $\{I \times \mathbb{R}: \Omega_1, \ldots, \Omega_N\}$.*

Proof. Let $\Phi_{h^*}: I \to \mathcal{G}(h^*)$ be the homeomorphism defined in Theorem 6.1 and Θ_u be the overlapping set generated by the IFS $\{I: u_1, \ldots, u_N\}$. Then, we have

$$\begin{aligned}
\Phi_{h^*}(\Theta_u) &= \Phi_{h^*}\left(\bigcup_{i\neq j} u_i(I) \cap u_j(I)\right)\\
&= \bigcup_{i\neq j} \Phi_{h^*}(u_i(I) \cap u_j(I))\\
&= \bigcup_{i\neq j} (\Phi_{h^*}(u_i(I)) \cap \Phi_{h^*}(u_j(I)))\\
&= \bigcup_{i\neq j} (\{(u_i(x), h^*(u_i(x))) : x \in I\} \cap \{(u_j(x), h^*(u_j(x))) : x \in I\})\\
&= \bigcup_{i\neq j} (\{(u_i(x), F_i(x, h^*(x))) : x \in I\} \cap \{(u_j(x), F_i(x, h^*(x))) : x \in I\})\\
&= \bigcup_{i\neq j} (\Omega_i(\mathcal{G}(h^*)) \cap \Omega_j(\mathcal{G}(h^*))) = \Theta_\Omega.
\end{aligned}$$

Since the IFS $\{I: u_1, \ldots, u_N\}$ is a self-similar IFS and satisfies the OSC, hence in view of [63, Theorem 2.1], we have $\mu_r(\Theta_u) = 0$ for each invariant measure μ_r generated by IFS $\{I: u_1, \ldots, u_N\}$. Now, using the fact that every overlapping set is a Borel set, Theorem 6.1 concludes the required result. □

Remark 6.5. Note that our result here is different from the result given by Morán and Rey[63] because there they have assumed the IFS should satisfy the self-similarity condition, but the IFS $\{I \times \mathbb{R}: \Omega_1, \ldots, \Omega_N\}$ is not self-similar. So, we observe that

FIF constructed for a given data set may not satisfy the self-similarity condition but still holds [63, Theorem 2.1].

Proposition 6.6. *For the Graph of FIF $\mathcal{G}(h^*)$, we get the following:*

(i) the overlapping set of the graph of h^* is $\Theta_\Omega = \{(x_1, y_1), \dots, (x_{N-1}, y_{N-1})\}$

(ii) the dynamical boundary of the graph of h^* is $\partial\mathcal{G}(h^*) = \{(x_0, y_0), (x_N, y_N)\}$

(iii) the inner boundary $\widehat{\Theta}_\Omega$ of the graph of h^* is dense in $\mathcal{G}(h^*)$.

Proof. Here, the graph of FIF is an attractor of the IFS $\{I \times \mathbb{R} : \ \Omega_1, \dots, \Omega_N\}$ and the overlapping set, the dynamical boundary and the inner boundary are defined with respect to this IFS.

(i) To prove this part, for $i = \Sigma_{N-1}$, let $(x_i, y_i) \in \{(x_1, y_1), \dots, (x_{N-1}, y_{N-1})\}$, then

$$\Omega_i(x_N, y_N) = (u_i(x_N), F_i(x_N, y_N)) = (x_i, y_i)$$

and

$$\Omega_{i+1}(x_0, y_0) = (u_{i+1}(x_0, y_0), F_{i+1}(x_0, y_0)) = (x_i, y_i).$$

This implies that $(x_i, y_i) \in \Theta_\Omega$, hence $\{(x_1, y_1), \dots, (x_{N-1}, y_{N-1})\} \subseteq \Theta_\Omega$.

Let if possible $\Theta_\Omega \neq \{(x_1, y_1), \dots, (x_{N-1}, y_{N-1})\}$, then there exists $(x^*, y^*) \in \Theta_\Omega$ such that $(x, y) \notin \{(x_1, y_1), \dots, (x_{N-1}, y_{N-1})\}$. From the definition of Θ_Ω, $x \in I_i \cap I_j$ for $i \neq j$, which is impossible. This completes the proof.

(ii) In view of (i), note that $\mathcal{F}_\Omega^{-1}(\Theta_\Omega) = \bigcup_{i=1}^N \Omega_i^{-1}(\Theta_\Omega) = \{(x_0, y_0), (x_N, y_N)\}$. Also for any $k \in \mathbb{N}$, $\mathcal{F}_\Omega^{-k}(\Theta_\Omega) = \{(x_0, y_0), (x_N, y_N)\}$, hence $\bigcup_{k=1}^\infty \mathcal{F}_\Omega^{-k}(\Theta_\Omega) = \{(x_0, y_0), (x_N, y_N)\}$, where $\mathcal{F}_\Omega$ is the Hutchinson operator. This completes the proof.

(iii) Since Θ_Ω is a compact subset of $\mathcal{G}(h^*)$, therefore $\lim_{k\to\infty} \mathcal{F}_\Omega^k(\Theta_\Omega) = \mathcal{G}(h^*)$. Also, observe that $\{\mathcal{F}_\Omega^k(\Theta_\Omega)\}_k$ is a sequence of increasing sets. As inner boundary $\widehat{\Theta}_\Omega$ is given by $\bigcup_{k=0}^{\infty} \mathcal{F}_\Omega^k(\Theta_\Omega)$. We need to prove that $\overline{\widehat{\Theta}}_\Omega = \mathcal{G}(h^*)$. For this, we have

$$\begin{aligned} H_d(\overline{\widehat{\Theta}}_\Omega, \mathcal{G}(h^*)) = H_d(\widehat{\Theta}_\Omega, \mathcal{G}(h^*)) &= H_d\left(\bigcup_{k=0}^{\infty} \mathcal{F}_\Omega^k(\Theta_\Omega), \mathcal{G}(h^*)\right) \\ &= H_d\left(\lim_{k\to\infty} \bigcup_{n=0}^{k} \mathcal{F}_\Omega^n(\Theta_\Omega), \mathcal{G}(h^*)\right) \\ &= \lim_{k\to\infty} H_d(\mathcal{F}_\Omega^k(\Theta_\Omega), \mathcal{G}(h^*)) \to 0. \end{aligned}$$

Hence, $\overline{\widehat{\Theta}}_\Omega = \mathcal{G}(h^*)$. This implies that the inner boundary is dense in $\mathcal{G}(h^*)$.

□

Remark 6.7. In view of Proposition 6.6 and the definition of a nonoverlapping set (Definition 1.5), the graph of FIF $\mathcal{G}(h^*)$ is nonoverlapping. Now using [7, Theorem 2.2], we can prove Lemma 6.4, i.e., $\mu_r^*(\Theta_\Omega) = 0$ for all invariant measures μ_r^* and using the same, we can also establish that $\mu_r^*(\partial\mathcal{G}(h^*)) = 0$, and $\mu_r^*(\widehat{\Theta}_\Omega) = 0$ for all invariant measures μ_r^* on $\mathcal{G}(h^*)$.

Remark 6.8. In view of Proposition 6.6 and Proposition 6.2, we can further prove that $\mu_r^*(\partial\mathcal{G}(h^*)) = 0$, and $\mu_r^*(\widehat{\Theta}_\Omega) = 0$ for all invariant measures μ_r^* on $\mathcal{G}(h^*)$.

Corollary 6.9. *Let $\boldsymbol{r} = (r_1, \dots, r_N)$ be a probability vector such that μ_r and μ_{r^*} be the invariant measures corresponding to $\boldsymbol{r}$ generated by $\{I : u_1, \dots, u_N\}$ and $\{I \times \mathbb{R} : \Omega_1, \dots, \Omega_N\}$, respectively. Let $\boldsymbol{C}$ be a countable subset of the $\mathcal{G}(h^*)$. Then, $\mu_r^*(\boldsymbol{C}) = 0$.*

Proof. We get this result using Theorem 6.1 and Proposition 6.2. □

Remark 6.10. Notice that we can also prove Lemma 6.4 by using the above corollary.

Definition 6.11. Let μ_1 and μ_2 be two Borel measures on $\mathbb{R}$ such that for any Borel set B, $\mu_2(B) = 0$ implies $\mu_1(B) = 0$, then μ_1 is said to be absolutely continuous with respect to μ_2 and denoted as $\mu_1 << \mu_2$. Further, if $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$, then μ_1 and μ_2 are said to be equivalent measures.

Theorem 6.12. *Let q_i be Lipschitz map and $\alpha_i(x) = \alpha_i$ for each $i \in \Sigma_N$. Set $a_{\min} := \min\{a_i : i \in \Sigma_N\}$ and $\alpha_{\max} := \max\{|\alpha_i| : i \in \Sigma_N\}$. If $\frac{\alpha_{\max}}{a_{\min}} < 1$, then there exists a probability vector $\overline{\boldsymbol{r}}$ such that corresponding to this $\overline{\boldsymbol{r}}$, the invariant measure $\boldsymbol{\mu}^*$ generated by the IFS $\{I \times \mathbb{R} : \Omega_1, \dots, \Omega_N\}$ is equivalent to the one-dimensional Hausdorff measure supported on $\mathcal{G}(h^*)$, i.e., $H^1|_{\mathcal{G}(h^*)}$.*

Proof. Since $\{I : u_1, \dots, u_N\}$ is a system of self-similar IFS and it satisfies the OSC, then using [63, Theorem 2.2] and (1.3), we have a probability vector $\overline{\boldsymbol{r}} = \{a_i : i \in \Sigma_N\}$ such that the invariant measure generated by the IFS $\{I : u_1, \dots, u_N\}$ corresponding to this probability vector is $\boldsymbol{\mu} = H^1|_I$. Let $\boldsymbol{\mu}^*$ be the invariant measure corresponding to $\overline{\boldsymbol{r}}$ generated by IFS $\{I \times \mathbb{R} : \Omega_1, \dots, \Omega_N\}$.

Let $\mathcal{B}$ be a Borel set of $\mathcal{G}(h^*)$ and Φ_{h^*} be the homeomorphism as defined in Theorem 6.1, then using Theorem 6.1, we have

$$\boldsymbol{\mu}^*(\mathcal{B}) = \boldsymbol{\mu}(\Phi_{h^*}^{-1}(\mathcal{B})) = H^1|_I(\Phi_{h^*}^{-1}(\mathcal{B})). \tag{6.1}$$

Using [47], we get h^* as a Lipschitz function. Let L_{h^*} be the Lipschitz constant of h^*. Then, for all $x, y \in I$, we have

$$\|\Phi_{h^*}(x) - \Phi_{h^*}(y)\|_2 = \|(x, h^*(x)) - (y, h^*(y))\|_2 \le (L_{h^*} + 1)|x - y|.$$

Also $|x - y| \leq \|\Phi_{h^*}(x) - \Phi_{h^*}(x)\|_2$, hence Φ_{h^*} is a bi-Lipschitz map, hence for any Borel set $\mathcal{B}$ of $\mathcal{G}(h^*)$, we have

$$H^1(\Phi_{h^*}^{-1}(\mathcal{B})) \leq H^1(\mathcal{B}) \leq (L_{h^*} + 1)H^1(\Phi_{h^*}^{-1}(\mathcal{B})). \tag{6.2}$$

From Equations (6.1) and (6.2),

$$\boldsymbol{\mu}^*(\mathcal{B}) \leq H^1(\mathcal{B}) \leq (L_{h^*} + 1)\boldsymbol{\mu}^*(\mathcal{B}). \tag{6.3}$$

This implies that $\boldsymbol{\mu}^*$ is absolutely continuous with respect to $H^1|_{\mathcal{G}(h^*)}$ and also $H^1|_{\mathcal{G}(h^*)}$ is absolutely continuous with respect to $\boldsymbol{\mu}^*$, hence the proof. $\square$

Note 6.13. Throughout this chapter, $\overline{\boldsymbol{r}}$ denotes the probability vector $\{a_i : i \in \Sigma_N\}$ such that $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$ denote the invariant measures corresponding to the probability vector $\overline{\boldsymbol{r}}$ generated by IFS $\{I : \ u_1, \dots, u_N\}$ and $\{I \times \mathbb{R} : \ \Omega_1, \dots, \Omega_N\}$, respectively.

Remark 6.14. From 6.3, it is evident that if the given data is linear and FIF is a constant map, then the invariant measure corresponding to the probability vector, given in Theorem 6.12, is the same as the one-dimensional Hausdorff measure supported on the graph of FIF.

Remark 6.15. In view of [63, Theorem 2.2], corresponding to probability vector $\overline{\boldsymbol{r}}$ the invariant measure generated by the IFS $\{I : \ u_1, \dots, u_N\}$ is one-dimensional Hausdorff measure supported on I. But the invariant measure generated by the IFS $\{I \times \mathbb{R} : \ \Omega_1, \dots, \Omega_N\}$ corresponding to the probability vector $\overline{\boldsymbol{r}}$ may not be equal to the one-dimensional Hausdorff measure supported on the graph of h^*.

To see this, consider a set of data points $\Delta = \{(0,0), (\frac{1}{2}, 1), (\frac{3}{4}, 1), (1,0)\}$. Let $u_1 : [0,1] \to [0, \frac{1}{2}]$, $u_2 : [0,1] \to [\frac{1}{2}, \frac{3}{4}]$, and $u_3 : [0,1] \to [\frac{3}{4}, 1]$ be homeomorphism maps defined as $u_1(x) = \frac{1}{2}x$, $u_2(x) = \frac{1}{4}x + \frac{1}{2}$, and $u_3(x) = \frac{1}{4}x + \frac{3}{4}$. Now for $i = 1, 2, 3$, take $F_i(x, y) = g(u_i(x))$, where g is the function whose graph is depicted in Fig 6.1.

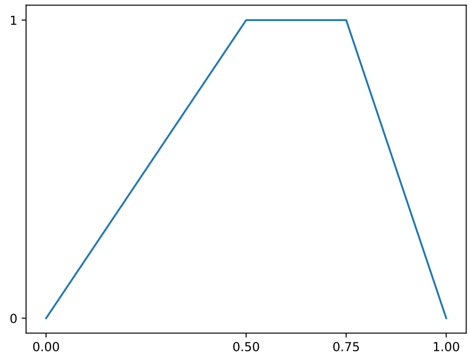

FIGURE 6.1: Graph of g.

Define $\Omega_i(x, y) = (u_i(x), F_i(x, y))$ for $i = 1, 2, 3$. Then, it is clear that FIF associated with the IFS $\{[0, 1] \times \mathbb{R} : \ \Omega_1, \Omega_2, \Omega_3\}$ is g.

Now let $\boldsymbol{\mu}^*$ be the invariant measure associated with this IFS corresponding to the probability vector $\overline{\boldsymbol{r}} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. To show that $\boldsymbol{\mu}^* \neq H^1|_{\mathcal{G}(g)}$, let Borel set $\mathcal{B}$ of $\mathcal{G}(g)$ be the line segment joining from $(\frac{3}{4}, 1)$ to $(1, 0)$. Then, we have

$$\boldsymbol{\mu}^*(\mathcal{B}) = \frac{1}{2}\boldsymbol{\mu}^*(\Omega_1^{-1}(\mathcal{B})) + \frac{1}{4}\boldsymbol{\mu}^*(\Omega_2^{-1}(\mathcal{B})) + \frac{1}{4}\boldsymbol{\mu}^*(\Omega_3^{-1}(\mathcal{B})) = \frac{1}{4}\boldsymbol{\mu}^*(\mathcal{G}(g)) = \frac{1}{4}.$$

However,

$$H^1|_{\mathcal{G}(g)}(\mathcal{B}) = \frac{H^1(\mathcal{B})}{H^1(\mathcal{G}(g))} = \frac{\frac{\sqrt{17}}{4}}{\frac{\sqrt{20}+1+\sqrt{17}}{4}} \approx 0.43 \neq 0.25 = \boldsymbol{\mu}^*(\mathcal{B}).$$

Theorem 6.16. *Let $\{I : \ u_1, \ldots, u_N\}$ and $\{I \times \mathbb{R} : \ \Omega_1, \ldots, \Omega_N\}$ be the IFSs and $\boldsymbol{r} = (r_1, \ldots, r_N)$ be a probabaility vector such that μ_r and μ_r^* be the invariant measures corresponding to $\boldsymbol{r}$ and generated by IFSs $\{I : \ u_1, \ldots, u_N\}$ and $\{I \times \mathbb{R} : \ \Omega_1, \ldots, \Omega_N\}$, respectively. Then, we have the following:*

(i) $\dim_H(\mu_r) \leq \dim_H(\mu_r^*)$

(ii) if for all $i \in \Sigma_N$*,* q_i *is a Lipschitz map,* $\alpha_i(x) = \alpha_i$ *and* $\frac{\alpha_{\max}}{a_{\min}} < 1$*, where* $a_{\min} := \min\{a_i : i \in \Sigma_N\}$ *and* $\alpha_{\max} := \max\{|\alpha_i| : i \in \Sigma_N\}$*. Then,*

$$\dim_H(\mu_r) = \dim_H(\mu_r^*).$$

Proof. Note that the homeomorphism map $\Phi_{h^*} : I \to \mathcal{G}(h^*)$ satisfies the condition

$$|x - y| \leq \|\Phi_{h^*}(x) - \Phi_{h^*}(y)\|_2 \text{ for all } x, y \in I.$$

(i) We know that

$$\begin{aligned} \dim_H(\mu_r) &= \inf\{\dim_H(A) : A \text{ is Borel subset of } I,\ \mu_r(A) > 0\} \text{ and} \\ \dim_H(\mu_r^*) &= \inf\{\dim_H(B) : B \text{ is a Borel subset of } \mathcal{G}(h^*),\ \mu_r^*(B) > 0\}. \end{aligned} \tag{6.4}$$

Now for any $B \subseteq \mathcal{G}(h^*)$, there exists an $A \subseteq I$ such that $\Phi_{h^*}(A) = B$. Then, using [36], we have $\dim_H(A) \leq \dim_H(B)$. We get the proof of the result using Theorem 6.1 and Equation (6.4).

(ii) Using [47], we get h^* is a Lipschitz map, and hence it is easy to prove that Φ_{h^*} is a bi-Lipschitz map. Then, by [36] and (6.4), we get $\dim_H(\mu_r) = \dim_H(\mu_r^*)$.

□

6.3 Fractal Transformation Between Two FIFs

Since $J^\infty = \{1, \ldots, N\}^\infty$ is a code space and the space (J^∞, d_{J^∞}) is a compact space with respect to the metric d_{J^∞}, where $d_{J^\infty} : J^\infty \times J^\infty \to \mathbb{R}$ such that for $\zeta, \theta \in J^\infty$

with $\zeta \neq \theta$, we have

$$d_{J^\infty}(\zeta, \theta) = 2^{-n}, \text{where } n \text{ is the least integer such that } \zeta_n \neq \theta_n.$$

Let $\{I : \ u_1, \ldots, u_N\}$ and $\{I \times \mathbb{R} : \ \Omega_1, \ldots, \Omega_N\}$ be the IFSs as defined in Subsection 1.3.1 such that I and $\mathcal{G}(h^*)$ are the attractors of these IFSs, respectively. Then, the following result is derived by Barnsley [9], which establishes the relation between the coding maps $\pi_\Omega : J^\infty \to \mathcal{G}(h^*)$ and $\pi_u : J^\infty \to I$.

Lemma 6.17. [9] *Let $\pi_\Omega : J^\infty \to \mathcal{G}(h^*)$ and $\pi_u : J^\infty \to I$ be the coding maps, then for any $\theta \in J^\infty$, we have*

$$\pi_\Omega(\theta) = (\pi_u(\theta), h^*(\pi_u(\theta)))\,.$$

Theorem 6.18. *Consider the IFSs $\{I : \ u_1, \ldots, u_N\}$ and $\{I \times \mathbb{R} : \ \Omega_1, \ldots, \Omega_N\}$. Then, the homeomorphism $\Phi_{h^*} : I \to \mathcal{G}(h^*)$ defined by $\Phi_{h^*}(x) = (x, h^*(x))$, is a fractal transformation.*

Proof. Fractal transformation between I and $\mathcal{G}(h^*)$ is defined as $T_{u\Omega} : I \to \mathcal{G}(h^*)$ such that $T_{u\Omega} = \pi_\Omega \circ \tau_u$. We need to show that $T_{u\Omega}(x) = \Phi_{h^*}(x)$ for all $x \in I$. For this, we have

$$\begin{aligned} T_{u\Omega}(x) =& (\pi_\Omega \circ \tau_u)(x) \\ =& \pi_\Omega(\theta), \qquad \text{where } \theta \in J^\infty \text{ such that } \pi_u(\theta) = x \\ =& (\pi_u(\theta), h^*(\pi_u(\theta))) = (x, h^*(x)) = \Phi_{h^*}(x). \end{aligned}$$

This completes the proof. □

Theorem 6.19. *For $i \in \Sigma_N$, let $\{I : \ u_1, \ldots, u_N\}$ be the IFS as discussed in Subsection 1.3.1. Let $F_i, F'_i : I \times \mathbb{R} \to \mathbb{R}$ be two continuous maps defined on $I \times \mathbb{R}$*

such that for all $x \in I,\ y, y^* \in \mathbb{R}$ *and* $0 \leq \rho_i,\ \rho_i' < 1$

$$|F_i(x,y) - F_i(x,y^*)| \leq \rho_i |y - y^*| \qquad |F_i'(x,y) - F_i'(x,y^*)| \leq \rho_i' |y - y^*|$$
$$F_i(x_0,y_0) = y_{i-1} \text{ and } F_i(x_N,y_N) = y_i \qquad F_i'(x_0,y_0) = y_{i-1} \text{ and } F_i'(x_N,y_N) = y_i.$$

Let $\Omega_i, \Omega_i' : I \times \mathbb{R} \to I \times \mathbb{R}$ *be maps defined as* $\Omega_i(x,y) = (u_i(x), F_i(x,y))$ *and* $\Omega_i'(x,y) = (u_i(x), F_i'(x,y))$*. Let* h^* *be the FIF generated by IFS* $\{I \times \mathbb{R} :\ \Omega_1, \ldots, \Omega_N\}$ *and* g^* *be the FIF generated by IFS* $\{I \times \mathbb{R} :\ \Omega_1', \ldots, \Omega_N'\}$*. Let* $T_{\Omega\Omega'}$ *be the fractal transformation from* $\mathcal{G}(h^*)$ *to* $\mathcal{G}(g^*)$*. Then,* $T_{\Omega\Omega'}$ *is a homeomorphism.*
Further, $T_{\Omega\Omega'} = \Phi_{g*} \circ \Phi_{h*}^{-1}$*, where* $\Phi_{h^*} : I \to \mathcal{G}(h^*)$ *and* $\Phi_{g^*} : I \to \mathcal{G}(g^*)$ *are homeomorphism maps defined as* $\Phi_{h^*}(x) = (x, h^*(x))$ *and* $\Phi_{g^*}(x) = (x, g^*(x))$*, respectively.*

Proof. Let $P_\Omega = \left\{\pi_\Omega^{-1}(x, h^*(x)) :\ x \in I\right\}$ and $P_{\Omega'} = \left\{\pi_{\Omega'}^{-1}(x, g^*(x)) :\ x \in I\right\}$ be the partitions of the code space J^∞. Then, in view of [7, Proposition 2.3], to prove $T_{\Omega\Omega'}$ is homeomorphism, it is sufficient to prove that $P_\Omega = P_{\Omega'}$. For this, let $\theta \in P_\Omega$, then using Lemma 6.17, we have $\pi_\Omega(\theta) = (x, h^*(x)) = (\pi_u(\theta), h^*(\pi_u(\theta)))$. Also we have, $\pi_{\Omega'}(\theta) = (\pi_u(\theta), g^*(\pi_u(\theta))) = (x, g^*(x))$, this implies that $\theta \in P_{\Omega'}$, hence $P_\Omega \subseteq P_{\Omega'}$. Similarly, we can prove that $P_{\Omega'} \subseteq P_\Omega$. This yields that $T_{\Omega\Omega'}$ is a homeomorphism. For further equality, let $(x, h^*(x)) \in \mathcal{G}(h^*)$, then we have

$$T_{\Omega\Omega'}(x, h^*(x)) = (\pi_{\Omega'} \circ \tau_\Omega)(x, h^*(x)) = \pi_{\Omega'}(\theta) = (x, g^*(x)) = \Phi_{g*} \circ \Phi_{h*}^{-1}(x, h^*(x)),$$

where $\theta \in J^\infty$ such that $\pi_u(\theta) = x$ and $\pi_{\Omega'}(\theta) = (\pi_u(\theta), h^*(\pi_u(\theta)))$. This completes the proof of the theorem. $\square$

6.4 Fractal Functions in Different Spaces

In this section, we shall define some spaces and present some results that give conditions for a FIF to be an element of those spaces.

6.4.1 $L^p(I)$-Space with Respect to Invariant Measures

Let $\boldsymbol{r} = (r_1, \ldots, r_N)$ be a probability vector and μ_r be the invariant measure corresponding to this probability vector generated by the IFS $\{I : \ u_1, \ldots, u_N\}$ such that

$$\mu_r = \sum_{i \in \Sigma_N} r_i \mu_r \circ u_i^{-1}.$$

For $1 \leq p < \infty$, we denote the space of all p-integrable functions on I associated to μ_r by $L^p(I, \mu_r)$, where

$$L^p(I, \mu_r) = \left\{ f : I \to \mathbb{R} : \int_I |f(x)|^p d\mu_r(x) < \infty \right\}.$$

Since the space $(L^p(I, \mu_r), \|\cdot\|_p)$ is a complete normed space with respect to $\|\cdot\|_p$, where

$$\|f\|_p = \left(\int_I |f(x)|^p d\mu_r(x) \right)^{1/p}.$$

For $p = \infty$, we define essential sup norm $\|\cdot\|_\infty$ with respect to measure μ_r such that $\|g\|_\infty = ess\sup|g|$. Then, the space $L^\infty(I, \|\cdot\|_\infty) = \{g : I \to \mathbb{R} : \ ess\sup|g| < \infty\}$ is a complete space.

Note 6.20. If $\mu_1 \neq \mu_2$, then $L^p(I, \mu_1)$ may not be equal to $L^p(I, \mu_2)$. For instance, let $\upsilon_1 : I \to \mathbb{R}$ and $\upsilon_2 : I \to \mathbb{R}$ be two maps defined as $\upsilon_1(x) = \frac{x}{2}$ and $\upsilon_2(x) = \frac{x}{2} + \frac{1}{2}$, respectively. Then, note that the IFS $\{I : \ \upsilon_1, \upsilon_2\}$ is self-similar and satisfies the OSC. Let $\boldsymbol{r}_1 = (\frac{1}{2}, \frac{1}{2})$ be a probability vector. Then, using [63, Theorem 2.2], invariant measure μ_1 corresponding to $\boldsymbol{r}_1$ is $\mu_1 = H^1|_I$ (one dimensional Hausdorff measure supported on I) and let $\boldsymbol{r}_2 = (\frac{1}{4}, \frac{3}{4})$ be another probability vector such that μ_2 is the invariant measure corresponding to the probability vector $\boldsymbol{r}_2$. We know that $\dim_H(\mu_1) = \dim_H(I) = 1$ and

$$\dim_H(\mu_2) = \frac{-\frac{1}{4}\log(4)+\frac{3}{4}\log(3/4)}{-\frac{1}{4}\log(2)-\frac{3}{4}\log(2)} = \frac{\frac{3}{4}\log(3)-2\log(2)}{-\log(2)} \approx 0.8112781245 < 1.$$

Therefore, for $\dim_H(\mu_2) < \gamma < 1$ there exists a set $D_\gamma \subset I$ such that $\dim_H D_\gamma = \gamma$ and $\mu_2(D_\gamma) > 0$ but $\mu_1(D_\gamma) = 0$. Now define a map

$$f^*(x) = \begin{cases} \infty, & \text{if } x \in D_\gamma \\ 0, & \text{otherwise.} \end{cases}$$

It is simple to prove that $f^* \in L^p(I, \mu_1)$ but not in $L^p(I, \mu_2)$.

Now consider the map $g^* : I \to \mathbb{R} \cup \{\infty\}$ such that

$$g^*(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Claim: $g^* \notin L^1(I, \mu_1)$ but $g^* \in L^1(I, \mu_2)$

We know that the

$$\int_I g^* d\mu_1 = \sup\left\{\int_I s d\mu_1 : \ s \text{ is a simple function such that } 0 \leq s \leq g^*\right\}.$$

Define $s_n(x) = \sum_{k=1}^{n} k\ \chi_{J_k}(x)$, where $J_k = \left(\frac{1}{k+1}, \frac{1}{k}\right]$ and χ_{J_k} is the characteristic function on J_k. Since $0 \le s_n \le g^*$, then

$$\begin{aligned}\int_I g^* d\mu_1 &\ge \int_I s_n d\mu_1 \\ &\ge \sum_{k=1}^{n} k\mu_1(J_k) \\ &\ge \sum_{k=1}^{n} k\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k+1} \to \infty.\end{aligned}$$

This implies that $\int_I g^* d\mu_1 = \infty$, hence $g^* \notin L^1(I, \mu_1)$. We will show that $g^* \in L^1(I, \mu_2)$. Define $s_n(x) = \sum_{k=1}^{n} k'_n\ \chi_{J_k}(x)$, where $J_k = \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]$ and χ_{J_k} is the characteristic function on J_k, and $k'_n = \frac{1}{d_{k,i}}$ is the number obtained by dividing the subinterval J_k containing x into n equi-length subintervals $J_{k,i}$ such that $x \in J_{k,i} = (c_{k,i}, d_{k,i}]$. Since $0 \le s_n(x) \le s_{n+1}(x)$ and $f(x) = \lim_{n\to\infty} s_n(x)$, the monotone convergence theorem dictates

$$\begin{aligned}\int_I g^* d\mu_2 &= \lim_{n\to\infty} \int_I s_n d\mu_2 \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} k'_n \mu_2(J_k) \\ &\le \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} 2^{k+1} \frac{3}{4^{k+1}} \le \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \frac{3}{2^{k+1}} = \frac{3}{2}.\end{aligned}$$

This completes our task.

Theorem 6.21. *If $\alpha_i \in L^\infty(I, \|\cdot\|_\infty)$ and $q_i \in L^p(I, \mu_r)$ for all $i \in \Sigma_N$, then the associated FIF $h^* \in L^p(I, \mu_r)$, provided $\sum_{i\in\Sigma_N} (\|\alpha_i\|_\infty^p r_i) < 1$.*

Proof. Let $L_0^p(I, \mu_r) = \{g \in L^p(I, \mu_r) :\ g(x_1) = y_1,\ g(x_N) = y_N\}$ be a subset of $L^p(I, \mu_r)$. It is easy to note that $L_0^p(I, \mu_r)$ is a closed subset of $L^p(I, \mu_r)$ and hence complete subspace with respect to the metric induced by $\|\cdot\|_p$. Define a Read-Bajraktarevic (RB) operator $T : L_0^p(I,\mu_r) \to L_0^p(I,\mu_r)$ such that

$$(Tg)(x) = \alpha_i(u_i^{-1}(x))g(u_i^{-1}(x)) + q_i(u_i^{-1}(x)) \text{ for all } x \in I_i, \text{ where } i \in \Sigma_N. \quad (6.5)$$

With assumptions on q_i and u_i, it is easy to find that T is well-defined. Now let $f, g \in L_0^p(I,\mu_r)$, then we have

$$\begin{aligned}
\|Tf - Tg\|_p^p &= \int_I |(Tf)(x) - (Tg)(x)|^p d\mu_r(x) \\
&= \sum_{i\in\Sigma_N} \int_{I_i} |\alpha_i(u_i^{-1}(x))f(u_i^{-1}(x)) - \alpha_i g(u_i^{-1}(x))|^p d\mu_r(x) \\
&= \sum_{i\in\Sigma_N} \int_{I_i} |\alpha_i(u_i^{-1}(x))(f-g)(u_i^{-1}(x))|^p d\mu_r(x) \\
&\le \sum_{i\in\Sigma_N} \|\alpha_i\|_\infty^p \int_{I_i} |(f-g)(u_i^{-1}(x))|^p d\mu_r(x) \\
&= \sum_{i\in\Sigma_N} \|\alpha_i\|_\infty^p \int_{I_i} |(f-g)(u_i^{-1}(x))|^p d\left(\sum_{k\in\Sigma_N} r_k \mu_r(u_k^{-1}(x))\right) \\
&= \sum_{i\in\Sigma_N} \|\alpha_i\|_\infty^p \int_{I_i} |(f-g)(u_i^{-1}(x))|^p r_i d(\mu_r(u_i^{-1}(x))) \\
&= \sum_{i\in\Sigma_N} \|\alpha_i\|_\infty^p r_i \int_I |(f-g)(x)|^p d\mu_r(x) = \sum_{i\in\Sigma_N} \|\alpha_i\|_\infty^p r_i \|f-g\|_p^p.
\end{aligned}$$

This implies that $\|Tf - Tg\|_p^p \le \sum_{i\in\Sigma_N}(\|\alpha_i\|_\infty^p r_i)\|f-g\|_p^p$, which further implies that T is a contraction map and hence there exists a unique fixed point h^* of T such that for all $x \in I_i$, where $i \in \Sigma_N$, we have $h^*(x) = \alpha_i(u_i^{-1}(x))h^*(u_i^{-1}(x)) + q_i(u_i^{-1}(x))$. This completes the proof. □

Remark 6.22. In [59], Massopust has discussed local FIFs in L^p spaces. There the measure is the one-dimensional Lebesgue measure. Here, we have proved the result for all invariant measures generated by IFS, so our result may be treated as a generalization of the previous result.

Now we shall define subsets of $\mathcal{C}^{\boldsymbol{m}}(I)$, which will help to find the dimension of FIF and will provide a more appropriate dimensional bound of the graph of fractal functions. It is well known that the space $(\mathcal{C}^{\boldsymbol{m}}(I), \|\cdot\|_{\boldsymbol{m}})$ is a Banach space with respect to the norm $\|\cdot\|_{\boldsymbol{m}}$, where $\|g\|_{\boldsymbol{m}} = \|g\|_\infty + \|g^{(1)}\|_\infty + \cdots + \|g^{(\boldsymbol{m})}\|_\infty$ for $g \in \mathcal{C}^{\boldsymbol{m}}(I)$ with $g^{(k)}$ as the k^{th} derivative of g.

6.4.2 Smooth Dimensional Space

Before defining the smooth dimensional space, let us define the following space and provide some required results to define the main space. For $\beta \in [1,2]$ and $\boldsymbol{m} \in \mathbb{N} \cup \{0\}$, let us define a set $\mathfrak{V}_\beta^{\boldsymbol{m}}(I) = \{g \in \mathcal{C}^{\boldsymbol{m}}(I) : \|g\|_{\beta_{\boldsymbol{m}}} < \infty\}$, where

$$\|g\|_{\beta_{\boldsymbol{m}}} = \|g\|_{\boldsymbol{m}} + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{g^{(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}},$$

where $R_{g^{(\boldsymbol{m})}}(U) = \sup_{x,y\in U}|g^{(\boldsymbol{m})}(x) - g^{(\boldsymbol{m})}(y)|$ for $U \subseteq I$. It is elementary to see that $(\mathfrak{V}_\beta^{\boldsymbol{m}}(I), \|\cdot\|_{\beta_{\boldsymbol{m}}})$ is a normed space.

Lemma 6.23. *The space $(\mathfrak{V}_\beta^{\boldsymbol{m}}(I), \|\cdot\|_{\beta_{\boldsymbol{m}}})$ is a Banach space.*

Proof. Let $\{g_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathfrak{V}_\beta^{\boldsymbol{m}}(I), \|\cdot\|_{\beta_{\boldsymbol{m}}})$. This implies that $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $(\mathcal{C}^{\boldsymbol{m}}(I), \|\cdot\|_{\boldsymbol{m}})$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to some $g^* \in \mathcal{C}^{\boldsymbol{m}}(I)$, hence we have

$$\|g^*\|_{\boldsymbol{m}} + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{g^{*(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}} \leq \limsup_{n\to\infty} \|g^*\|_{\beta_{\boldsymbol{m}}}.$$

This implies $g^* \in \mathfrak{V}_\beta^{\boldsymbol{m}}(I)$. Further,

$$\|g_n - g^*\|_{\beta_{\boldsymbol{m}}} \leq \limsup_{n\to\infty} \|g_n - g_k\|_{\beta_{\boldsymbol{m}}}.$$

Hence, $\{g_n\}_{n\in\mathbb{N}}$ converges to g^* in $\|\cdot\|_{\beta_{\boldsymbol{m}}}$. This completes the proof. □

Remark 6.24. If we choose $\boldsymbol{m} = 0$, this space will coincide with the space defined in [47]. Hence, this result generalizes the result in [47].

Lemma 6.25. *Let $\beta \in [1, 2]$. For all $t \in \mathbb{N}$ and $g \in \mathcal{C}^{\boldsymbol{m}}(I)$, we have*

$$\|g\|_{\beta_{\boldsymbol{m}}+\frac{1}{t+1}} \geq \|g\|_{\beta_{\boldsymbol{m}}+\frac{1}{t}}.$$

Hence, $\left\{\mathfrak{V}^{\boldsymbol{m}}_{\beta+\frac{1}{t}}(I)\right\}_{t\in\mathbb{N}}$ is a decreasing sequence of Banach spaces.

Proof. For $t \in \mathbb{N}$ and $g \in \mathcal{C}^{\boldsymbol{m}}(I)$, we have

$$\begin{aligned}\|g\|_{\beta_{\boldsymbol{m}}+\frac{1}{t}} =& \|g\|_{\boldsymbol{m}} + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{g^{(\boldsymbol{m})}}(U)}{2^{n(\beta+\frac{1}{t}-1)}}\\ \leq& \|g\|_{\boldsymbol{m}} + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{g^{(\boldsymbol{m})}}(U)}{2^{n(\beta+\frac{1}{t+1}-1)}} = \|g\|_{\beta_{\boldsymbol{m}}+\frac{1}{t+1}}.\end{aligned}$$

This completes the proof. □

Lemma 6.26. [47, Remark 3.7] *If $|I_i| = \frac{1}{2^{n_i}}$ for some $n_i \in \mathbb{N}$ such that $\sum_{i\in\Sigma_N} \frac{1}{2^{n_i}} = 1$, then for $m \geq \max_{i\in\Sigma_N}\{n_i\}$, we have*

$$\sum_{|U|=2^{-n}} R_g(U) = \sum_{i\in\Sigma_N} \sum_{\substack{|U|=2^{-n}\\ U\subset I_i}} R_g(U).$$

Theorem 6.27. *Let $q_i \in \mathfrak{V}^{\boldsymbol{m}}_\beta(I)$ and $\alpha_i(x) = \alpha_i$ for all $i \in \Sigma_N$ and let $|I_i| = \frac{1}{2^{n_i}}$ for some $n_i \in \mathbb{N}$ such that $\sum_{i\in\Sigma_N} \frac{1}{2^{n_i}} = 1$. Then, $h^* \in \mathfrak{V}^{\boldsymbol{m}}_\beta(I)$, given that $\sum_{i\in\Sigma_N} |\frac{N\alpha_i}{a_i^{\boldsymbol{m}}}| < 1$.*

Proof. Let us define ${}_0\mathfrak{V}^{\boldsymbol{m}}_\beta(I) = \{g \in \mathfrak{V}^{\boldsymbol{m}}_\beta(I) : g(x_0) = y_0 \text{ and } g(x_N) = y_N\}$. It is an elementary thing to observe that ${}_0\mathfrak{V}^{\boldsymbol{m}}_\beta(I)$ is a complete metric space with respect to the metric induced by the norm $\|\cdot\|_{\beta_{\boldsymbol{m}}}$. Denote $\alpha_{\max} = \max\{|\alpha_i| : i \in \Sigma_N\}$. Now define an RB operator $\mathcal{T} : {}_0\mathfrak{V}^{\boldsymbol{m}}_\beta(I) \to {}_0\mathfrak{V}^{\boldsymbol{m}}_\beta(I)$ such that

$$(\mathcal{T}g)(x) = q_i(u_i^{-1}(x)) + \alpha_i g(u_i^{-1}(x)) \text{ for all } x \in I_i \text{ with } i \in \Sigma_N.$$

Because of the assumptions, note that $\mathcal{T}$ is well-defined. Now let $g, h \in {}_0\mathfrak{V}^{\boldsymbol{m}}_\beta(I)$, then in view of Lemma 6.26, we have

$$\begin{aligned}
&\|(\mathcal{T}g) - (\mathcal{T}h)\|_{\beta_{\boldsymbol{m}}} \\
=&\|(\mathcal{T}g) - (\mathcal{T}h)\|_{\boldsymbol{m}} + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{(\mathcal{T}g-\mathcal{T}h)^{(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}} \\
=&\|(\mathcal{T}g) - (\mathcal{T}h)\|_\infty + \cdots + \|(\mathcal{T}g)^{(\boldsymbol{m})} - (\mathcal{T}h)^{(\boldsymbol{m})}\|_\infty + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{(\mathcal{T}g-\mathcal{T}h)^{(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}} \\
\leq& \max_{i\in\Sigma_N}|\alpha_i|\|g-h\|_\infty + \cdots + \max_{i\in\Sigma_N}\Big|\frac{\alpha_i}{a_i^{\boldsymbol{m}}}\Big|\|g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})}\|_\infty \\
&+ \sum_{i\in\Sigma_N}\Big|\frac{N\alpha_i}{a_i^{\boldsymbol{m}}}\Big| \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{(g-h)^{(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}} \\
\leq& \sum_{i\in\Sigma_N}\Big|\frac{N\alpha_i}{a_i^{\boldsymbol{m}}}\Big| \left(\|g-h\|_\infty + \cdots + \|g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})}\|_\infty + \sup_{n\in\mathbb{N}} \frac{\sum_{|U|=2^{-n}} R_{(g-h)^{(\boldsymbol{m})}}(U)}{2^{n(\beta-1)}} \right) \\
=& \sum_{i\in\Sigma_N}\Big|\frac{N\alpha_i}{a_i^{\boldsymbol{m}}}\Big| \|g-h\|_{\beta_{\boldsymbol{m}}}.
\end{aligned}$$

This implies that $\mathcal{T}$ is a contraction map. This completes the proof. □

Let us define a set $\mathcal{C}^{\boldsymbol{m}}_\beta(I)$ such that

$$\mathcal{C}^{\boldsymbol{m}}_\beta(I) = \{g \in \mathcal{C}^{\boldsymbol{m}}(I) : \overline{\dim}_B(\mathcal{G}(g^{(\boldsymbol{m})})) \leq \beta\} \text{ for } \beta \in [1,2]. \tag{6.6}$$

Proposition 6.28. *If $\mathcal{C}^{\boldsymbol{m}}_\beta(I)$ is defined as in (6.6), then we have*

$$\mathcal{C}^{\boldsymbol{m}}_\beta(I) = \bigcap_{n\in\mathbb{N}} \mathfrak{V}^{\boldsymbol{m}}_{\beta+\frac{1}{n}}(I).$$

Moreover, $(\mathcal{C}^{\boldsymbol{m}}_{\beta}(I), d_{\beta_{\boldsymbol{m}}})$ *is a complete metric space, where*

$$d_{\beta_{\boldsymbol{m}}}(g,h) = \sum_{n\in\mathbb{N}} \min\left\{2^{-n}, \|g-h\|_{\beta_{\boldsymbol{m}}+\frac{1}{n}}\right\}.$$

Proof. Using Lemma 6.23, Lemma 6.25 and [38, Lemma 3.2], we get $\left(\bigcap_{n\in\mathbb{N}} \mathfrak{V}^{\boldsymbol{m}}_{\beta+\frac{1}{n}}(I), d_{\beta_{\boldsymbol{m}}}\right)$ is a complete metric space.

Let $g \in \mathcal{C}^{\boldsymbol{m}}_{\beta}$, then $\overline{\dim}_B(\mathcal{G}(g^{(\boldsymbol{m})})) \le \beta$. Using [36, Proposition 11.1], for each $n \in \mathbb{N}$, one can prove that $\|g\|_{\beta_{\boldsymbol{m}}+\frac{1}{n}} < \infty$, hence $g \in \bigcap_{n\in\mathbb{N}} \mathfrak{V}^{\boldsymbol{m}}_{\beta+\frac{1}{n}}(I)$.

Conversely, suppose $g \in \bigcap_{n\in\mathbb{N}} \mathfrak{V}^{\boldsymbol{m}}_{\beta+\frac{1}{n}}(I)$, then for all $n \in \mathbb{N}$, we have

$$\sup_{k\in\mathbb{N}} \frac{\sum_{|U|=2^{-k}} R_{g^{(\boldsymbol{m})}}(U)}{2^{k(\beta+\frac{1}{n}-1)}} < \infty.$$

This implies that

$$\frac{\sum_{|U|=2^{-k}} R_{g^{(\boldsymbol{m})}}(U)}{2^k} \le C_n 2^{-\beta-\frac{1}{n}}, \text{ where } C_n \text{ depends only on } g^{(\boldsymbol{m})} \text{ and } n.$$

By [36, Proposition 11.1], we get $\overline{\dim}_B(\mathcal{G}g^{(\boldsymbol{m})}) \le \beta+\frac{1}{n}$ for all n, hence $\overline{\dim}_B\mathcal{G}(g^{(\boldsymbol{m})}) \le \beta$. This completes the proof. □

Remark 6.29. The utility of the space $\mathcal{C}^{\boldsymbol{m}}_{\beta}(I)$ can be seen as it gives a more appropriate bound for the dimension of the graph of some functions.

To see this, let $g : I \to \mathbb{R}$ be a map defined as $g(x) = \frac{2}{3}x^{3/2}$. Then, it is clear that $g \in \mathcal{C}^1(I)$ such that $g^{(1)}(x) = \sqrt{x}$.

It is easy to observe that $g^{(1)}(x) = \sqrt{x}$ is a Hölder map with Hölder exponent $\frac{1}{2}$, hence we get an upper bound of $\overline{\dim}_B(\mathcal{G}(g^{(1)})) \le 2 - \frac{1}{2} = 1.5$. Now it is trivial to prove that $g \in \mathfrak{V}^{\boldsymbol{m}}_{1+\frac{1}{n}}(I)$ for each $n \in \mathbb{N}$, hence in return $g \in \mathcal{C}^1_1(I)$. This implies that $\overline{\dim}_B(\mathcal{G}(g^{(1)})) \le 1$, hence $\overline{\dim}_B(\mathcal{G}(g^{(1)})) = 1$.

Here we witnessed that the space $\mathcal{C}^{\boldsymbol{m}}_{\beta}(I)$ gives the exact dimension of the graph of the function $\sqrt{x}$ while the classical result due to Hölder space [36] only provides an upper bound 1.5. Hence, our result refines the previously established results.

Corollary 6.30. *For all $i \in \Sigma_N$, let $q_i \in \mathcal{C}^{\boldsymbol{m}}_{\beta}(I)$ and $\alpha_i(x) = \alpha_i$. Let $|I_i| = \frac{1}{2^{n_i}}$ for some $n_i \in \mathbb{N}$ such that $\sum_{i\in\Sigma_N} \frac{1}{2^{n_i}} = 1$. Then, $\overline{\dim}_B(\mathcal{G}(h^{*(\boldsymbol{m})})) \leq \beta$, provided $\sum_{i\in\Sigma_N} |\alpha_i| < 1$.*

6.4.3 Smooth Convex-Lipschitz Space

Definition 6.31. [60] Let $g : [a, b] \to \mathbb{R}$ be a map defined on interval $[a, b] \subseteq \mathbb{R}$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a map defined on $\mathbb{R}^+$. If there exists a constant L_ω such that for $a \leq x \leq x + y \leq b$ and $0 \leq \delta \leq 1$, we have

$$|\boldsymbol{D}(x, y, \delta)| := |g(x + \delta y) - (\delta g(x + y) + (1 - \delta)g(x))| \leq L_\omega \omega(y).$$

Then, the map g is considered convex-Lipschitz of order ω.

For $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ and $\boldsymbol{m} \in \mathbb{N} \cup \{0\}$, let us define a set

$$\mathfrak{C}^{\boldsymbol{m}}_{\omega}(I) = \{g \in \mathcal{C}^{\boldsymbol{m}}(I) : \; g^{(\boldsymbol{m})} \text{ is convex-Lipschitz of order } \omega\}.$$

Define a norm $\|g\|_\omega = \|g\|_{\boldsymbol{m}} + [g^{(\boldsymbol{m})}]^*$ on $\mathfrak{C}^{\boldsymbol{m}}_{\omega}(I)$, where

$$[g^{(\boldsymbol{m})}]^* = \sup_{0\leq x<x+y\leq 1} \frac{|g^{(\boldsymbol{m})}(x + \delta y) - (\delta g^{(\boldsymbol{m})}(x + y) + (1 - \delta)g^{(\boldsymbol{m})}(x))|}{\omega(y)}.$$

Then, it is easy to notice that $(\mathfrak{C}^{\boldsymbol{m}}_{\omega}(I), \|g\|_\omega)$ is a normed space.

Lemma 6.32. *The space $(\mathfrak{C}^{\boldsymbol{m}}_{\omega}(I), \|\cdot\|_\omega)$ is a Banach space.*

Proof. Let $\{g_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathfrak{C}^{\boldsymbol{m}}_\omega(I), \|g\|_\omega)$. Then, for a given $\epsilon > 0$, we have $n_0 \in \mathbb{N}$ such that

$$\|g_n - g_l\|_\omega < \epsilon \text{ for all } n, l \geq n_0,$$

$$\text{i.e., } \|g_n - g_l\|_{\boldsymbol{m}} < \epsilon \text{ for all } n, l \geq n_0,$$

since $(\mathcal{C}^{\boldsymbol{m}}(I), \|\cdot\|_{\boldsymbol{m}})$ is a complete space, there exists a $g_* \in \mathcal{C}^{\boldsymbol{m}}(I)$ such that $g_n \to g_*$ and $g_n^{(\boldsymbol{m})} \to g_*^{(\boldsymbol{m})}$ as $n \to \infty$ uniformly, then using [29, Lemma 2.2], we have

$$\begin{aligned}\|g_n - g_*\|_\omega = \|g_n - g_*\|_{\boldsymbol{m}} + [(g_n - g_*)^{(\boldsymbol{m})}]^* &\leq \liminf_{l\to\infty} \left\{\|g_n - g_l\|_{\boldsymbol{m}} + [(g_n - g_l)^{(\boldsymbol{m})}]^*\right\} \\ &\leq \sup_{l\geq n_0} \|g_n - g_l\|_\omega < \epsilon.\end{aligned}$$

This implies that $g_* = g_* - g_{n_0} + g_{n_0} \in \mathfrak{C}^m_\omega(I)$ and $\|g_n - g_*\|_\omega < \epsilon$ for all $n \geq n_0$. This completes the proof of the lemma. □

Corollary 6.33. *Let $g \in \mathfrak{C}^{\boldsymbol{m}}_\omega(I)$ and $y \in \mathbb{R}^+$, then under the assumptions of* [60, Theorem 3], *we have*

(i) if $\omega(y) = y^\beta$, then $\dim_H(\mathcal{G}(g^{(\boldsymbol{m})})) \leq \overline{\dim}_B(\mathcal{G}(g^{(\boldsymbol{m})})) \leq 2 - \beta$

(ii) if $\omega(y) = y \ln \frac{1}{y}$, then $\dim_H(\mathcal{G}(g^{(\boldsymbol{m})})) = \dim_B(\mathcal{G}(g^{(\boldsymbol{m})})) = 1$.

Theorem 6.34. *Let $q_i \in \mathfrak{C}^{\boldsymbol{m}}_\omega(I)$ and $\alpha_i(x) = \alpha$ for all $i \in \Sigma_N$. If*

$$\max\left\{\max_{i\in\Sigma_N}|\frac{\alpha}{a_i^{\boldsymbol{m}}}|, \max_{i\in\Sigma_N}|\frac{N\alpha}{a_i^{\boldsymbol{m}}}| \sup_{0<y\leq 1}\frac{\omega(\frac{y}{a_i})}{\omega(y)}\right\} < 1.$$

Then, we have the following:

(i) $h^* \in \mathfrak{C}^{\boldsymbol{m}}_\omega(I)$

(ii) if $\omega(y) = y^{\beta}$, *then* $\dim_H(\mathcal{G}(h^{*(\boldsymbol{m})})) \le \overline{\dim_B}(\mathcal{G}(h^{*(\boldsymbol{m})})) \le 2 - \beta$ *and* *if* $\omega(y) = y \ln \frac{1}{y}$, *then* $\dim_H(\mathcal{G}(h^{*(\boldsymbol{m})})) = \dim_B(\mathcal{G}(h^{*(\boldsymbol{m})})) = 1$.

Proof. (i). Define a set $\mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I) = \{g \in \mathfrak{C}^{\boldsymbol{m}}_{\omega}(I) :\ g(x_0) = y_0 \text{ and } g(x_N) = y_N\}$. It is elementary to observe that $\mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I)$ is a closed subset of $\mathfrak{C}^{\boldsymbol{m}}_{\omega}(I)$, hence complete with respect to norm $\|\cdot\|_{\omega}$. Now define an RB operator $\mathfrak{T} : \mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I) \to \mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I)$ such that

$$(\mathfrak{T}g)(x) = q_i(u_i^{-1}(x)) + \alpha_i g(u_i^{-1}(x)) \text{ for all } x \in I_i \text{ with } i \in \Sigma_N.$$

Under the assumptions, $\mathfrak{T}$ is well defined. Now for $g, h \in \mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I)$, we have

$$\begin{aligned}
&\|\mathfrak{T}g - \mathfrak{T}h\|_{\omega}\\
=&\|\mathfrak{T}g - \mathfrak{T}h\|_{\boldsymbol{m}} + [(\mathfrak{T}g)^{(\boldsymbol{m})} - (\mathfrak{T}h)^{(\boldsymbol{m})}]^*\\
=&\sup_{x\in I}\Big(|(\mathfrak{T}g)(x) - (\mathfrak{T}h)(x)| + |(\mathfrak{T}g)^{(1)}(x) - (\mathfrak{T}h)^{(1)}(x)| + \cdots + |(\mathfrak{T}g)^{(\boldsymbol{m})}(x) - (\mathfrak{T}h)^{(\boldsymbol{m})}(x)|\Big)\\
&+ \sup_{0\le x<x+y\le 1}\frac{1}{\omega(y)}\Big|((\mathfrak{T}g)^{(\boldsymbol{m})} - (\mathfrak{T}h)^{(\boldsymbol{m})})(x+\delta y) - (\delta((\mathfrak{T}g)^{(\boldsymbol{m})} - (\mathfrak{T}h)^{(\boldsymbol{m})})(x+y)\\
&\qquad +(1-\delta)((\mathfrak{T}g)^{(\boldsymbol{m})} - (\mathfrak{T}h)^{(\boldsymbol{m})})(x))\Big|\\
\le&\max_{i\in\Sigma_N}\sup_{x\in I_i}\Big(|\alpha||(g-h)(u_i^{-1}(x))| + \cdots + |\frac{\alpha}{a_i^{\boldsymbol{m}}}||(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(u_i^{-1}(x))|\Big) + \max_{i\in\Sigma_N}\\
&\sup_{x_{i-1}\le x<x+y\le x_i}\frac{\omega(\frac{y}{a_i})}{\omega(y)}\left(\frac{1}{\omega(\frac{y}{a_i})}\left|\frac{\alpha}{a_i^{\boldsymbol{m}}}\right|\Big(\Big|(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(u_i^{-1}(x+\delta y)) - \Big(\delta(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})\right.\\
&\qquad\left.(u_i^{-1}(x+y)) + (1-\delta)(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(u_i^{-1}(x))\Big)\Big|\Big)\right)\\
\le&\max_{i\in\Sigma_N}|\frac{\alpha}{a_i^{\boldsymbol{m}}}|\left[\sup_{x\in I}\Big(|(g-h)(x)| + \cdots + |(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(x)|\Big)\right] + N\max_{i\in\Sigma_N}|\frac{\alpha}{a_i^{\boldsymbol{m}}}|\sup_{0<y\le 1}\frac{\omega(\frac{y}{a_i})}{\omega(y)}\\
&\sup_{0\le x'<x'+y'\le 1}\frac{1}{\omega(y')}\left(\left|\frac{\alpha}{a_i^{\boldsymbol{m}}}\right|\Big[\Big|(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(x'+\delta y') - \Big(\delta(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(x'+y')\right.\\
&\qquad\left.+(1-\delta)(g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})})(x')\Big)\Big|\Big]\right)\\
\le&\max_{i\in\Sigma_N}|\frac{\alpha}{a_i^{\boldsymbol{m}}}|\|g-h\|_{\boldsymbol{m}} + \max_{i\in\Sigma_N}|\frac{N\alpha}{a_i^{\boldsymbol{m}}}|\sup_{0<y\le 1}\frac{\omega(\frac{y}{a_i})}{\omega(y)}[g^{(\boldsymbol{m})} - h^{(\boldsymbol{m})}]^*\\
\le&\max\left\{\max_{i\in\Sigma_N}|\frac{\alpha}{a_i^{\boldsymbol{m}}}|, \max_{i\in\Sigma_N}|\frac{N\alpha}{a_i^{\boldsymbol{m}}}|\sup_{0<y\le 1}\frac{\omega(\frac{y}{a_i})}{\omega(y)}\right\}\|g-h\|_{\omega},
\end{aligned}$$

where $x' = \frac{x-b_i}{a_i}$ and $y' = \frac{y}{a_i}$ such that $u_i(x') = x$, $u_i(x'+y') = x+y$ and $u_i(x'+\delta y') = x+\delta y$. Hence, in view of the given assumptions, $\mathfrak{T}$ is a contraction map on $\mathfrak{C}^{\boldsymbol{m}}_{\omega_*}(I)$. Using the Banach contraction principle, we get the proof.

(ii). The proof will follow by using (i) and Corollary 6.33.

$\square$

Remark 6.35. If we choose $\boldsymbol{m} = 0$, this space will be the same as defined in [29]. Hence, Theorem 6.34 generalizes the [29, Theorem 2.7]. It should also be noted that these spaces are very useful in computing the Hausdorff dimension of Weierstarss-type functions, see, for instance, [60].

6.5 Conclusion

In this chapter, we have studied a few properties of invariant measures associated with the IFS whose attractor is the graph of a FIF satisfying a given set of data points (Lemma 6.4, Proposition 6.6, Corollary 6.9, Theorem 6.12, Theorem 6.16). Further, we have given a fractal transformation between the unit interval I and graph of FIF $\mathcal{G}(h^*)$ (Theorem 6.18). Also, we have defined a fractal transformation between the graphs of two different FIFs satisfying a given set of data points (Theorem 6.19). Later, in Section 6.4, we have defined function spaces (Subsections 6.4.1, 6.4.2, and 6.4.3). Further, we have provided some results that give conditions under which a FIF becomes an element of those spaces (Theorem 6.21, Corollary 6.30, Theorem 6.34).

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