Chapter 3

On T_0 and T_1 fuzzy soft topological spaces

3.1 Introduction

Separation axioms in a fuzzy soft topological space have been introduced and studied earlier by Mahanta and Das[72]. In fuzzy topological spaces, there exist several definitions of fuzzy T_0 and fuzzy T_1 separation axioms(cf. [5, 71, 109, 110], etc.). The definition given by Lowen and Srivastava [71] turns out to be 'categorically right' definition of fuzzy T_0 separation axiom.

In this chapter, we have extended this definition to the case of a fuzzy soft topological space. Further, we have introduced T_1 separation axiom in a fuzzy soft toplogical space as an extension of the definition of T_1-ness in a fuzzy topological space given by Srivastava et al.[110]. We have given a complete comparison of our definitions with those of Mahanta and Das[72]. It has been proved that our definitions satisfy the basic desirable properties e.g., hereditary, productive and projective.

Throughout the chapter we mean a fuzzy soft topological space in the sense of Definition 1.21.

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3.2 Fuzzy soft T_0 topological spaces

Lowen and Srivastava[71] have given the following definition of T_0 separation axiom in fuzzy topological spaces.

Definition 3.1. [71] A fuzzy topological space (X, τ) is said to be *fuzzy* T_0 if for every pair of distinct points $x, y \in X$, there exists a fuzzy open set U in X such that $U(x) \neq U(y)$.

Motivated by this, we give the following definition of fuzzy soft T_0 separation axiom in a fuzzy soft topological space:

Definition 3.2. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then (X, τ) is said to be *fuzzy soft* T_0 if for each pair $(x_1, e_1), (x_2, e_2) \in X \times E, (x_1, e_1) \neq (x_2, e_2)$, there exists $f_A \in \tau$ such that $f_A(e_1)(x_1) \neq f_A(e_2)(x_2)$.

Mahanta and Das[72] had given the following definition of fuzzy soft T_0 topological spaces:

Definition 3.3. [72] A fuzzy soft topological space (X, τ) is said to be *fuzzy soft* T_0 if for every pair of disjoint fuzzy soft sets e_{h_A}, e_{g_B}(where e_{h_A} is a fuzzy soft set over X such that $e_{h_A}(e) \neq 0_X$ and $e_{h_A}(e') = 0_X$, for $e' \neq e$, similarly e_{g_B} is defined), there exists a fuzzy soft open set p_A over X containing one but not the other.

The above two definitions of a fuzzy soft T_0 space are independent as exhibited through the following examples.

Example 3.1. *Definition 3.2* $\not\Rightarrow$ *Definition 3.3*

Let $X = \{a, b\}$ and E be the parameter set which consists of only one element, say $E = \{e\}$ and $\tau = \{p_A : E \to I^X \mid p_A(e)(a) = 1/2, p_A(e)(b) = 0\} \cup \{0_E, 1_E\}$. Then (X, τ) is fuzzy soft T_0 in the sense of Definition 3.2, since for $(a, e), (b, e) \in X \times E$, there exists $p_A \in \tau$ such that $p_A(e)(a) \neq p_A(e)(b)$. But it fails to be a fuzzy soft T_0 space in the sense of Definition 3.3, since for the two disjoint fuzzy soft sets e_{h_A} and e_{g_B} such that

$$h_A(e)(x) = \begin{cases} 3/4, & \text{if } x = a \\ 0, & \text{if } x = b \end{cases}$$

and

$$g_B(e)(x) = \begin{cases} 0, & \text{if } x = a \\ 1, & \text{if } x = b, \end{cases}$$

there does not exist any non trivial fuzzy soft open set over X containing any of them.

Example 3.2. *Definition 3.3 $\not\Rightarrow$ Definition 3.2*

Let $X = \{a, b\}$ and E be the parameters set which consists of two elements, say $E = \{e_1, e_2\}$ and $\tau = \{p_A : E \to I^X \mid p_A(e_1) = p_A(e_2) = \chi_{\{a\}}\} \cup \{0_E, 1_E\}$. Then (X, τ) is fuzzy soft T_0 in the sense of Definition 3.3. To show this, consider the disjoint fuzzy soft sets $(e_1)_{h_A}$ and $(e_1)_{g_B}$. Since $h_A(e_1) \cap g_B(e_1) = 0_X$, we may assume that $h_A(e_1) = a_\lambda$ and $g_B(e_1) = b_s$, where λ, $s \in (0, 1]$. Then there exists $p_A \in \tau$ such that p_A contains $(e_1)_{h_A}$ but does not contain $(e_1)_{g_B}$. Similarly, we can deal with the case of disjoint fuzzy soft open sets of the form $(e_2)_{h_A}$ and $(e_2)_{g_B}$. But it fails to be a fuzzy soft T_0 space in the sense of Definition 3.2, since for $(a, e_1) \neq (a, e_2)$, there does not exist any fuzzy soft open set f_A such that $f_A(e_1)(a) \neq f_A(e_2)(a)$.

From now onwards we mean a fuzzy soft T_0 topological space in the sense of Definition 3.2.

Theorem 3.4. *Let (Y, σ) be an indiscrete enriched fuzzy soft topological space relative to the parameters set K. Then (X, τ) is fuzzy soft T_0 implies that every fuzzy soft continuous mapping $(\varphi, \psi) : (Y, \sigma) \to (X, \tau)$ is constant.*

Proof. First suppose that (X, τ) is fuzzy soft T_0. Then for each pair $(x_1, e_1), (x_2, e_2) \in X \times E, (x_1, e_1) \neq (x_2, e_2)$, there exists $f_A \in \tau$ such that $f_A(e_1)(x_1) \neq f_A(e_2)(x_2)$. We have to show that every fuzzy soft continuous mapping $(\varphi, \psi) : (Y, \sigma) \to (X, \tau)$ is constant. For this, suppose on the contrary that there exists a fuzzy soft continuous mapping $(\varphi, \psi) : (Y, \sigma) \to (X, \tau)$ which is not constant. Then,
Case 1: If φ is not constant and ψ is constant.
Let $\varphi(y_1) \neq \varphi(y_2)$, for some $y_1, y_2 \in Y$ and $\psi(k) = e$, for each $k \in K$. Then, for $(\varphi(y_1), e)$, $(\varphi(y_2), e) \in X \times E$, there exists $f_A \in \tau$ such that $f_A(e)(\varphi(y_1)) \neq f_A(e)(\varphi(y_2))$.

Now, for $k \in K$, we have

$$\begin{aligned}(\varphi,\psi)^{-1}f_A(k)(y_1) &= f_A(\psi(k))(\varphi(y_1))\\ &= f_A(e)(\varphi(y_1))\\ &\neq f_A(e)(\varphi(y_2))\\ &= f_A(\psi(k))(\varphi(y_2))\\ &= (\varphi,\psi)^{-1}f_A(k)(y_2),\end{aligned}$$

which implies that $(\varphi,\psi)^{-1}f_A$ is not a constant fuzzy soft set over Y. Therefore (φ,ψ) is not fuzzy soft continuous, a contradiction.

Case 2: If φ is constant and ψ is not constant.

Let $\varphi(y) = x, \forall y \in Y$ and $\psi(k_1) \neq \psi(k_2)$, for some $k_1, k_2 \in K$. Then, for $(x, \psi(k_1))$, $(x, \psi(k_2)) \in X \times E$, there exists $f_A \in \tau$ such that $f_A(\psi(k_1))(x) \neq f_A(\psi(k_2))(x)$. Now, for $y \in Y$, we have

$$\begin{aligned}(\varphi,\psi)^{-1}f_A(k_1)(y) &= f_A(\psi(k_1))(x)\\ &\neq f_A(\psi(k_2))(x)\\ &= f_A(\psi(k_2))(\varphi(y))\\ &= (\varphi,\psi)^{-1}f_A(k_2)(y),\end{aligned}$$

which implies that $(\varphi,\psi)^{-1}f_A$ is not a constant fuzzy soft set over Y. Therefore (φ,ψ) is not fuzzy soft continuous, a contradiction.

Case 3: If φ and ψ both are not constant.

Let $\varphi(y_1) \neq \varphi(y_2)$ and $\psi(k_1) \neq \psi(k_2)$, for some $y_1, y_2 \in Y$ and $k_1, k_2 \in K$. Then, for $(\varphi(y_1), \psi(k_1))$, $(\varphi(y_2), \psi(k_2)) \in X \times E$, there exists $f_A \in \tau$ such that $f_A(\psi(k_1))(\varphi(y_1)) \neq f_A(\psi(k_2))(\varphi(y_2))$. Now,

$$\begin{aligned}(\varphi,\psi)^{-1}f_A(k_1)(y_1) &= f_A(\psi(k_1))(\varphi(y_1))\\ &\neq f_A(\psi(k_2))(\varphi(y_2))\\ &= (\varphi,\psi)^{-1}f_A(k_2)(y_2),\end{aligned}$$

which implies that $(\varphi,\psi)^{-1}f_A$ is not a constant fuzzy soft set over Y. Therefore (φ,ψ) is not fuzzy soft continuous, a contradiction.

□

T_0-ness in fuzzy soft topological spaces satisfies the hereditary property as shown in the following proposition.

Proposition 3.5. *Fuzzy soft subspace of a fuzzy soft T_0 space is fuzzy soft T_0.*

Proof. Let (X, τ) be a fuzzy soft T_0 space relative to the parameters set E and $G \subseteq E$. Then, for each pair $(x_1, e_1), (x_2, e_2) \in X \times E$, $(x_1, e_1) \neq (x_2, e_2)$, there exists $f_A \in \tau$ such that $f_A(e_1)(x_1) \neq f_A(e_2)(x_2)$. In particular, for $(x_1, g), (x_2, g') \in X \times G$, $(x_1, g) \neq (x_2, g')$, there exists $f_A \in \tau$ such that $f_A(g)(x_1) \neq f_A(g')(x_2)$. So, $f_A \mid_G (g)(x_1) \neq f_A \mid_G (g')(x_2)$. This implies that (X, τ_G) is fuzzy soft T_0. □

In the following theorem it is proved that T_0-ness in fuzzy soft topological spaces satisfies the productive and projective properties.

Theorem 3.6. *If $\{(X_i, \tau_i) : i \in \Omega\}$ is a family of fuzzy soft topological spaces relative to the parameters sets E_i, respectively. Then the product fuzzy soft topological space $(X, \tau) = \prod_{i\in\Omega}(X_i, \tau_i)$ is fuzzy soft T_0 iff each coordinate fuzzy soft topological space (X_i, τ_i) is fuzzy soft T_0.*

Proof. First, let us assume that (X_i, τ_i) is fuzzy soft T_0, for each $i \in \Omega$. To show that (X, τ) is fuzzy soft T_0, choose $(x, e), (y, e') \in X \times E$, $(x, e) \neq (y, e')$, where $x = \prod\limits_{j\in\Omega} x_j$, $y = \prod\limits_{j\in\Omega} y_j$, $e = \prod\limits_{j\in\Omega} e_j$ and $e' = \prod\limits_{j\in\Omega} e'_j$. Then $x_i \neq y_i$ or $e_k \neq e'_k$ for some i, $k \in \Omega$. Let $x_i \neq y_i$. Since (X_i, τ_i) is fuzzy soft T_0, for (x_i, e_i), $(y_i, e'_i) \in X_i \times E_i$, there exists $f_{A_i} \in \tau_i$ such that $f_{A_i}(e_i)(x_i) \neq f_{A_i}(e'_i)(y_i)$. Set $f_A = \prod\limits_{j\in\Omega} f^1_{A_j}$ such that $f^1_{A_j} = 1_{E_j}$, for $j \neq i$ and $f^1_{A_i} = f_{A_i}$. Then f_A is a fuzzy soft open set such that $f_A(e)(x) \neq f_A(e')(y)$ implying that (X, τ) is fuzzy soft T_0. The other cases can be handled similarly.

Conversely, assume that (X, τ) is fuzzy soft T_0. To show (X_i, τ_i) is fuzzy soft T_0, choose $(x_i, e_i), (y_i, e'_i) \in X_i \times E_i$, $(x_i, e_i) \neq (y_i, e'_i)$. Then $x_i \neq y_i$ or $e_i \neq e'_i$. Let $x_i \neq y_i$. Now construct two points $x = \prod\limits_{j\in\Omega} x'_j$ and $y = \prod\limits_{j\in\Omega} y'_j$ in X, where $x'_j = y'_j$, for $j \neq i$ and $x'_i = x_i$, $y'_i = y_i$ and two points $e^1 = \prod\limits_{j\in\Omega} e^1_j$ and $e^2 = \prod\limits_{j\in\Omega} e^2_j$ in E, where $e^1_j = e^2_j$, for $j \neq i$ and $e^1_i = e_i$, $e^2_i = e'_i$. Then, since (X, τ) is fuzzy soft T_0, for $(x, e^1), (y, e^2) \in X \times E$, there exists $f_A \in \tau$ such that $f_A(e^1)(x) \neq f_A(e^2)(y)$. Also, since each fuzzy soft open set can be written as a union of basic fuzzy soft open sets, so we can write f_A in the following form:

$$f_A = \bigcup_{k\in T}\prod_{j\in\Omega}(f_{A_j})_k,$$

where T is an arbitrary set.

Now, if we assume that

$$\begin{aligned}
&\prod_{j\in\Omega}(f_{A_j})_k(e^1)(x) = \prod_{j\in\Omega}(f_{A_j})_k(e^2)(y), \quad \text{for each } k\in T\\
&\Rightarrow \inf_j(f_{A_j})_k(e_j^1)(x_j') = \inf_j(f_{A_j})_k(e_j^2)(y_j'), \quad \text{for each } k\in T\\
&\Rightarrow \sup_k\inf_j(f_{A_j})_k(e_j^1)(x_j') = \sup_k\inf_j(f_{A_j})_k(e_j^2)(y_j')\\
&\Rightarrow (\bigcup_{k\in T}\prod_{j\in\Omega}(f_{A_j})_k)(e^1)(x) = (\bigcup_{k\in T}\prod_{j\in\Omega}(f_{A_j})_k)(e^2)(y)\\
&\Rightarrow f_A(e^1)(x) = f_A(e^2)(y), \text{ a contradiction.}
\end{aligned}$$

Therefore, there exists some $k\in T$ such that

$$\begin{aligned}
\prod_{j\in\Omega}(f_{A_j})_k(e^1)(x) &\neq \prod_{j\in\Omega}(f_{A_j})_k(e^2)(y)\\
\Rightarrow \inf_j(f_{A_j})_k(e_j^1)(x_j') &\neq \inf_j(f_{A_j})_k(e_j^2)(y_j').
\end{aligned}$$

Since, for $j\neq i$, $x_j' = y_j'$, $e_j^1 = e_j^2$, so we have

$$(f_{A_i})_k(e_i)(x_i) \neq (f_{A_i})_k(e_i')(y_i),$$

which implies that (X_i,τ_i) is fuzzy soft T_0. The other cases can be handled similarly. □

3.3 Fuzzy soft T_1 topological spaces

In this section, we introduce and study fuzzy soft T_1 topological spaces. The following definition of fuzzy soft T_1 topological spaces is motivated by the definition 5.1 of fuzzy T_1−topological spaces given by Srivastava et al.[110].

Definition 3.7. Let (X,τ) be a fuzzy soft topological space relative to the parameters set E. Then (X,τ) is said to be *fuzzy soft* T_1 if for each pair $(x_1,e_1),(x_2,e_2)\in$

$X \times E, (x_1, e_1) \neq (x_2, e_2)$, there exist $f_A,\ g_B \in \tau$ such that $f_A(e_1)(x_1) = 1,\ f_A(e_2)(x_2) = 0,\ g_B(e_1)(x_1) = 0,\ g_B(e_2)(y_2) = 1$.

Earlier Mahanta and Das[72] have introduced fuzzy soft T_1 topological spaces as follows:

Definition 3.8. [72] A fuzzy soft topological space (X, τ) is said to be *fuzzy soft* T_1 if for every pair of disjoint fuzzy soft sets e_{h_A}, e_{g_B}(where e_{h_A} is a fuzzy soft set over X such that $e_{h_A}(e) \neq 0_X$ and $e_{h_A}(e') = 0_X$, for $e' \neq e$, similarly e_{g_B} is defined), there exist fuzzy soft open sets p_A and q_A over X such that $e_{h_A} \subseteq p_A$ and $e_{g_B} \nsubseteq p_A$; $e_{h_A} \nsubseteq q_A$ and $e_{g_B} \subseteq q_A$.

We observe that Definition 3.7 $\Rightarrow$ Definition 3.8, to show this we first prove the following proposition:

Proposition 3.9. *The following statements are equivalent in a fuzzy soft topological space (X, τ) relative to the parameters set E:*

1. *(X, τ) is fuzzy soft T_1.*
2. *$e_{\{x\}}$ is fuzzy soft closed, for each $e \in E$ and $x \in X$, where $e_{\{x\}}$ denotes the fuzzy soft set over X such that*

$$e_{\{x\}}(e')(x') = \begin{cases} 1, & \text{if } e' = e,\ x' = x \\ 0, & \text{otherwise.} \end{cases}$$

Proof. *(1)$\Rightarrow$(2)*

First, assume that (X, τ) is fuzzy soft T_1. Then, to show that $e_{\{x\}}$ is fuzzy soft closed, equivalently, $e^c_{\{x\}}$ is fuzzy soft open, choose any fuzzy soft point $e'_{y_\lambda} \in e^c_{\{x\}}$. Then,

Case 1: If $y = x$ and $e' \neq e$.
Then, for $(x, e), (x, e') \in X \times E$, there exist $f_A,\ g_B \in \tau$ such that $f_A(e)(x) = 1,\ f_A(e')(x) = 0,\ g_B(e)(x) = 0,\ g_B(e')(x) = 1$. Now $e'_{x_\lambda} \in g_B \sqsubseteq e^c_{\{x\}}$.
Case 2: If $y \neq x$ and $e' = e$.
Then, for $(x, e),\ (y, e) \in X \times E$, there exist $f_A,\ g_B \in \tau$ such that $f_A(e)(x) = 1,\ f_A(e)(y) = 0,\ g_B(e)(x) = 0,\ g_B(e)(y) = 1$. Now $e_{y_\lambda} \in g_B \sqsubseteq e^c_{\{x\}}$.
Case 3: If $y \neq x$ and $e' \neq e$.

Then, for (x, e), $(y, e') \in X \times E$, there exist f_A, $g_B \in \tau$ such that $f_A(e)(x) = 1$, $f_A(e')(y) = 0$, $g_B(e)(x) = 0$, $g_B(e')(y) = 1$. Now $e'_{y_\lambda} \in g_B \sqsubseteq e^c_{\{x\}}$.
Therefore, $e^c_{\{x\}}$ is fuzzy soft open.

(2)⇒(1)

Suppose that $e_{\{x\}}$ is fuzzy soft closed for each $e \in X$, and $x \in X$. To show that (X, τ) is fuzzy soft T_1, choose (x', e'), $(x'', e'') \in X \times E$, $(x', e') \neq (x'', e'')$. Now consider the fuzzy soft open sets $(e'_{\{x'\}})^c$ and $(e''_{\{x''\}})^c$. Then, $(e'_{\{x'\}})^c(e')(x') = 0$, $(e'_{\{x'\}})^c(e'')(x'') = 1$, $(e''_{\{x''\}})^c(e')(x') = 1$ and $(e''_{\{x''\}})^c(e'')(x'') = 0$. □

Now we prove that Definition 3.7 ⇒ Definition 3.8, as follows:

Let (X, τ) be a fuzzy soft T_1 space in the sense of Definition 3.7.

⇒ $e_{\{x\}}$ is fuzzy soft closed, for each $e \in E$ and $x \in X$ (in view of Proposition 3.9).
⇒ $(e_{\{x\}})^c$ is fuzzy soft open, for each $e \in E$ and $x \in X$.

Now choose any two disjoint fuzzy soft sets e_{h_A} and e_{g_B}. Since $e_{h_A}(e) \neq 0_X$ and $e_{g_B}(e) \neq 0_X$, there exist a, $b \in X$ such that $e_{h_A}(e)(a) \neq 0$ and $e_{g_B}(e)(b) \neq 0$. Since $h_A(e) \cap g_B(e) = 0_X$, so $a \neq b$. Now consider the fuzzy soft open sets $(e_{\{b\}})^c$, $(e_{\{a\}})^c$. Then $e_{h_A} \subseteq (e_{\{b\}})^c$ and $e_{g_B} \nsubseteq (e_{\{b\}})^c$; $e_{h_A} \nsubseteq (e_{\{a\}})^c$ and $e_{g_B} \subseteq (e_{\{a\}})^c$, showing that (X, τ) is fuzzy soft T_1 in the sense of Definition 3.8.

But the converse is not true which is exhibited from the following example.

Example 3.3. *Let $X = \{a, b\}$, E be the parameters set which consists of two elements, say $E = \{e_1, e_2\}$ and $\tau = \{0_E, 1_E\} \cup \{p_A : E \to I^X, q_A : E \to I^X \mid p_A(e_1) = p_A(e_2) = \chi_{\{a\}}, q_A(e_1) = q_A(e_2) = \chi_{\{b\}}\}$. Then (X, τ) is a fuzzy soft T_1 space in the sense of Definition 3.8. For this, let $(e_1)_{h_A}$ and $(e_1)_{g_B}$ be disjoint fuzzy soft sets. Since $h_A(e_1) \cap g_B(e_1) = 0_X$, we may assume that $h_A(e_1) = a_\lambda$ and $g_B(e_1) = b_s$, where λ, $s \in (0, 1]$. Then for $(e_1)_{h_A}$ and $(e_1)_{g_B}$, there exist $p_A, q_A \in \tau$ such that p_A contains $(e_1)_{h_A}$ but does not contain $(e_1)_{g_B}$ and q_A contains $(e_1)_{g_B}$ but does not contain $(e_1)_{h_A}$. Similarly, we can deal with the case of disjoint fuzzy soft open sets of the form $(e_2)_{h_A}$ and $(e_2)_{g_B}$. But it fails to be a fuzzy soft T_1 space in the sense of Definition 3.7, since for $(a, e_1) \neq (a, e_2)$, there does not exist any fuzzy soft open set f_A such that $f_A(e_1)(a) = 1$ and $f_A(e_2)(a) = 0$.*

From now onwards we mean a fuzzy soft T_1 topological space in the sense of Definition 3.7.

In the following proposition, we show that T_1-ness in fuzzy soft topological spaces satisfies the hereditary property.

Proposition 3.10. *Fuzzy soft subspace of a fuzzy soft T_1 space is fuzzy soft T_1.*

Proof. Let (X, τ) be a fuzzy soft T_1 space relative to the parameters set E and $G \subseteq E$. Then, for each pair (x_1, e_1), $(x_2, e_2) \in X \times E, (x_1, e_1) \neq (x_2, e_2)$, there exist $f_A, g_B \in \tau$ such that

$$f_A(e_1)(x_1) = 1,\ f_A(e_2)(x_2) = 0,\ g_B(e_1)(x_1) = 0,\ g_B(e_2)(x_2) = 1.$$

In particular, for (x_1, g), $(x_2, g') \in X \times G$, $(x_1, g) \neq (x_2, g')$, there exist $f_A, g_B \in \tau$ such that

$$\begin{aligned} &f_A(g)(x_1) = 1,\ f_A(g')(x_2) = 0,\ g_B(g)(x_1) = 0,\ g_B(g')(x_2) = 1 \\ &\Rightarrow f_A \mid_G (g)(x_1) = 1,\ f_A \mid_G (g')(x_2) = 0,\ g_B \mid_G (g)(x_1) = 0,\ g_B \mid_G (g')(x_2) = 1. \end{aligned}$$

This implies that (X, τ_G) is fuzzy soft T_1. □

In the following theorem we prove that T_1-ness in fuzzy soft topological spaces, satisfies the productive and projective properties. The proof is based on the proof of the corresponding result given in [108].

Theorem 3.11. *If $\{(X_i, \tau_i); i \in \Omega\}$ is a family of fuzzy soft topological spaces relative to the parameters sets E_i respectively. Then the product fuzzy soft topological space $(X, \tau) = \prod_{i\in\Omega}(X_i, \tau_i)$ is fuzzy soft T_1 iff each coordinate fuzzy soft topological space (X_i, τ_i) is fuzzy soft T_1.*

Proof. Let (X_i, τ_i) be fuzzy soft T_1, for each $i \in \Omega$. Let (x, e), $(y, e') \in X \times E$, $(x, e) \neq (y, e')$, where $x = \prod\limits_{j\in\Omega} x_j$, $y = \prod\limits_{j\in\Omega} y_j$, $e = \prod\limits_{j\in\Omega} e_j$ and $e' = \prod\limits_{j\in\Omega} e'_j$. Then there exist some i, $k \in \Omega$ such that $x_i \neq y_i$ or $e_k \neq e'_k$. Let $x_i \neq y_i$. Now since (X_i, τ_i) is fuzzy soft T_1, for (x_i, e_i), (y_i, e'_i), there exist $f_{A_i}, g_{B_i} \in \tau_i$ such that $f_{A_i}(e_i)(x_i) = 1$, $f_{A_i}(e'_i)(y_i) = 0$, $g_{B_i}(e_i)(x_i) = 0$, $g_{B_i}(e'_i)(y_i) = 1$. Now consider the fuzzy soft open sets $f_A = (p_{X_i}, q_{E_i})^{-1} f_{A_i}$ and $g_B = (p_{X_i}, q_{E_i})^{-1} g_{B_i}$. Then, $f_A(e)(x) = ((p_{X_i}, q_{E_i})^{-1} f_{A_i})(e)(x) = f_{A_i}(q_{E_i}(e))(p_{X_i}(x)) = f_{A_i}(e_i)(x_i) = 1$ and

$f_A(e')(y) = f_{A_i}(e'_i)(y_i) = 0$. Similarly, we get $g_B(e')(y) = 1, g_B(e)(x) = 0$. Hence (X, τ) is fuzzy soft T_1. The other cases can be handled similarly.

Conversely, let us assume that (X, τ) is fuzzy soft T_1. To show that (X_i, τ_i) is fuzzy soft T_1, choose (x_i, e_i), $(y_i, e'_i) \in X_i \times E_i$ such that $(x_i, e_i) \neq (y_i, e'_i)$. Then $x_i \neq y_i$ or $e_i \neq e'_i$. Let $x_i \neq y_i$. Now consider two points $x = \prod_{j\in\Omega} x'_j$ and $y = \prod_{j\in\Omega} y'_j$ in X, where $x'_j = y'_j, \text{for } j \neq i$ and $x'_i = x_i$, $y'_i = y_i$ and two points $e^1 = \prod_{j\in\Omega} e^1_j$ and $e^2 = \prod_{j\in\Omega} e^2_j$ in E, where $e^1_j = e^2_j, \text{for } j \neq i$ and $e^1_i = e_i$, $e^2_i = e'_i$. Since the product space (X, τ) is fuzzy soft T_1, for (x, e^1), $(y, e^2) \in X \times E$, there exist $f_A, g_B \in \tau$ such that $f_A(e^1)(x) = 1$, $f_A(e^2)(y) = 0$, $g_B(e^1)(x) = 0$, $g_B(e^2)(y) = 1$. Since $f_A(e^1)(x) = 1$, so for each $r \in (0, 1)$, $e^1_{x_r} \in f_A$ and similarly $e^2_{y_s} \in g_B$, for each $s \in (0, 1)$. Now since f_A and g_B are fuzzy soft open, so for $e^1_{x_r} \in f_A$ and $e^2_{y_s} \in g_B$, we can find basic fuzzy soft open sets $\prod_{j\in\Omega} f^r_{A_j}$ and $\prod_{j\in\Omega} g^s_{B_j}$ such that

$$e^1_{x_r} \in \prod_{j\in\Omega} f^r_{A_j} \sqsubseteq f_A,$$

and

$$e^2_{y_s} \in \prod_{j\in\Omega} g^s_{B_j} \sqsubseteq g_B$$

$$\Rightarrow \quad r < (\prod_{j\in\Omega} f^r_{A_j})(e^1)(x) \leqslant f_A(e^1)(x), \quad \text{for each } r \in (0,1)$$

$$\Rightarrow \quad r < \inf_{j\in\Omega} f^r_{A_j}(e^1_j)(x'_j) \leqslant 1, \quad \text{for each } r \in (0,1) \tag{3.1}$$

$$\Rightarrow \quad 1 \leqslant \sup_{0<r<1} \inf_{j\in\Omega} f^r_{A_j}(e^1_j)(x'_j) \leqslant 1$$

$$\Rightarrow \quad \sup_{0<r<1} \inf_{j\in\Omega} f^r_{A_j}(e^1_j)(x'_j) = 1$$

$$\Rightarrow \quad (\bigcup_{0<r<1} \prod_{j} f^r_{A_j})(e^1)(x) = 1 \tag{3.2}$$

Next, since

$$f^r_{A_j}(e^1_j)(x'_j) \leqslant \sup_r f^r_{A_j}(e^1_j)(x'_j), \quad \text{for each } r \in (0,1)$$

$$\Rightarrow \quad \inf_{j\in\Omega} f^r_{A_j}(e^1_j)(x'_j) \leqslant \inf_{j\in\Omega} \sup_r f^r_{A_j}(e^1_j)(x'_j), \quad \text{for each } r \in (0,1)$$

$$\Rightarrow \quad \sup_r \inf_{j\in\Omega} f^r_{A_j}(e^1_j)(x'_j) \leqslant \inf_{j\in\Omega} \sup_r f^r_{A_j}(e^1_j)(x'_j)$$

$$\Rightarrow \quad (\bigcup_r \prod_{j\in\Omega} f^r_{A_j})(e^1)(x) \leqslant (\prod_{j\in\Omega} \bigcup_r f^r_{A_j})(e^1)(x),$$

$$\Rightarrow \quad (\prod_{j\in\Omega} \bigcup_r f^r_{A_j})(e^1)(x) = 1, \quad \text{(using 3.2)}$$

$$
\begin{aligned}
&\Rightarrow \quad \inf_j \sup_r f^r_{A_j}(e^1_j)(x'_j) = 1 \\
&\Rightarrow \quad \sup_r f^r_{A_j}(e^1_j)(x'_j) = 1, \quad \text{for each } j \in \Omega \\
&\Rightarrow \quad (\bigcup_r f^r_{A_i})(e_i)(x_i) = 1.
\end{aligned}
$$

Further, since

$$
\begin{aligned}
& f_A(e^2)(y) = 0 \\
\Rightarrow \quad & (\prod_{j\in\Omega} f^r_{A_j})(e^2)(y) = 0, \quad \text{for each } r \in (0,1) \\
\Rightarrow \quad & \inf_{j\in\Omega} f^r_{A_j}(e^2_j)(y'_j) = 0, \quad \text{for each } r \in (0,1)
\end{aligned}
$$

Next, since $x'_j = y'_j$ and $e^1_j = e^2_j$ for $j \neq i$, so

$$
\begin{aligned}
f^r_{A_j}(e^2_j)(y'_j) &= f^r_{A_j}(e^1_j)(x'_j), \quad j \neq i, \quad \text{for each } r \in (0,1) \\
&> 0 \quad (\text{ using } 3.1)
\end{aligned}
$$

Therefore, $f^r_{A_i}(e'_i)(y_i) = 0$, for each $r \in (0,1)$. Put $f_{A_i} = \bigcup_{0<r<1} f^r_{A_i}$. Then $f_{A_i} \in \tau_i$ such that $f_{A_i}(e_i)(x_i) = 1$ and $f_{A_i}(e'_i)(y_i) = 0$. Similarly, we can obtain another fuzzy soft open set g_B such that $g_{B_i}(e_i)(x_i) = 0$ and $g_{B_i}(e'_i)(y_i) = 1$. The other cases can be handled similarly. □

Theorem 3.12. *Let (X,τ) be a fuzzy soft topological space. Then (X,τ) is fuzzy soft Hausdorff $\Rightarrow$ (X,τ) is fuzzy soft T_1 $\Rightarrow$ (X,τ) is fuzzy soft T_0.*

Proof. First, assume that (X,τ) is fuzzy soft Hausdorff. Then to show that (X,τ) is fuzzy soft T_1, choose $(x,e),(y,e') \in X \times E, (x,e) \neq (y,e')$. Consider e_{x_λ} and e'_{y_s}, which are distinct fuzzy soft points over X. Next, since (X,τ) is fuzzy soft Hausdorff, there exist f^λ_A and $g^s_B \in \tau$ such that

$$
\begin{aligned}
& e_{x_\lambda} \in f^\lambda_A,\ e'_{y_s} \in g^s_B \text{ and } f^\lambda_A \sqcap g^s_B = 0_E && (3.3) \\
\Rightarrow\ & \lambda < f^\lambda_A(e)(x) \text{ and } s < g^s_B(e')(y) \\
\Rightarrow\ & 1 \leq \sup_{0<\lambda<1} f^\lambda_A(e)(x) \text{ and } 1 \leq \sup_{0<s<1} g^s_B(e')(y) \\
\Rightarrow\ & \sup_{0<\lambda<1} f^\lambda_A(e)(x) = 1 \text{ and } \sup_{0<s<1} g^s_B(e')(y) = 1. && (3.4)
\end{aligned}
$$

Now, set $f_A = \bigcup_{0<\lambda<1} f_A^\lambda$. Then,

$$\begin{aligned} f_A(e)(x) &= \sup_{0<\lambda<1} f_A^\lambda(e)(x) \\ &= 1 \qquad \text{(using 3.4)}. \end{aligned}$$

Next, since

$$\begin{aligned} & f_A^\lambda \sqcap g_B^s = 0_E, \quad \text{for each } \lambda \in (0,1), \text{ for each } s \in (0,1) \\ \Rightarrow\ & \min\{f_A^\lambda(e_1)(y), g_B^s(e_1)(y)\} = 0, \quad \text{for each } \lambda \in (0,1), s \in (0,1) \text{ and } e_1 \in E \\ \Rightarrow\ & f_A^\lambda(e')(y) = 0, \quad \text{for each } \lambda \in (0,1) \text{ (using 3.3)} \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned} f_A(e')(y) &= \sup_{0<\lambda<1} f_A^\lambda(e')(y) \\ &= 0, \qquad \text{(using 3.5)} \end{aligned}$$

Similarly, we can construct a fuzzy soft open set g_B such that $g_B(e)(x) = 0$, $g_B(e')(y) = 1$.

Next, assume that (X, τ) is fuzzy soft T_1. So, for $(x_1, e_1), (x_2, e_2) \in X \times E$, $(x_1, e_1) \neq (x_2, e_2)$, there exist f_A, $g_B \in \tau$ such that $f_A(e_1)(x_1) = 1$, $f_A(e_2)(x_2) = 0$, $g_B(e_1)(x_1) = 0$, $g_B(e_2)(y_2) = 1$. Then, clearly (X, τ) is fuzzy soft T_0. □

The converse of the above Theorem 3.12 does not hold good as can be seen in the following counter examples.

Example 3.4. *Let X be an infinite set and E be a parameters set. Suppose*

$$\tau = \{f_A : E \to I^X\} \cup \{0_E\},$$

where f_A is defined as follows:

$$f_A(e)(x) = \begin{cases} 1, & \textit{except for finitely many pairs } (e, x) \\ 0, & \textit{otherwise.} \end{cases}$$

Then τ is a fuzzy soft topology over X as shown below:

1. 0_E, $1_E \in \tau$.

2. *If $f_A, g_B \in \tau$, then*

$$\begin{aligned}(f_A \sqcap g_B)(e)(x) &= min\{f_A(e)(x), g_B(e)(x)\} \\ &= \begin{cases} 1, & \text{except for finitely many pairs } (e,x) \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

   Hence $f_A \sqcap g_B \in \tau$.

3. *If $(f_A)_i \in \tau$, for $i \in \Omega$. Then*

$$\begin{aligned}(\bigsqcup_{i\in\Omega}(f_A)_i)(e)(x) &= \sup_{i\in\Omega}(f_A)_i(e)(x) \\ &= \begin{cases} 1, & \text{except for finitely many pairs } (e,x) \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

   Hence $\bigsqcup_{i\in\Omega} f_{A_i} \in \tau$.

Now this fuzzy soft topological space is fuzzy soft T_1 as $e_{\{x\}}$ is fuzzy soft closed, for each $e \in E$ and $x \in X$. But (X, τ) is not fuzzy soft T_2, since there does not exist any pair of non trivial disjoint fuzzy soft open sets.

Example 3.5. *Let $X = \{a, b\}$ and E be a parameter set which consists of only one element say, $E = \{e\}$ and $\tau = \{f_A : E \to I^X \mid f_A(e) = \chi_{\{a\}}\} \cup \{0_E, 1_E\}$. Then (X, τ) is a fuzzy soft topological space which is fuzzy soft T_0, since for $(a, e), (b, e) \in X \times E$, there exists $f_A \in \tau$ such that $f_A(e)(a) \neq f_A(e)(b)$ but it is not fuzzy soft T_1, since there does not exist any fuzzy soft open set g_B such that $g_B(e)(b) = 1$ and $g_B(e)(a) = 0$.*

3.4 Conclusion

In this chapter, we have introduced T_0 and T_1 separation axioms in fuzzy soft topological spaces. We have also given a complete comparison of our definitions with those given by Mahanta and Das[72]. Further, we have proved that these axioms satisfy productive, projective and hereditary properties. A characterization for a fuzzy soft T_1 topological space in terms of the fuzzy soft set $e_{\{x\}}$, where

$$e_{\{x\}}(e')(x') = \begin{cases} 1, & \text{if } e' = e,\ x' = x \\ 0, & \text{otherwise.} \end{cases}$$

has been obtained. It has also been shown that $T_2 \Rightarrow T_1 \Rightarrow T_0$ but none of the implications is reversible.