

MULTISTABILITY ANALYSIS AND
SYNCHRONIZATION OF ARTIFICIAL NEURAL
NETWORKS



*The thesis submitted in partial fulfilment
for the Award of Degree
DOCTOR OF PHILOSOPHY*

by

Shiv Shankar Chouhan

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI -221005

Roll No: 18121518

February 2024

Chapter 6

Summary of the thesis and future work

6.1 Summary

This thesis is dedicated to the problems of multistability and synchronization of neural networks with multi-dimensional neurons. When an neural network is applied for associative memories and pattern recognition, the coexistence of multiple stable or multistable equilibrium points is required. It should be noted that the study of multistability analysis is quite different from that of monostability analysis. In monostability analysis, we need only to investigate the global stability of the equilibrium point of neural networks due to the uniqueness of equilibrium point. In contrast, the multistability analysis of neural networks is much more complicated than the monostability analysis. The notion of “multistability” of the neural networks is used to describe the coexistence of multiple stable equilibrium points. Here,

different equilibrium points may possess different dynamical properties, either stable or unstable, or even cannot be determined. The major challenge in this issue is how to determine the number of equilibrium points and at the same time find out under what conditions how many will be stable or unstable. On the other hand, the systems which are to be synchronized are called master (drive) and slave (response) systems. The synchronization analysis is obtained through the stability analysis of the error system obtained by subtracting the state vectors of the master system from those of the slave system. As seen from the in previous chapters, multi-dimensional neurons have large amounts of storage capacity, for which those can store a lot of information. Therefore, in this thesis, it is only focused on such types of neural networks whose neurons are multi-dimensional. In the first chapter, there are almost everything from the origin of the motivation of artificial neural networks to their architecture, training process, and mathematical modeling. Some basic stability definitions, synchronization concepts, and DDEs are also included in this chapter. We finished the first chapter by introducing quaternion and octonion algebra, which are used in the mathematical models of the thesis chapters.

In the second chapter, the coexistence of locally multistable equilibrium points for n -neuron delayed quaternion-valued neural networks have been investigated with continuous piecewise nonlinear activation functions. In the neural networks model, quaternion variables are taken as state variables of the neurons, moreover a type of continuous piecewise nonlinear activation functions taken in our model, which have not been studied by any researcher till date. By state space decomposition, the n - dimensional state space \mathbb{H}^n is decomposed into 5^{4n} disjoint regions Ω^α and by applying sufficient conditions, it is found that the considered delayed quaternion valued neural networks can have 5^{4n} equilibrium points each of those lies in one of the 5^{4n} regions Ω^α , 3^{4n} of which are locally stable. Finally, two numerical examples

have been performed where the second example has shown that the designed delayed quaternion valued neural networks work efficiently for the application of associative memory.

In the third chapter, the multiple μ -stability analysis of n -dimensional quaternion-valued neural networks is studied with unbounded time-varying delays and two general classes of activation functions. Through this chapter, it is dealt with the modern concept of μ -stability which is a generalization of exponential stability, power stability, log-stability and log-log stability. With this, some sufficient conditions have been established those ensure that the considered quaternion-valued neural networks model with unbounded time varying delays can have 3^{4n} equilibrium points, 2^{4n} of which are locally μ -stable.

In the fourth chapter, the multistability analysis of octonion-valued neural networks with time-varying delays is studied. Here, the multistability result of quaternion valued neural networks is extended to octonion valued neural networks. In the mathematical model, the state variables, synaptic connections strength, activation functions and external inputs are considered as octonions. Based on geometrical properties of activation functions, 3^{8n} disjoint regions are constructed in \mathbb{O}^n . Then, by using the inequality technique, several sufficient conditions are obtained to ensure the existence of 3^{8n} equilibrium points of the system, each of which is located in one of the regions, and 2^{8n} of those are locally exponentially stable. Through this chapter it is investigated that, the n -neurons octonion valued neural networks can have 2^{8n} locally stable equilibrium points, which are obviously more as compared to respective complex valued neural networks and quaternion valued neural networks. Here, the superiority of octonion valued neural networks have been discussed, where it is shown that the octonion valued neural networks require only 72 neurons to

store and retrieve the 12 by 12-pixel image, which clearly indicates the high storage capacity of octonion valued neural networks.

In the fifth chapter, the fixed-time synchronization of neural networks is studied. Fixed-time synchronization is the extension of the finite-time synchronization. The trajectory of the solution curve reaches the equilibrium point in finite time together with the advantage that this time is independent of the initial condition. Here, the octonion-valued neural networks have been explored. The octonion-valued neural networks are decomposed into four equivalent real-valued neural networks. Then using some lemmas, several sufficient conditions for the fixed time synchronization of octonion valued neural networks are obtained. Moreover, the settling time is also estimated for the fixed time synchronization. Finally, a numerical example is performed to illustrate and validate the theoretical results.

6.2 Future work

In previous chapters, it is seen that the neural networks consisting of multidimensional neurons are capable of receiving large amounts of information due to their high storage capacity. Therefore, through this thesis the focus was given on studying the multistability and synchronization of neural networks with multi-dimensional neurons such as quaternion and octonion valued neurons. In Chapters 2 and 3, quaternion-valued neural networks are decomposed into four equivalent real-valued neural networks. While, in Chapters 4 and 5, the octonion valued neural networks are decomposed into eight equivalent real valued neural networks. In future research work, the multistability and synchronization of quaternion and octonion valued neural networks without decomposing the considered models into the equivalent real

valued neural networks will be studied. Furthermore, a drive will be given to investigate the following potential problems in future research work.

- (1). Multistability analysis of quaternion and octonion valued neural networks with discontinuous activation functions.
- (2). Fixed time synchronization of quaternion valued neural networks with discontinuous activation functions and its application to image encryption.
- (3). Application to associative memory for storing and retrieving the multiple images by using multiple stable equilibrium points of quaternion valued neural networks.
