Chapter 2

A Numerical Approximation for Generalized Fractional Sturm-Liouville Problem with Application

In this chapter, a GFSLP in terms of generalized fractional derivative (B-operator) is considered. Section 2.1 provide the introduction on GFSLP. The main problem and well-posedness of considered problem is defined in Section 2.2. Section 2.3 consists of the main scheme in which the discretization of GFSLP is implemented. Further, it is observed that eigenvalues obtained by proposed scheme are real, and the corresponding eigenfunctions are orthogonal. In Section 2.4 , theoretical convergence order for eigenfunctions and eigenvalues is discussed. In Section 2.5, the numerical results are presented. Firstly, the eigenvalues and associated eigenfunctions of the considered cases for modified power and Prabhakar kernels are obtained. Further, the convergence order of the numerical method is discussed. In Section 2.6, numerically obtained eigenvalues and eigenfunctions are used to construct an approximate solution of the one-dimensional fractional diffusion equation defined in a bounded domain.

2.1 Introduction

In [42], Klimek and Agrawal studied the GFSLPs with A and B operators and extended the properties of the regular SLPs for the GFSLPs. A and B operators were initially discussed in [17] and they gained attention in [62, 63, 60]. In particular cases, these operators reduce to several other fractional differential and integral operators introduced in the literature. Thus, these operators may be considered as the generalization of fractional integrals and derivatives. This generalization motivates us to develop a numerical scheme for a class of GFSLPs defined using B-operators. This chapter aims to examine numerical eigenvalues and eigenfunctions associated with the GFSLPs defined in terms of the B-operators. To compute the eigenvalues and eigenfunctions numerically, we have used the modified power kernel [22] and the Prabhakar kernel [24, 64, 25] in the B-operator. Further, convergence order of the proposed scheme is discussed.

Now, we define a GFSLP in terms of B-operator with mixed boundary conditions.

2.2 Statement of the Problem

Here, we consider a GFSLP using the B-operator as follows:

$$B^{\alpha}_{P_2}\left(p(x)B^{\alpha}_{P_1}y(x)\right) + q(x)y(x) = \lambda r(x)y(x),\ x\in[a,b], \tag{2.1}$$

subjected to the mixed boundary conditions,

$$y(a) = 0, \qquad p(x) B^{\alpha}_{P_1} y(x)|_{x=b} = 0, \tag{2.2}$$

where $\alpha \in (0,1)$, and $p(x), q(x), r(x)$ are continuous functions defined on $[a,b]$ such that $r(x) > 0$ in $[a,b]$. The parameter λ is the eigenvalue, and a non-trivial solution for Eqs.(2.1)-(2.2) is known as eigenfunction corresponding to eigenvalue λ.

In $L^2_w(a,b)$-space, we obtain orthogonal eigenfunctions y_{λ_1} and y_{λ_2} corresponding to distinct eigenvalues λ_1, λ_2 respectively, i.e.

$$\int_a^b w(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0, \qquad \lambda_1 \neq \lambda_2. \tag{2.3}$$

Note: Here, we consider $P_1 = (a,b;p_1,q_1)$ and $P_2 = (a,b;p_2,q_2)$.

2.2.1 Well-Posedness

Zayernouri et al. [21] discussed well-posedness of the FSLP which was further extended for tempered FSLP in [65]. In a similar way, we establish the well-posedness of considered GFSLP given by Eqs.(2.1)-(2.2). Let us define the operator L for $\Omega = [a,b]$ such that

$$L(\cdot) = B^{\alpha}_{P_2}\Big(p(x) B^{\alpha}_{P_1}(\cdot)\Big) + q(x). \tag{2.4}$$

We define the bilinear form,

$$B(y,z) = (Ly, z)_{\Omega}, \text{ for some } \ z(x), \tag{2.5}$$

and using Eq.(1.27), we have

$$B(y,z) = -\int_a^b p(x)B^{\alpha}_{P_1}y(x)A^{\alpha}_{P_2^*}z(x)dx + A^{\alpha-1}_{P_2}z(x)[p(x)B^{\alpha}_{P_1}y(x)]|_a^b + \int_a^b q(x)y(x)z(x)dx. \tag{2.6}$$

If we further assume that $z(a) = 0$, $A^{\alpha-1}_{P_2}z(x)[p(x)B^{\alpha}_{P_1}y(x)]|_{x=b} = 0$ and, if $p_1 = q_2, q_1 = p_2$, then we get,

$$B(y,z) = -\Big(p(x)B^{\alpha}_{P_1}y(x), B^{\alpha}_{P_1}z(x)\Big)_{\Omega} + (q(x)y(x), z(x))_{\Omega}. \tag{2.7}$$

Now, let

$$U = \{y \in C(\Omega)\big|\ \|B^{\alpha}_{P_1}y\|_{L^2(\Omega,p(x))} < \infty, \|y(x)\|_{L^2(\Omega,q(x))} < \infty \text{ and } y(a) = 0, A^{\alpha-1}_{P_2}z(x)[p(x)B^{\alpha}_{P_1}y(x)]|_{x=b} = 0\}. \tag{2.8}$$

$$V = \{z \in C(\Omega)\big|\ \|B^{\alpha}_{P_1}z\|_{L^2(\Omega,p(x))} < \infty, \|z(x)\|_{L^2(\Omega,q(x))} < \infty \text{ and } z(a) = 0\}. \tag{2.9}$$

As $U \subset V$ therefore, we use the Galerkin method and choose $U = U \cap V$ for the trial and test function y and z respectively. Hence, taking $Ly = \lambda wy$ with weight function $w > 0$. We observe that U is a Hilbert space and bilinear form $B(y,z)$ is linear and continuous. Further following the conclusion by Lemma 2.4 [66], there are positive constants C_1 and C_2 such that for any $\omega \in L^2([a,b])$,

$$C_1\int_a^b B^{\alpha}_{P_1}\omega(x)B^{\alpha}_{P_2}\omega(x)dx \leq \|B^{\alpha}_{P}\omega(x)\|^2 \leq C_2\int_a^b B^{\alpha}_{P_1}\omega(x)B^{\alpha}_{P_2}\omega(x)dx, \tag{2.10}$$

in which $B^{\alpha}_{P}\omega(x)$ can be either $B^{\alpha}_{P_1}$ or $B^{\alpha}_{P_2}$. Hence, $B(y,z) \geq C\|y\|_U$. The bilinear form $B(y,z)$ is coercive when $p(x) = 1$ and by Lax-Milgram lemma, the considered GFSLP is well-posed.

2.3 Numerical Solution

Here, we discuss a numerical scheme for the GFSLP defined by Eqs.(2.1)-(2.2) using the matrix approach introduced initially in [2] and further studied in [65, 67]. In this scheme, the GFSLP is transformed into an algebraic eigenvalue problem and by solving the obtained eigenvalue problem, eigenvalues and eigenvectors are estimated. To approximate the B-operator, a finite difference dicretization scheme is used and presented as follows:

2.3.1 Numerical Approximation of the Problem

To find the numerical solution, we take equispaced nodes x_i in $[a, b]$ with step $\Delta x = (b-a)/N$, $x_i = a + i\Delta x$ for $i = 0, 1, 2, ..., N$ and $x_0 = a, x_N = b$ and denote $y_i = y(x_i)$. Now, we approximate $B^{\alpha}_{P_1}$ using finite difference method. The operator $B^{\alpha}_{P_1} y(x)$ at nodes x_i, $i = 0, 1, ..., N$ is approximated as follows,

$$B^{\alpha}_{P_1} y(x) = K^{1-\alpha}_{P_1} Dy(x) = p_1 \int_a^x k_{1-\alpha}(x,t)Dy(t)dt + q_1 \int_x^b k_{1-\alpha}(t,x)Dy(t)dt, \quad (2.11)$$

$$= p_1 S^{\alpha} y(x) + q_1 T^{\alpha} y(x). \quad (2.12)$$

Now, approximating $S^{\alpha}y(x)$ and $T^{\alpha}y(x)$ at nodes $x_i = 0, 1, 2, ..., N$,

$$S^{\alpha} y(x)|_{x=x_0} = 0, \quad (2.13)$$

$$S^{\alpha} y(x)|_{x=x_i} = \int_{x_0}^{x_i} k_{1-\alpha}(x_i,t)Dy(t)dt,$$

$$\cong \sum_{j=0}^{i-1} \frac{y_{j+1} - y_j}{\Delta x} \int_{x_j}^{x_{j+1}} k_{1-\alpha}(x_i,t)dt. \quad (2.14)$$

After simplifying Eq.(2.14), we get

$$S^{\alpha}y(x)|_{x=x_i} \cong \sum_{j=0}^{i-1}(y_{j+1}-y_j)\int_0^1 k_{1-\alpha}(x_i, x_j+\zeta\Delta x)d\zeta. \tag{2.15}$$

We can rewrite Eq.(2.15) in the final form as,

$$S^{\alpha}y(x)|_{x=x_i} \cong \sum_{j=0}^{i} v_l(\Delta x, i, j)y_j, \tag{2.16}$$

where

$$v_l(\Delta x, i, j) = \begin{cases} 0 & \text{for } i=0, \\ C(\Delta x, i, 0) & \text{for } i>0,\ j=0, \\ C(\Delta x, i, j-1) - C(\Delta x, i, j) & \text{for } i>0,\ j=1,2,...,i-1, \\ C(\Delta x, i, i-1) & \text{for } i>0,\ j=i, \end{cases} \tag{2.17}$$

and

$$C(\Delta x, i, j) = \int_0^1 k_{1-\alpha}(x_i, x_j+\zeta\Delta x)d\zeta. \tag{2.18}$$

At each node $x_i,\ i=0,1,2,...,N$, approximation given by Eq.(2.16) can be written in matrix form as follows

$$Y_L^{\alpha} = V_L^{\alpha}Y, \tag{2.19}$$

where

$$
V_L^\alpha = \begin{bmatrix}
v_{l_{(0,0)}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
v_{l_{(1,0)}} & v_{l_{(1,1)}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
v_{l_{(2,0)}} & v_{l_{(2,1)}} & v_{l_{(2,2)}} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
v_{l_{(i,0)}} & v_{l_{(i,1)}} & v_{l_{(i,2)}} & v_{l_{(i,3)}} & \cdots & v_{l_{(i,j)}} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
v_{l_{(N,0)}} & v_{l_{(N,1)}} & v_{l_{(N,2)}} & v_{l_{(N,3)}} & \cdots & v_{l_{(N,j)}} & \cdots & v_{l_{(N,N)}}
\end{bmatrix},
$$

$$
Y_L^\alpha = \begin{bmatrix}
S^\alpha y(x)|_{x=x_0} \\
S^\alpha y(x)|_{x=x_1} \\
S^\alpha y(x)|_{x=x_2} \\
\vdots \\
S^\alpha y(x)|_{x=x_i} \\
\vdots \\
S^\alpha y(x)|_{x=x_N}
\end{bmatrix}, Y = \begin{bmatrix}
y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_N
\end{bmatrix}.
$$

In a similar way, $T^\alpha y(x)$ is approximated at nodes $x_i,\ i = 0, 1, ..., N$ as follows,

$$
\begin{aligned}
T^\alpha y(x)|_{x=x_i} =& \int_{x_i}^{x_N} k_{1-\alpha}(t, x_i) Dy(t) dt, \\
\cong& \sum_{j=i}^{N-1} \frac{y_{j+1} - y_j}{\Delta x} \int_{x_j}^{x_{j+1}} k_{1-\alpha}(t, x_i) dt, \\
\cong& \sum_{j=i}^{N-1} (y_{j+1} - y_j) \int_0^1 k_{1-\alpha}(x_j + \zeta \Delta x, x_i) d\zeta, \qquad (2.20) \\
T^\alpha y(x)|_{x=x_N} =& 0. \qquad (2.21)
\end{aligned}
$$

Combining Eq.(2.20) and Eq.(2.21), we get,

$$T^{\alpha}y(x)|_{x=x_i} \cong \sum_{j=i}^{N} v_u(\Delta x, i, j)y_j, \tag{2.22}$$

where

$$v_u(\Delta x, i, j) = \begin{cases} 0 & \text{for } i = N, \\ D(\Delta x, i, N) & \text{for } i < N, j = N, \\ D(\Delta x, i, j) - D(\Delta x, i, j-1) & \text{for } i < N, j = i+1, i+2...., N-1, \\ D(\Delta x, i, i) & \text{for } i < N, j = i, \end{cases} \tag{2.23}$$

and

$$D(\Delta x, i, j) = \int_0^1 k_{1-\alpha}(x_j + \zeta\Delta x, x_i)d\zeta. \tag{2.24}$$

At each node x_i, Eq.(2.22) takes the form

$$Y_U^{\alpha} = V_U^{\alpha} Y, \tag{2.25}$$

where,

$$V_U^{\alpha} = \begin{bmatrix} v_{u_{(0,0)}} & v_{u_{(0,1)}} & v_{u_{(0,2)}} & v_{u_{(0,3)}} & \cdots & v_{u_{(0,j)}} & \cdots & v_{u_{(0,N)}} \\ 0 & v_{u_{(1,1)}} & v_{u_{(1,2)}} & v_{u_{(1,3)}} & \cdots & v_{u_{(1,j)}} & \cdots & v_{u_{(1,N)}} \\ 0 & 0 & v_{u_{(2,2)}} & v_{u_{(2,3)}} & \cdots & v_{u_{(2,j)}} & \cdots & v_{u_{(2,N)}} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & v_{u_{(i,j)}} & \cdots & v_{u_{(i,N)}} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & v_{u_{(N,N)}} \end{bmatrix},$$

$$Y_U^\alpha = \begin{bmatrix} T^\alpha y(x)|_{x=x_0} \\ T^\alpha y(x)|_{x=x_1} \\ T^\alpha y(x)|_{x=x_2} \\ \vdots \\ T^\alpha y(x)|_{x=x_i} \\ \vdots \\ T^\alpha y(x)|_{x=x_N} \end{bmatrix},$$

and V_U^α is an upper triangular matrix.

Following the similar approach, we can find the approximation for $B_{P_2}^\alpha$. Now, we introduce the following diagonal matrices to solve Eqs.(2.1)-(2.2):

$P = diag(p(x_0), p(x_1), p(x_2), ...p(x_N))$, $\quad Q = diag(q(x_0), q(x_1), q(x_2), ...q(x_N))$,

$R = diag(r_0, r_1, r_2, ...r_N)$,

where

$$r_k = \frac{\int_{x_k}^{x_{k+1}} r(x)dx}{x_{k+1} - x_k},\ k < N. \tag{2.26}$$

We now use the matrix approximations given by Eq.(2.19) and Eq.(2.25) to get discretized version of the GFSLP (Eq.(2.1)) and obtain the following,

$$[(p_1 V_L^\alpha + q_1 V_U^\alpha) P (p_2 V_L^\alpha + q_2 V_U^\alpha)] Y + QY = \lambda RY. \tag{2.27}$$

$$R^{-1} V Y = \lambda Y, \tag{2.28}$$

where $V = (p_1 V_L^\alpha + q_1 V_U^\alpha) P (p_2 V_L^\alpha + q_2 V_U^\alpha + Q)$, $Y = [y_0, y_1, y_2, ...y_N]^T$. The elements of matrix V can be written as follows:

$$V_{i,k} = \sum_{j=\max(i,k)}^{N} (p_1 v_{l_{i,j}} + q_1 v_{u_{i,j}})(p(x_j))(p_2 v_{l_{j,k}} + q_2 v_{u_{j,k}}) + q(x_i)\delta_{i,k}, \quad i,k = 0,1,2,...,N. \tag{2.29}$$

After applying boundary conditions, Eq.(2.29) reduces to

$$V_{i,k} = \sum_{j=\max(i,k)}^{N} (p_1 v_{l_{i,j}} + q_1 v_{u_{i,j}})(p(x_j))(p_2 v_{l_{j,k}} + q_2 v_{u_{j,k}}) + q(x_i)\delta_{i,k}, \quad i,k = 1,2,...,N-1. \tag{2.30}$$

Since, $v_{u_{k,j}} = v_{l_{j,k}}$, $k, j = 1,2,...,N-1$, and for $p_1 = q_2, q_1 = p_2$, we have, $V_{i,k} = V_{k,i}$. Thus, V is a symmetric matrix.

We shall now establish that in the respective N-dimensional vector space, the eigenfunctions associated to distinct eigenvalues are orthogonal under the scalar product:

$$\langle Y, X \rangle_R = \sum_{i=0}^{N-1} Y_i r_i X_i = \langle X, Y \rangle_R. \tag{2.31}$$

Let λ_j and λ_k be two eigenvalues of $R^{-1}V$ and the corresponding eigenvectors are Y_j and Y_k respectively, then

$$\langle Y_j, R^{-1}VY_k \rangle_R = \lambda_k \langle Y_j, Y_k \rangle_R, \tag{2.32}$$

$$\langle Y_k, R^{-1}VY_j \rangle_R = \lambda_j \langle Y_k, Y_j \rangle_R. \tag{2.33}$$

Subtracting Eq.(2.33) from Eq.(2.32), we get,

$$\langle Y_j, R^{-1}VY_k \rangle_R - \langle Y_k, R^{-1}VY_j \rangle_R = (\lambda_k - \lambda_j)\langle Y_j, Y_k \rangle_R. \tag{2.34}$$

Using the scalar product definition given in Eq.(2.33) and symmetricity of matrix V, we get,

$$\langle Y_j, R^{-1}VY_k\rangle_R - \langle Y_k, R^{-1}VY_j\rangle_R = 0.$$

This proves that the eigenfunctions corresponding to distinct eigenvalues are orthogonal, i.e.

$$\langle Y_k, Y_j\rangle_R = 0, \quad k \neq j. \tag{2.35}$$

This completes the proof.

Now, we establish the convergence analysis of the proposed scheme using a variational formulation.

2.4 Theoretical Convergence Analysis

A variational formulation of the considered GFSLP given by Eqs.(2.1)-(2.2) is designed in Section 2.4 as follows:

Find $y \in U$ and $\lambda \in \mathbb{R}$ such that

$$B(y, z) = (Ly, z)_\Omega = \lambda(y, z)_{\Omega,r(x)} \quad \forall z \in U. \tag{2.36}$$

We approximate U by U_h consisting of piecewise linear functions. For a given uniform partition $h = (b-a)/N$, let y_h be the solution of the approximated problem:

$$B(y, z) = \lambda_h(y_h, z)_{\Omega,r(x)} \quad \forall z \in U_h. \tag{2.37}$$

2.4.1 Convergence of Eigenfunctions

Theorem 2.1. *Let $y(x) \in C^2(\Omega)$ then the truncation error $T^k(y,h,\alpha)$ in approximation of $B^{\alpha}_{P_1}$ satisfies,*

$$T^k(y,h,\alpha) = |B^{\alpha}_{P_1} y(x) - B^{\alpha}_{P_1} y_h(x)| \leq \frac{Ch^{2-\alpha}}{8} \max_{0 \leq x \leq x_N} |y''(x)|, \tag{2.38}$$

and bound for the solution $y(x)$ is given as,

$$\|y(x) - y_h(x)\|_{L^2} \leq \frac{C_3 h^{2-\alpha}}{8} \max_{0 \leq x \leq x_N} |y''(x)|, \tag{2.39}$$

where C and C_3 are constants.

Proof. Here, we first find the truncation error $T^k(y,h,\alpha)$ and further establish the bound for the solution. Let,

$$T^k(y,h,\alpha) = |B^{\alpha}_{P_1} y(x) - B^{\alpha}_{P_1} y_h(x)|, \tag{2.40}$$

where $y_h(x)$ is a linear interpolating function between points (x_j, y_j) and (x_{j+1}, y_{j+1}), defined as,

$$y_h(x) = \frac{(x - x_{j+1})}{(x_j - x_{j+1})} y_j + \frac{(x - x_j)}{(x_{j+1} - x_j)} y_{j+1}, \tag{2.41}$$

so we have,

$$y_h'(x) = \frac{y_{j+1} - y_j}{h}, \tag{2.42}$$

and

$$y(x) - y_h(x) = \frac{y''(\xi_j)}{2}(x - x_j)(x - x_{j+1}), \tag{2.43}$$

where $\xi_j \in (x_j, x_{j+1})$.

Since,

$$\begin{aligned}
|B^{\alpha}_{P_1} y(x) - B^{\alpha}_{P_1} y_h(x)| &= \bigg| p_1 \int_a^x k_{1-\alpha}(x,s)(y(s) - y_h(s))' ds \\
&\qquad + q_1 \int_x^b k_{1-\alpha}(s,x)(y(s) - y_h(s))' ds \bigg|, \\
&\leq p_1 \bigg| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} k_{1-\alpha}(x_k,s)(y(s) - y_h(s))' ds \bigg| \\
&\qquad + q_1 \bigg| \sum_{j=k}^{N-1} \int_{x_j}^{x_{j+1}} k_{1-\alpha}(s,x_k)(y(s) - y_h(s))' ds \bigg| \\
&= p_1 \bigg| \sum_{j=1}^{k-1} \int_{x_j}^{x_{j+1}} \frac{d}{ds} k_{1-\alpha}(x_k,s)(y(s) - y_h(s)) ds \bigg| \\
&\qquad + q_1 \bigg| \sum_{j=k}^{N-1} \int_{x_j}^{x_{j+1}} \frac{d}{ds} k_{1-\alpha}(s,x_k)(y(s) - y_h(s)) ds \bigg| \\
&\leq p_1 \bigg| \frac{h^2}{8} \sum_{j=1}^{k-1} y''(\xi_j) \int_{x_j}^{x_{j+1}} \frac{d}{ds} k_{1-\alpha}(x_k,s) ds \bigg| \\
&\qquad + q_1 \bigg| \frac{h^2}{8} \sum_{j=k}^{N-1} y''(\xi_j) \int_{x_j}^{x_{j+1}} \frac{d}{ds} k_{1-\alpha}(s,x_k) ds \bigg| \\
&\leq p_1 \frac{h^2}{8} \sum_{j=1}^{k-1} y''(\xi_j) |k_{1-\alpha}(x_k,x_{j+1}) - k_{1-\alpha}(x_k,x_j)| \\
&\qquad + q_1 \frac{h^2}{8} \sum_{j=k}^{N-1} y''(\xi_j) |(k_{1-\alpha}(x_{j+1},x_k) - k_{1-\alpha}(x_j,x_k))|.
\end{aligned} \tag{2.44}$$

Since, the Prabhakar kernel [64, 25] and the modified power kernel [22] satisfy, $|k_{1-\alpha}(x_k,x_{j+1}) - k_{1-\alpha}(x_k,x_j)| \leq C_1 h^{-\alpha}$ and $|k_{1-\alpha}(x_{j+1},x_k) - k_{1-\alpha}(x_j,x_k)| \leq C_2 h^{-\alpha}$, therefore,

$$T^k(y,h,\alpha) = |B^\alpha_{P_1}(y(x) - y_h(x))| \le \frac{Ch^{2-\alpha}}{8} \max_{0\le x\le x_N} |y''(x)|, \tag{2.45}$$

and

$$\|B^\alpha_{P_1} y(x) - B^\alpha_{P_1} y_h(x)\|_{L^2} = \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} |B^\alpha_{P_1}(y(x) - y_h(x))|^2 dx \right)^{1/2}, \tag{2.46}$$

using Eq.(2.45), we have

$$\begin{aligned} \|B^\alpha_{P_1} y(x) - B^\alpha_{P_1} y_h(x)\|_{L^2} &\le \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\frac{Ch^{2-\alpha}}{8} \max_{0\le x\le x_k} |y''(x)| \right)^2 dx \right)^{1/2} \\ &\le \frac{Ch^{2-\alpha}}{8} \max_{0\le x\le x_N} |y''(x)| \left(\int_0^1 dx \right)^{1/2} \\ &\le \frac{Ch^{2-\alpha}}{8} \max_{0\le x\le x_N} |y''(x)|. \end{aligned} \tag{2.47}$$

Now, using Poincare-Writinger inequality, the error in energy norm can be estimated in a standard way [26, 27] as follows:

$$\|y(x) - y_h(x)\|_{L^2} \le C \|B^\alpha_{P_1} y(x) - B^\alpha_{P_1} y_h(x)\|_{L^2},$$

thus, Eq. (2.47) implies

$$\|y(x) - y_h(x)\|_{L^2} \le \frac{C_3 h^{2-\alpha}}{8} \max_{0\le x\le x_N} |y''(x)|. \tag{2.48}$$

□

2.4.2 Convergence of Eigenvalues

Let $\lambda^{(k)}, k \in \mathbb{N}$ denote the eigenvalues of Eq.(2.36) with the natural numbering $\lambda^{(1)} \leq \lambda^{(2)} \leq \lambda^{(3)} \leq ... \leq \lambda^{(k)}$ and $y^{(k)}$ represents the corresponding eigenfunction. Let $E^{(k)} = \text{span}\ \{y^{(k)}\}$ denotes the associated eigenspaces and $U^{(k)}$ be the set of all subspaces to U with dimension equals to k, such that $E = \oplus_{i=1}^{k} E^{(i)}$, so that $z = \sum_{i=1}^{k} c_i y^{(i)}$.

We define the Rayleigh quotient for Eq.(2.36) as

$$\lambda^{(k)} = \min_{E \in U^{(k)}} \max_{z \in E} \frac{B(z,z)}{(z,z)_\Omega}. \tag{2.49}$$

Let $E_h^{(k)} = \text{span}\ \{y_h^k\}$ denotes the associated eigenspaces and assume that $U_h^{(k)}$ be the set of all subspaces to U_h with dimension equal to k, such that $E_h = \oplus_{i=1}^{k} E_h^{(i)}$. The properties of Rayleigh quotient can be applied to approximate eigenvalues as follows:

$$\begin{aligned}
\lambda_h^{(k)} &= \min_{E_h \in U_h^{(k)}} \max_{y_h \in E_h} \frac{B(y_h, y_h)}{(y_h, y_h)_\Omega}, \\
\lambda_h^{(k)} &\leq \max_{y_h \in E_h} \frac{\|B_{P_1}^{\alpha} y_h\|_{L^2(\Omega)}^2}{\|y_h\|_{L^2(\Omega)}^2} = \max_{y \in U_h^{(k)}} \frac{\|B_{P_1}^{\alpha} y_h\|_{L^2(\Omega)}^2}{\|y_h\|_{L^2(\Omega)}^2},
\end{aligned}$$

$$\begin{aligned}
\lambda_h^{(k)} &\leq \max_{y \in U_h^{(k)}} \frac{\|B_{P_1}^{\alpha} y\|_{L^2(\Omega)}^2}{\|y_h\|_{L^2(\Omega)}^2} = \max_{y \in U_h^{(k)}} \frac{\|B_{P_1}^{\alpha} y\|_{L^2(\Omega)}^2}{\|y\|_{L^2(\Omega)}^2} \frac{\|y\|_{L^2(\Omega)}^2}{\|y_h\|_{L^2(\Omega)}^2}, \\
\lambda_h^{(k)} &\leq \lambda^{(k)} \max_{y \in U_h^{(k)}} \frac{\|y\|_{L^2(\Omega)}^2}{\|y_h\|_{L^2(\Omega)}^2},
\end{aligned} \tag{2.50}$$

and

$$\|y_h\|_{L^2(\Omega)} \geq \|y\|_{L^2(\Omega)} - \|y - y_h\|_{L^2(\Omega)}. \tag{2.51}$$

From Eq.(2.48) and Eq.(2.51), we have

$$\|y_h\|_{L^2(\Omega)} \geq \left(1 - C_1 h^{2-\alpha} \frac{\max_{0\leq x\leq x_k} |y''(x)|}{\|y\|_{L^2(\Omega)}}\right) \|y\|_{L^2(\Omega)}. \tag{2.52}$$

Thus, Eq.(2.50) implies

$$\begin{aligned} \lambda_h^k &\leq \lambda^{(k)} \max_{y\in U_h^{(k)}} \frac{1}{\left(1 - C_1 h^{2-\alpha} \frac{\max_{0\leq x\leq x_k} |y''(x)|}{\|y\|_{L^2(\Omega)}}\right)^2} \\ \lambda_h^k &\approx \lambda^{(k)} \left(1 + 2MC_1 h^{2-\alpha} \max_{y\in U_h^{(k)}} \frac{1}{\|y\|_{L^2(\Omega)}}\right), \end{aligned} \tag{2.53}$$

where $M = \max_{0\leq x\leq x_N} |y''(x)|$.

2.5 Numerical Examples

This section consists of four different cases and their numerical results. Firstly, we have calculated the eigenvalues and eigenfunctions numerically for four different cases to validate the theory provided in Section 2.4. The eigenvalues are provided through tables, and graph of normalized eigenfunctions are demonstrated through figures. Further, the numerical convergence rate are validated for the considered test cases.

2.5.1 Evaluation of Eigenvalues and Eigenfunctions

In the first two cases, we consider the modified power kernel $k_\alpha(t,x) = \frac{1}{\Gamma(\alpha)[m+(1-m)(t-x)^{1-\alpha}]}$ [22] which leads to the power kernel for $m = 0$. In the other two cases, we take the Prabhakar kernel $k_\alpha(t,x) = (t-x)^{\alpha-1} E_{\rho,\alpha}^{\gamma}(\omega(t-x)^{\rho})$ [24, 25], where $E_{\rho,\alpha}^{\gamma}(\cdot)$ denotes the three parameter Mittag-Leffler function defined as $E^{\gamma}_{\rho,\alpha}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\alpha)} \frac{z^k}{k!}$, $\rho,\ \alpha,\ \gamma \in \mathbb{C}$. We choose the following cases to perform simulation numerically.

Case 1. $p(x) = 1,\ q(x) = 0,\ r(x) = 1,$

Case 2. $p(x) = 1,\ q(x) = 5sin(\pi x),\ r(x) = \frac{1}{(1+x)^2},$

Case 3. $p(x) = 1 + 2x^2, q(x) = 5sin(\pi x), r(x) = 2e^x,$

Case 4. $p(x) = (5x^2 + 1)/2,\ q(x) = 0,\ r(x) = 1.$

Here, in Tables 2.1-2.3, the numerical approximations of first ten eigenvalues for Case 1 and Case 2 are presented. For this, we have considered following parameters: $a = 0$, $b = 1$, $N = 200$, $\alpha \in \{0.4, 0.6, 0.75, 0.8, 0.9\}$, $m \in \{0, 0.01, 0.1\}$, and for Case 3 and Case 4, we take $N = 200$ and different values of α i.e. $\alpha \in \{0.3, 0.5, 0.6, 0.7, 0.8, 0.9\}$. The results for Case 3 and Case 4 are presented in Table 2.4. The obtained eigenvalues set up the increasing sequence. From Tables 2.1-2.4, we observe that the eigenvalue increases as we increase fractional order α and decreases with increasing values of m for fixed α as shown in Tables 2.1-2.3.

Further, we have plotted the normalized eigenfunctions associated with the first four eigenvalues for above mentioned cases in Figs 2.1-2.4. It is noticed that y_k^{th} eigenfunction of GFSLP has $(k-1)$ zeros in the given interval which is similar to a result of integer order SLPs. We have normalized the approximated eigenfunctions through the formula given as:

$$\int_0^1 r(x) y_k(x) y_k(x) = 1. \tag{2.54}$$

Table 2.1: First 10 eigenvalues for Case 1 for $P_1 = (a, b; 1, 0), P_2 = (a, b; 0, 1)$, different values of m and α.

$\alpha = 0.6$			$\alpha = 0.8$			$\alpha = 0.9$		
$m = 0$ [67]	$m = 0.01$	$m = 0.1$	$m = 0$ [67]	$m = 0.01$	$m = 0.1$	$m = 0$ [67]	$m = 0.01$	$m = 0.1$
1.6107	1.2978	0.7545	1.9728	0.7143	0.2679	2.2053	0.2507	0.0075
6.5160	4.7767	2.1117	12.1073	3.4007	0.8838	16.4627	1.3028	0.2621
11.9392	8.2134	3.0698	27.3053	6.4745	1.3283	41.2108	2.5625	0.3962
17.9685	11.7637	3.8776	46.9623	9.8143	1.6877	75.6964	3.9455	0.5007
24.3034	15.2626	4.5474	70.2924	13.2260	1.9725	119.1122	5.3662	0.5805
31.0125	18.7835	5.1381	97.1557	16.7141	2.2112	171.2069	6.8160	0.6448
37.9582	22.2578	5.6536	127.1610	20.20231	2.4098	231.5478	8.2634	0.6965
45.1892	25.7302	6.1197	160.2504	23.7058	2.5810	299.9951	9.7091	0.7395
52.6215	29.1613	6.5376	196.1747	27.1834	2.7279	376.2586	11.1374	0.7751
60.2916	32.5819	6.9216	234.9058	30.6482	2.8570	460.2441	12.5512	0.8954

Table 2.2: First 10 eigenvalues for Case 2 for $P_1 = (a, b; 1, 0), P_2 = (a, b; 0, 1)$, different values of m and α.

$\alpha = 0.4$			$\alpha = 0.75$			$\alpha = 0.9$		
$m = 0$	$m = 0.01$	$m = 0.1$	$m = 0$	$m = 0.01$	$m = 0.1$	$m = 0$	$m = 0.01$	$m = 0.1$
7.9469	7.5177	5.2621	5.2232	9.4142	3.6553	14.6020	5.6013	1.0513
13.3838	12.6838	8.1860	27.6479	14.4962	5.6320	41.1373	8.9933	1.2240
15.8882	15.0362	9.8202	52.6554	21.7689	6.4546	93.1808	10.7075	1.3192
19.1013	17.5994	11.1137	83.1409	29.8498	6.9355	164.9178	14.2679	1.4151
22.4341	20.4637	12.0827	118.2403	37.8337	7.2229	255.2111	17.2339	1.5121
25.5044	23.0906	12.7907	157.7324	45.8655	7.4195	363.4749	19.9803	1.6105
28.3962	25.5168	13.3528	201.0848	53.8879	7.5799	488.9160	22.7425	1.7104
31.1887	27.8098	14.0405	248.1800	61.9098	7.7351	631.1722	25.5091	1.8118
33.9081	30.0135	14.7235	298.6738	69.8750	7.8955	789.7025	28.2414	1.9148
36.5828	32.1560	15.3589	352.5051	77.7975	8.0555	964.2548	30.9313	2.1091

TABLE 2.3: First 10 eigenvalues for Case 1 for $P_1 = (a, b; 1/2, 1/2), P_2 = (a, b; 1/2, 1/2)$, different values of m and α.

$\alpha = 0.3$			$\alpha = 0.6$			$\alpha = 0.8$		
$m = 0$	$m = 0.01$	$m = 0.1$	$m = 0$	$m = 0.01$	$m = 0.1$	$m = 0$	$m = 0.01$	$m = 0.1$
1.0959	1.0863	1.0056	0.7898	0.7578	0.5949	0.3370	0.2996	0.1959
2.0883	2.0519	1.7664	2.5423	2.3726	1.5576	1.4844	1.2342	0.6188
2.7919	2.728301	2.2468	4.4807	4.0832	2.3433	3.1591	2.4602	0.9954
3.4100	3.3175	2.6411	6.6349	5.9147	3.0425	5.3668	3.9230	1.3280
3.9538	3.8339	2.9688	8.9182	7.7922	1.9725	8.0572	5.5399	1.6107
4.4599	4.3114	3.2603	11.3538	9.7272	4.2021	11.2587	7.2966	1.8562
4.9293	4.7516	3.5188	13.9023	11.6888	4.6958	14.9587	9.1510	2.0677
5.3761	5.1686	3.7578	16.5818	13.6911	5.1487	19.1932	11.0968	2.2539
5.7993	5.5640	3.9773	19.3711	15.7126	5.5623	23.9626	11.1374	2.4174
6.2081	5.9403	4.1819	22.2810	17.7627	5.9463	29.3025	15.1187	2.5624

2.5.2 Numerical Convergence Order

Here, we determine the convergence order of the presented numerical method. As, it is difficult to calculate the exact eigenvalues analytically, and therefore we obtain the approximate eigenvalues numerically, and estimate the numerical orders of convergence. For this, we have used formulas for experimental rate of convergence of the approximated k^{th} eigenvalue and eigenfunction for fixed α and N,

$$erc_\lambda(N, k, \alpha) = \log_2 \frac{\lambda_k^{(N,\alpha)} - \lambda_k^{(N/2,\alpha)}}{\lambda_k^{(2N,\alpha)} - \lambda_k^{(N,\alpha)}}, \tag{2.55}$$

$$erc_y(N, k, \alpha) = \log_2 \frac{\|y_{k,N} - y_{k,N/2}\|_{L^2_w}}{\|y_{k,2N} - y_{k,N}\|_{L^2_w}}. \tag{2.56}$$

TABLE 2.4: First 10 eigenvalues for Case 3 and Case 4 for $P_1 = (a, b; 1, 0), P_2 = (a, b; 0, 1)$ and different values of α.

$p(x) = 1 + 2x^2, q(x) = 5sin(\pi x), r(x) = 2e^x$				$p(x) = (5x^2 + 1)/2, q(x) = 0, r(x) = 1$			
$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 0.9$
2.9021	3.0082	3.0627	3.1215	1.5877	2.7257	4.1391	4.5187
3.4097	4.3256	5.6650	7.5200	2.4344	4.6082	11.2143	15.4177
7.0927	10.0825	14.5983	21.3917	3.4046	8.5456	29.5968	44.8291
9.3020	14.4576	22.8215	36.3297	3.9391	10.7387	47.6631	77.9095
12.6827	20.8309	34.6642	58.0456	4.6760	14.3673	73.3085	125.4235
15.5701	26.8553	46.8299	81.9918	5.1158	16.9934	99.7804	178.0600
19.0799	34.1135	61.5687	111.4146	5.7478	20.4703	131.8272	242.6089
22.3314	41.2788	76.8999	143.4161	6.1376	23.3384	165.1907	312.9060
26.0010	49.3581	94.3201	180.238	6.7059	26.7519	203.1106	393.7502
29.5162	57.4554	112.4416	219.7927	7.0665	29.7631	242.5647	480.6118

In Case 1, we take $k_\alpha(t, x) = \frac{1}{\Gamma(\alpha)[m+(1-m)(t-x)^{1-\alpha}]}$, $q(x) = 0$, $p(x) = 1$, $r(x) = 1$, and vary $N = \{10, 20, 40, 80, 160\}, m = \{0, 0.01, 0.1\}$ to calculate the eigenvalues and the convergence order for the problem Eqs.(2.1)-(2.2). The obtained values of the convergence order are shown through Table 2.5. In Table 2.6, we present the convergence order for $q(x) = 0$, $p(x) = (5x^2 + 1)/2$, $r(x) = 1$, $k_\alpha(t, x) = (t - x)^{\alpha-1}E^{\gamma}_{\rho,\alpha}(\omega(t - x)^\rho)$ and $N = \{50, 100, 200, 400, 800\}$. From Tables 2.5 and 2.6, one can observe that the convergence order do not depend on the functions p, q, r, α and m. It tends to 1 for all cases.

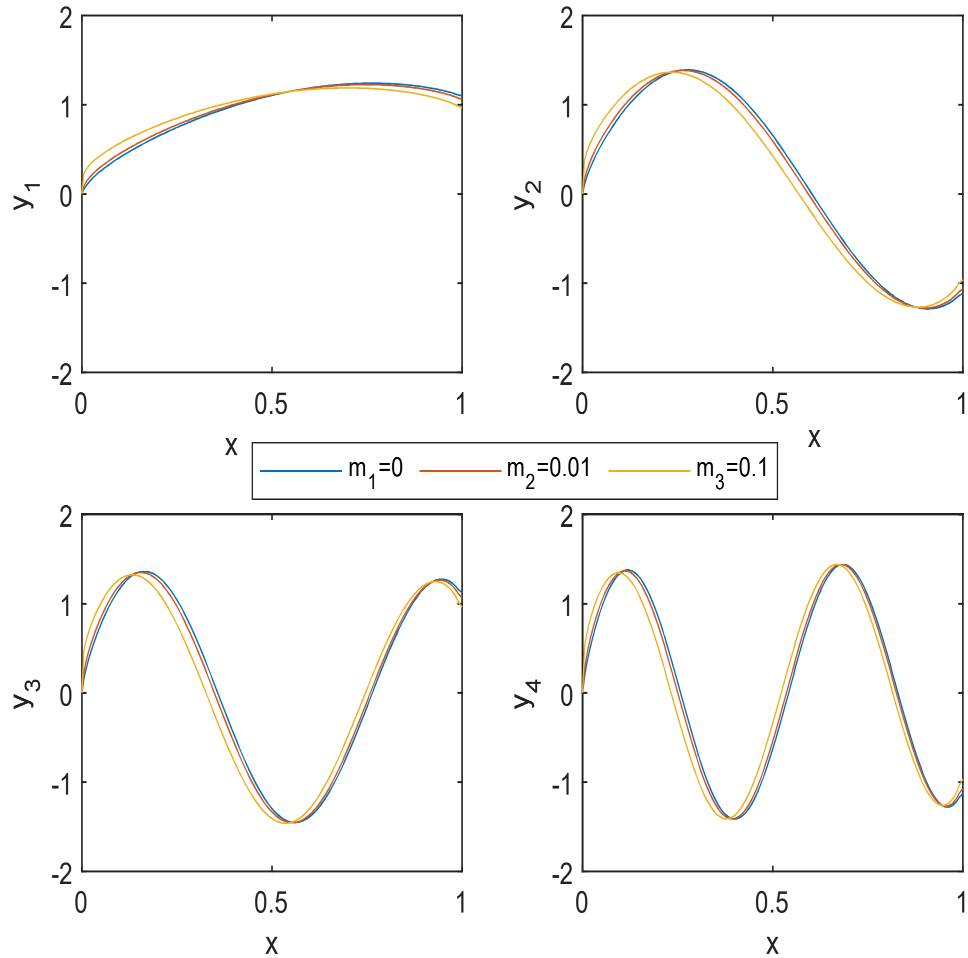

FIGURE 2.1: Plot of the eigenfunctions y_i, $i = 1, 2, 3, 4$ corresponding to the first four eigenvalues for different values of m and for fixed $\alpha = 0.6$ with $p(x) = 1, q(x) = 0, r(x) = 1$.

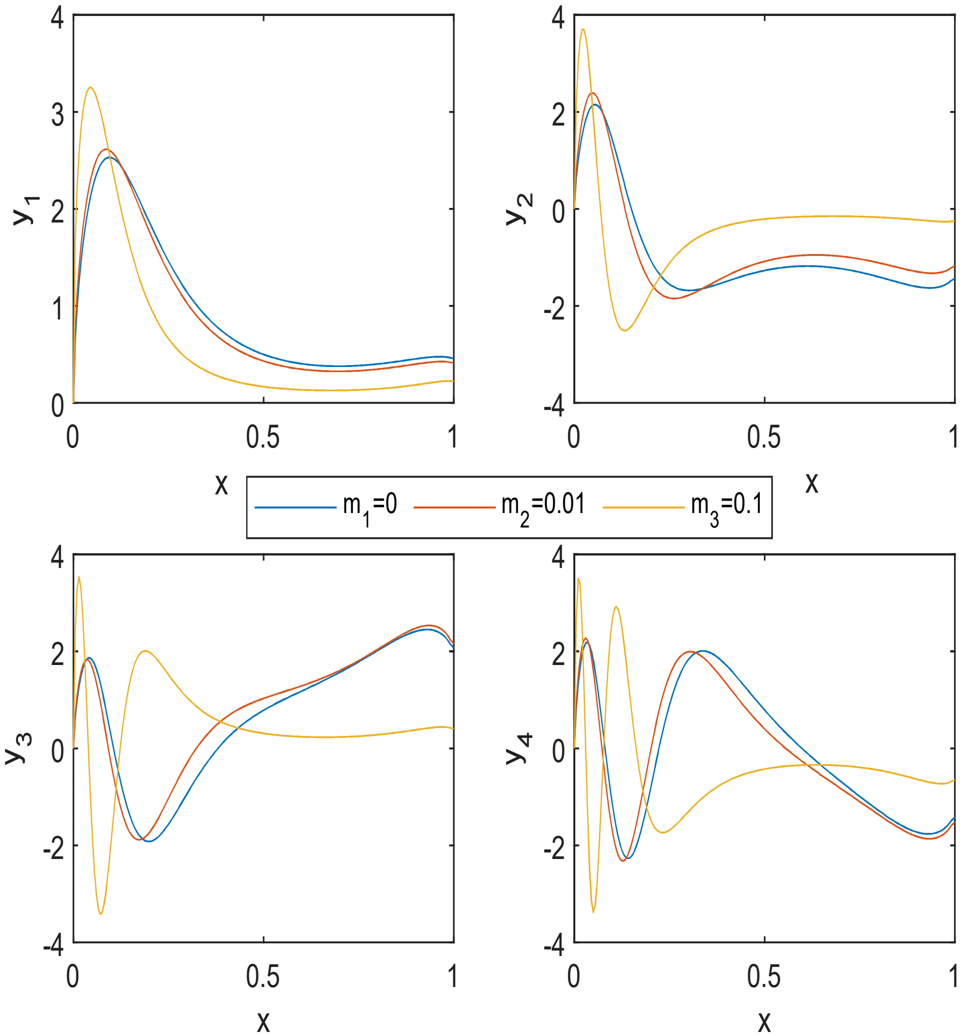

FIGURE 2.2: Plot of the eigenfunctions $y_i,\ i = 1, 2, 3, 4$ corresponding to the first four eigenvalues for different values of m and for fixed $\alpha = 0.4$ for $p(x) = 1, q(x) = 5sin(\pi x)$ and $r(x) = \frac{1}{(1+x)^2}$.

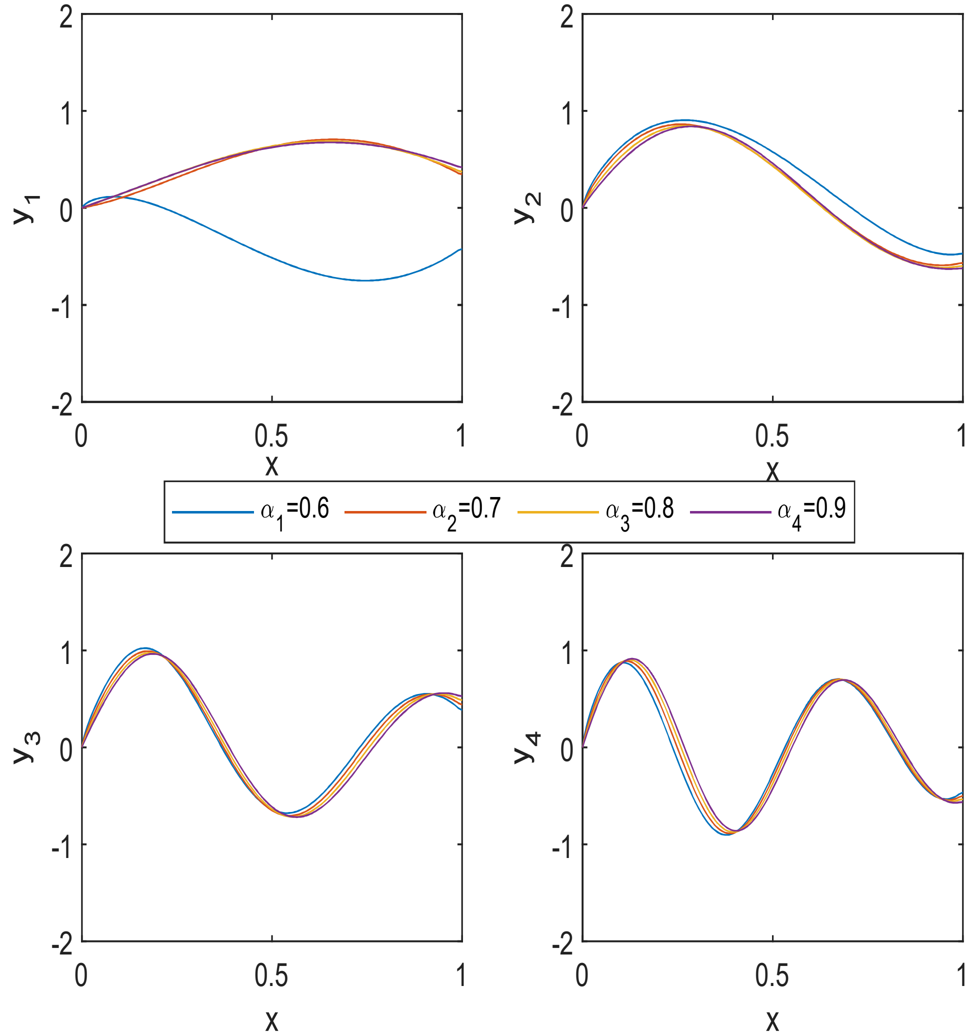

FIGURE 2.3: Plot of the eigenfunctions $y_i,\ i = 1, 2, 3, 4$ corresponding to the first four eigenvalues for different values of α for $p(x) = 1+2x^2, q(x) = 5sin(\pi x), r(x) = 2e^x$.

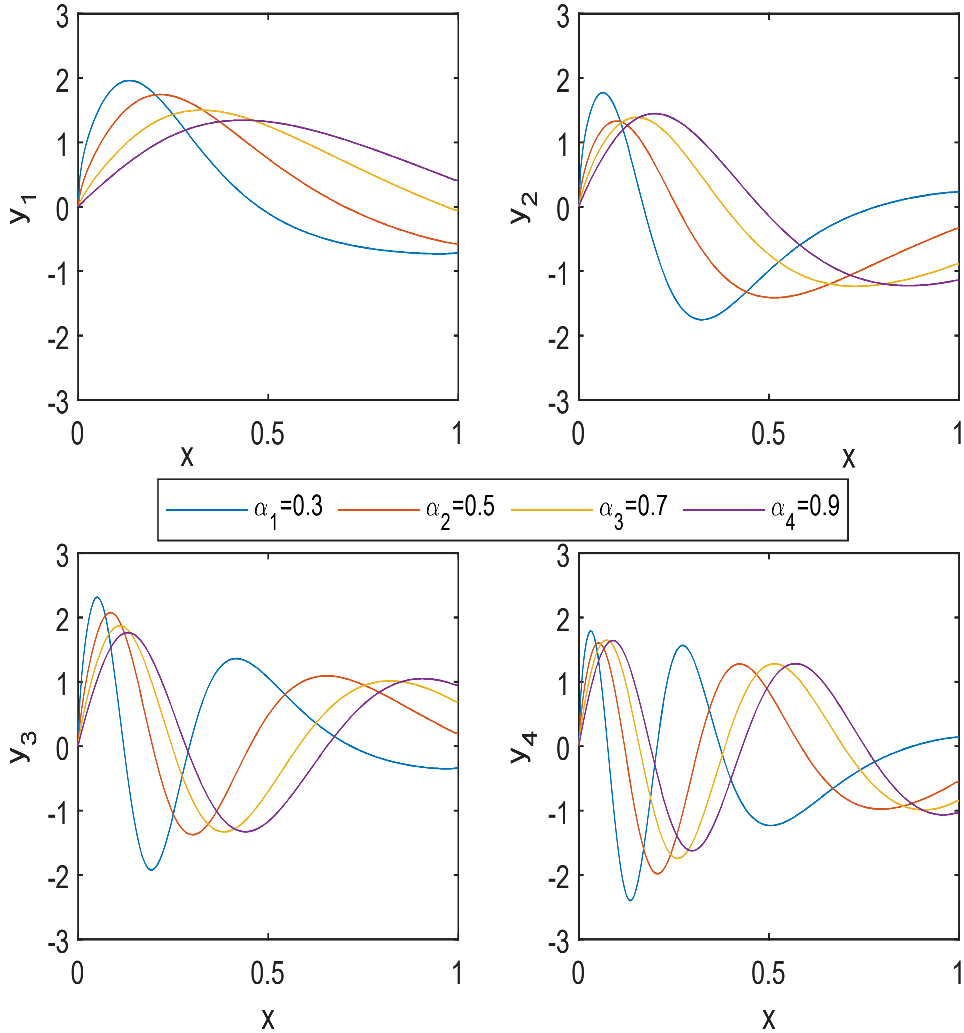

FIGURE 2.4: Plot of the eigenfunctions $y_i,\ i = 1, 2, 3, 4$ corresponding to the first four eigenvalues for different values of α for $p(x) = (5x^2+1)/2,\ q(x) = 0,\ r(x) = 1$.

TABLE 2.5: First five eigenvalues and experimental orders of convergence for different values of m and for $\alpha = 0.6$, $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, and $k_\alpha(t,x) = \frac{1}{\Gamma(\alpha)(m+(1-m)(t-x)^{1-\alpha})}$.

k	N	$m=0$			$m=0.01$			$m=0.1$		
		λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y
1	10	1.8206	-	-	1.4562	-	-	0.8258	-	-
	20	1.7064	1.07	0.80	1.3694	1.09	0.78	0.7865	1.11	0.80
	40	1.6522	1.04	0.85	1.3287	1.06	0.84	0.7683	1.06	0.85
	80	1.6261	1.03	0.89	1.3092	1.04	0.87	0.7596	1.04	0.89
	160	1.6133	-	-	1.2997	-	-	0.7553	-	-
2	10	7.6410	-	-	5.5187	-	-	2.3275	-	-
	20	7.0358	1.02	0.85	5.1078	1.08	0.85	2.2036	1.19	0.88
	40	6.7375	1.07	0.89	4.9140	1.12	0.89	2.1493	1.16	0.92
	80	6.5975	1.08	0.91	4.8252	1.12	0.91	2.1250	1.11	0.94
	160	6.5288	-	-	4.7845	-	-	2.1139	-	-
3	10	14.4556	-	-	9.7722	-	-	3.4329	-	-
	20	13.1703	0.87	0.88	8.9308	0.98	0.88	3.2203	1.20	0.92
	40	12.4664	1.05	0.90	8.5054	1.14	0.90	3.1281	1.25	0.94
	80	12.1267	1.10	0.93	8.3132	1.18	0.93	3.0896	1.21	0.95
	160	11.9689	-	-	8.2288	-	-	3.0729	-	-
4	10	22.1092	-	-	14.2638	-	-	4.3934	-	-
	20	20.2222	0.58	0.89	13.0126	0.76	0.89	4.0963	1.12	0.92
	40	18.9579	0.99	0.91	12.2746	1.12	0.91	3.9594	1.31	0.94
	80	18.3207	1.10	0.93	11.9354	1.22	0.93	3.9042	1.29	0.96
	160	18.0240	-	-	11.7896	-	-	3.8816	-	-
5	10	29.8505	-	-	18.6040	-	-	5.1946	-	-
	20	27.8328	0.08	0.90	17.1540	0.4	0.90	4.8399	0.93	0.93
	40	25.9074	0.90	0.91	16.0525	1.06	0.91	4.6548	1.34	0.95
	80	24.8790	1.08	0.93	15.5261	1.23	0.93	4.5811	1.35	0.96
	160	24.3938	-	-	15.3017	-	-	4.5523	-	-

TABLE 2.6: First five eigenvalues and experimental orders of convergence corresponding to different values of N, α and $p(x) = (5x^2+1)/2,\ q(x) = 0,\ r(x) = 1,$ and $k_\alpha(t,x) = (t-x)^{\alpha-1}E^{\gamma}_{\rho,\alpha}(\omega(t-x)^{\rho})$.

k	N	$\alpha = 0.6$			$\alpha = 0.8$			$\alpha = 0.9$		
		λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y
1	50	3.3443	-	-	4.2341	-	-	4.6374	-	-
	100	3.2983	1.12	0.92	4.1695	1.09	0.92	4.5573	1.05	0.94
	200	3.2771	1.11	0.94	4.1392	1.08	0.95	4.5187	1.05	0.96
	400	3.2673	1.10	0.95	4.1249	1.08	0.96	4.5000	1.04	0.97
	800	3.2627	-	-	4.1181	-	-	4.4909	-	-
2	50	6.3004	-	-	11.6863	-	-	16.0727	-	-
	100	6.1660	1.13	0.90	11.3672	1.06	0.91	15.6332	1.03	0.93
	200	6.1048	1.12	0.93	11.2143	1.06	0.93	15.4178	1.02	0.95
	400	6.0766	1.11	0.95	11.1408	1.05	0.95	15.3119	1.02	0.96
	800	6.0635	-	-	11.1055	-	-	15.2598	-	-
3	50	13.4419	-	-	30.964	-	-	46.7603	-	-
	100	13.0802	1.15	0.89	30.0445	1.04	0.90	45.4767	0.99	0.92
	200	12.9170	1.17	0.94	29.5968	1.06	0.93	44.8290	1.01	0.94
	400	12.8445	1.17	0.96	29.3827	1.07	0.95	44.5079	1.02	0.96
	800	12.8124	-	-	29.2812	-	-	44.3501	-	-
4	50	18.6148	-	-	50.3188	-	-	81.6633	-	-
	100	17.9570	1.13	0.89	48.5452	1.00	0.89	79.1942	0.94	0.91
	200	17.6568	1.17	0.94	47.6632	1.05	0.93	77.9096	0.99	0.94
	400	17.5230	1.17	0.97	47.2377	1.07	0.95	77.2673	1.02	0.96
	800	17.4637	-	-	47.0353	-	-	76.9443	-	-
5	50	26.3367	-	-	77.7522	-	-	131.6029	-	-
	100	25.2701	1.10	0.86	74.8205	0.96	0.86	127.6083	0.87	0.89
	200	24.7722	1.17	0.94	73.3086	1.03	0.92	125.4235	0.96	0.93
	400	24.5505	1.19	0.97	72.5701	1.07	0.95	124.3022	1.00	0.95
	800	24.4535	-	-	72.2183	-	-	123.7430	-	-

2.6 Application to Fractional Diffusion Equation

In classical theory, we use orthogonal system of functions for finding the solution of diffusion equation in bounded domain. As we proved that the eigenfunctions of the GFSLP are orthogonal in nature, therefore, the system of the eigenfuctions and eigenvalues of the GFSLP is advantageous to find the solution in bounded domain for the fractional diffusion equation. Fractional diffusion processes appear in many fields of science and engineering such as heat movement in materials, fluid pressure in porous media etc. [68]. Thus, the study of FSLPs in a bounded domain is productive for fractional diffusive processes.

In this section, we shall study the solution of one dimensional generalized fractional diffusion equation defined in terms of the B-operator as follows:

$$B^{\alpha}_{P_1,t}u(x,t) = -p(x)B^{\alpha}_{P_2,x}B^{\alpha}_{P_1,x}u(x,t), \tag{2.57}$$

$$u(x,0) = K^{\alpha}_{P_1}K^{\alpha}_{P_2}\frac{1}{p(x)}g(x), \quad g \in L^2_{1/p}, \tag{2.58}$$

$$u(0,t) = 0, \; B^{\alpha}_{P_1}u(x,t)|_{x=b} = 0. \tag{2.59}$$

Now, we represent an approximate weak solution in terms of a finite summation of the eigenfunctions and calculate the approximation error.

Theorem 2.2. *Consider the generalized fractional diffusion equation given by Eqs.(2.57)-(2.59) and* $\alpha \in (0,1)$. *This equation has a unique and real-valued weak solution in* $[0,b] \times (0,\infty)$ *which has the following form:*

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\eta_k}\langle g,\psi_k\rangle_{1/p}(-\eta_k)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\psi_k(x), \tag{2.60}$$

where the coefficient $\langle g, \psi_k \rangle_{1/p}$ *is given by*

$$\langle g, \psi_k \rangle_{1/p} = \int_a^b \frac{1}{p(x)} g(x)\psi_k(x)dx. \tag{2.61}$$

Note: Here, we obtain the weak solution of the generalized fractional diffusion equation given by Eqs.(2.57)-(2.59) for the Prabhakar fractional kernel and $p_1 = q_2 = 1, p_2 = q_1 = 0$, *in the* B*-operator. Similar approach can be adopted to find the solution for the choice of other kernels and values of parameter set in the* B*-operator.*

Proof. To prove this theorem, we follow similar steps as in [51, 69]. Let $\psi_k(x)$ be the eigenfunctions of the fractional Sturm-Liouville operator (FSLO), $L_x = p(x)B^{\alpha}_{P_2,x}B^{\alpha}_{P_1,x}$ and the corresponding eigenvalues are η_k, we expand the solution in form as follows:

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\psi_k(x). \tag{2.62}$$

Using the orthonormality of the eigenfunctions, we get

$$B^{\alpha}_{P_1} b_k(t) = -\eta_k b_k(t),\ k \in \mathbb{N}. \tag{2.63}$$

To solve the fractional differential equation given by Eq.(2.63), we have fixed the kernel as the Prabhakar fractional kernel [24, 25] defined as

$$k_\alpha(t,x) = (t-x)^{\alpha-1} E^{\gamma}_{\rho,\alpha}(\omega(t-x)^{\rho}). \tag{2.64}$$

Taking Laplace transform of both side of Eq.(2.63), we have

$$L\{B^{\alpha}_{P_1} b_k(t)\} = -L\{\eta_k b_k(t)\}. \tag{2.65}$$

As we know,

$$B_{P_1}^{\alpha} b_k(t) = D(b_k(t)) * k_{1-\alpha}(t). \tag{2.66}$$

Taking Laplace transform on both side of Eq.(2.66) and using property of convolution, we have

$$\begin{aligned} L\{B_{P_1}^{\alpha} b_k(t)\} &= L\{k_{1-\alpha}(t)\} L\{D(b_k(t))\} \\ &= s^{-(1-\alpha)}(1-\omega s^{-\rho})^{-\gamma}(sF(s)-c_k), \end{aligned} \tag{2.67}$$

where $F(s) = L\{b_k(t)\}$ and $b_k(0) = c_k$.

Now substituting Eq.(2.67) in Eq.(2.65), we have

$$s^{-(1-\alpha)}(1-\omega s^{-\rho})^{-\gamma}(sF(s)-c_k) = -\eta_k F(s). \tag{2.68}$$

Simplifying Eq.(2.68), we have

$$F(s) = \frac{c_k}{s}\left[1 + \eta_k s^{-\alpha}(1-\omega s^{-\rho})^{\gamma}\right]^{-1},$$

further, we get

$$F(s) = c_k \sum_{m=0}^{\infty} (-\eta_k)^m s^{-m\alpha-1}(1-\omega s^{-\rho})^{\gamma m}. \tag{2.69}$$

Taking inverse Laplace transform [25], we get

$$b_k(t) = \sum_{m=0}^{\infty} c_k(-\eta_k)^m t^{m\alpha} E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho}), \tag{2.70}$$

where constants c_k are derived using initial condition Eq.(2.58) and orthonormality of basis $\psi_k,\ k \in \mathbb{N}$,

$$c_k = \frac{1}{\eta_k}\langle g, \psi_k\rangle_{1/p}. \tag{2.71}$$

Thus, the solution of Eqs.(2.57)-(2.59) will be of the form

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\eta_k}\langle g, \psi_k\rangle_{1/p}(-\eta_k)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\psi_k(x). \tag{2.72}$$

We check now that obtained series is indeed a weak solution to Eq. (2.57). To prove this it is enough to show that for each eigenfunction ψ_k from the countable basis of $L^2_{1/p}$ space, we have

$$\big\langle (B^{\alpha}_{P_1,t} + L_x)u, \psi_k\big\rangle_{1/p} = 0,\ t \geq 0. \tag{2.73}$$

Using the series form of solution Eq.(2.72), we have for any $k \in \mathbb{N}$

$$\begin{aligned}\big\langle (B^{\alpha}_{P_1,t} + L_x)u, \psi_k\big\rangle_{1/p} &= B^{\alpha}_{P_1,t}\langle u, \psi_k\rangle_{1/p} + \langle L_x u, \psi_k\rangle_{1/p}\\ &= \frac{1}{\eta_k}B^{\alpha}_{P_1,t}\langle g, \psi_k\rangle_{1/p}\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho}) + \langle u, L_x\psi_k\rangle_{1/p}\\ &= -\langle g, \psi_k\rangle_{1/p}\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho}) + \langle g, \psi_k\rangle_{1/p}\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\\ &= 0.\end{aligned} \tag{2.74}$$

Now, we will use the finite linear combination of the eigenfunctions to approximate the exact solution as follows

$$u(x,t) = u_M(x,t) + R_M(x,t), \tag{2.75}$$

where $u_M(x,t)$ and $R_M(x,t)$ are the theoretical approximation and error of the approximation respectively. Here, $u_M(x,t)$ is a finite linear combination of the eigenfunctions defined as,

$$u_M(x,t) = \sum_{k=1}^{M-1}\sum_{m=0}^{\infty}\frac{1}{\eta_k}\langle g,\psi_k\rangle_{1/p}(-\eta_k)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\psi_k(x), \tag{2.76}$$

and $R_M(x,t)$ is given as

$$R_M(x,t) = \sum_{k=M}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\eta_k}\langle g,\psi_k\rangle_{1/p}(-\eta_k)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\psi_k(x). \tag{2.77}$$

$\square$

Theorem 2.3. *The bound for the approximation error $R_M(t,\cdot)$ given by Eq.(2.77) is obtained as:*

$$\|R_M\| \leq \frac{C_1}{\eta_M}, \tag{2.78}$$

where $\|R_M\| = sup\|R_M(\cdot,t)\|_{1/p}$ and C_1 is a constant.

Proof. According to the definition of norm, we get

$$\|R_M(x,t)\|_{1/p}^2 = \sum_{k=M}^{\infty}\frac{1}{\eta_k^2}|\langle g,\psi_k\rangle_{1/p}|^2\left(\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\psi_k(x)\right)^2,$$

since, the series $\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\psi_k(x)$ is infact a repeated series:

$$\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha}\sum_{r=0}^{\infty}\frac{(\omega t^{\rho})\Gamma(r-m\gamma)}{r!\Gamma(\rho r+m\alpha+1)}. \tag{2.79}$$

Since, the three paramter Mittag Leffler function is an entire function, in order to prove the absolute convergence of the series it is sufficient to show that for each

$r \in \mathbb{N} \cup \{0\}$, series $\sum_{m=0}^{\infty}(-\eta_k)^m t^{m\alpha} \sum_{r=0}^{\infty} \frac{(\omega t^\rho)\Gamma(r-m\gamma)}{r!\Gamma(\rho r+m\alpha+1)}$ is convergent (Using Ratio test of convergence of series). Thus series (2.79) is uniformly convergent, hence bounded.

This implies,

$$\|R_M(x,t)\|_{1/p}^2 \leq \sum_{k=M}^{\infty} \frac{1}{\eta_k^2} |\langle g, \psi_k \rangle_{1/p}|^2 C^2 \tag{2.80}$$

$$\leq \frac{C_1^2}{\eta_M^2}. \tag{2.81}$$

Now, applying the properties of the norm and the Schwarz-Bunyakovsky inequality, we get

$$\|R_M(x,t)\| \leq \frac{C_1}{\eta_M}. \tag{2.82}$$

□

2.6.1 Error Estimate of the Approximation

To find the exact solution of Eqs.(2.57)-(2.59), we have followed similar steps as in [69]. However, it is very challenging to find the exact solution of such complex problems. So, the numerical method formulated in this chapter is used to get the numerical eigenvalues and the corresponding eigenfunctions denoted as λ_j, $\psi_{j,N}$ respectively, for $j = 1, 2...., M-1, M \in \mathbb{N}$.

Let us assume $N > M$, and approximate the exact solution as

$$\tilde{u}_{M,N}(x,t) = \sum_{j=1}^{M-1} \frac{1}{\lambda_j} \langle g, \psi_{j,N} \rangle_{1/p} \left(\sum_{m=0}^{\infty} (-\lambda_j)^m t^{m\alpha} E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^\rho) \right) \psi_{j,N}(x). \tag{2.83}$$

Now, we will compute the $L^2_{1/p}$ error appearing through the passage from u_M to $\tilde{u}_{M,N}$

$$\begin{aligned}
\|u_M(.,t)-\tilde{u}_{M,N}(.,t)\|_{1/p} &= \Bigg\| \sum_{j=1}^{M-1} \frac{1}{\eta_j}\langle g,\psi_j\rangle_{1/p}\Bigg(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\Bigg)\psi_j(x) \\
&\quad - \sum_{j=1}^{M-1} \frac{1}{\lambda_j}\langle g,\psi_{j,N}\rangle_{1/p}\Bigg(\sum_{m=0}^{\infty}(-\lambda_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\Bigg)\psi_{j,N}(x)\Bigg\|_{1/p} \\
&\le \Bigg\| \sum_{j=1}^{M-1} \left(\frac{1}{\eta_j}-\frac{1}{\lambda_j}\right)\langle g,\psi_{j,N}\rangle_{1/p}\Bigg(\sum_{m=0}^{\infty}(-\lambda_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\Bigg)\psi_{j,N}(x)\Bigg\|_{1/p} \\
&\quad + \Bigg\| \sum_{j=1}^{M-1} \frac{1}{\eta_j}\langle g,\psi_{j,N}\rangle_{1/p}\Bigg(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho}) \\
&\quad - \sum_{m=0}^{\infty}(-\lambda_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\Bigg)\psi_{j,N}(x)\Bigg\|_{1/p} \\
&\quad + \Bigg\| \sum_{j=1}^{M-1} \frac{1}{\eta_j}\langle g,\psi_j-\psi_{j,N}\rangle_{1/p}\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\psi_j(x)\Bigg\|_{1/p} \\
&\quad + \Bigg\| \sum_{j=1}^{M-1} \frac{1}{\eta_j}\langle g,\psi_{j,N}\rangle_{1/p}\Bigg(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\Bigg)(\psi_j(x)-\psi_{j,N}(x))\Bigg\|_{1/p},
\end{aligned} \tag{2.84}$$

and now define $\Delta\psi_{j,N}$ and $\Delta w_{j,N}$

$$\Delta\psi_{j,N} = |\psi_j(x)-\psi_{j,N}(x)|,\ \Delta w_{j,N} = |\eta_j-\lambda_j|. \tag{2.85}$$

Now, we approximate all terms of right hand side of Eq.(4.63) separately. Since,

$$\begin{aligned}
&\Bigg\| \sum_{j=1}^{M-1} \left(\frac{1}{\eta_j}-\frac{1}{\lambda_j}\right)\langle g,\psi_{j,N}\rangle_{1/p}\sum_{m=0}^{\infty}(-\lambda_j)^m t^{m\alpha}E^{-m\gamma}_{\rho,m\alpha+1}(\omega t^{\rho})\psi_{j,N}(x)\Bigg\|_{1/p} \\
&\qquad \le C\frac{\max_{j<M}(\Delta w_{j,N})}{\eta_1\lambda_1}\Bigg\|\sum_{j=1}^{M-1}\langle g,\psi_{j,N}\rangle_{1/p}\Bigg\|_{1/p} \\
&\qquad \le C\frac{\max_{j<M}(\Delta w_{j,N})}{\eta_1\lambda_1}\|g\|_{1/p}.
\end{aligned} \tag{2.86}$$

Similarly,

$$\left\|\sum_{j=1}^{M-1}\frac{1}{\eta_j}\langle g,\psi_{j,N}\rangle_{1/p}\left(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\right)-\sum_{m=0}^{\infty}(-\lambda_j)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\right\|_{1/p}$$
$$\leq C_1\frac{\max_{j<M}(\Delta w_{j,N})}{\eta_1}\|g\|_{1/p}, \tag{2.87}$$

$$\left\|\sum_{j=1}^{M-1}\frac{1}{\eta_j}\langle g,\psi_j-\psi_{j,N}\rangle_{1/p}\left(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})\right)\psi_j(x)\right\|_{1/p}$$
$$\leq C\max_{j<M}\|\Delta\psi_{j,N}\|_{1/p}\|g\|_{1/p}\left(\sum_{j=1}^{\infty}\frac{1}{\eta_j^2}\right)^{\frac{1}{2}}, \tag{2.88}$$

$$\left\|\sum_{j=1}^{M-1}\frac{1}{\eta_j}\langle g,\psi_{j,N}\rangle_{1/p}(\sum_{m=0}^{\infty}(-\eta_j)^m t^{m\alpha}E_{\rho,m\alpha+1}^{-m\gamma}(\omega t^{\rho})(\psi_j(x)-\psi_{j,N}(x))\right\|_{1/p}$$
$$\leq \frac{C\max_{j<M}\|\Delta\psi_{j,N}\|_{1/p}}{\eta_1}\|g\|_{1/p}, \tag{2.89}$$

which implies that

$$\begin{aligned}\|u_M(\cdot,t)-\tilde{u}_{M,N}(\cdot,t)\|_{1/p} &\leq \left(\frac{C}{\eta_1\lambda_1}+\frac{C_1}{\eta_1}\right)\max_{j<M}(\Delta w_{j,N})\|g\|_{1/p}\\ &\quad+\left(C\left(\sum_{j=1}^{\infty}\frac{1}{\eta_j^2}\right)^{\frac{1}{2}}+\frac{C}{\eta_1}\right)\max_{j<M}\|\Delta\psi_{j,N}\|_{1/p}\eta_1\|g\|_{1/p}.\end{aligned} \tag{2.90}$$

Further we get,

$$\begin{aligned}\|u(\cdot,t)-\tilde{u}_{M,N}(\cdot,t)\|_{1/p} &\leq \|u(\cdot,t)-u_M(\cdot,t)\|_{1/p}+\|u_M(\cdot,t)-u_{M,N}(\cdot,t)\|_{1/p}\\ &\leq \frac{C}{\eta_1}+\left(\frac{C}{\eta_1\lambda_1}+\frac{C_1}{\eta_1}\right)\max_{j<M}(\Delta w_{j,N})\|g\|_{1/p}\\ &\quad+\left(C\left(\sum_{j=1}^{\infty}\frac{1}{\eta_j^2}\right)^{\frac{1}{2}}+\frac{C}{\eta_1}\right)\max_{j<M}\|\Delta\psi_{j,N}\|_{1/p}\eta_1\|g\|_{1/p}.\end{aligned} \tag{2.91}$$

Thus, it is noted that estimation of the error depends on both M and N. We fix the value of M to discover a satisfactory level of theoretical approximation error then pick the value of N to achieve an error small enough from the theoretical scheme.

2.7 Conclusions

Here, we have studied a numerical approximation of the GFSLP with the mixed boundary conditions. Firstly, we have proved the well-posedness of the considered GFSLP. Then, a numerical method to solve the proposed GFSLP is described. For this, we have used the finite difference method to discretize the B-operator involved in GFSLP. Further, the system of algebraic equations is generated and the set of approximated eigenvalues and corresponding eigenfunctions are calculated. The obtained numerical results are shown through Tables 2.1-2.4 for the test cases and the convergence order of the proposed scheme is shown through Table 2.5 and Table 2.6 which is close to one. The proposed scheme can also be extended in finding the solution of other FSLPs by varying the kernel in the B-operator. To establish an application, we have obtained the solution of the fractional diffusion equation given by Eqs.(2.57)-(2.59) defined in terms of the B-operator with the Prabhakar kernel.

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