

Chapter 3

Multiple μ -stability analysis of time-varying delayed quaternion-valued neural networks

3.1 Introduction

This chapter investigated the multiple μ -stability analysis of n -dimensional quaternion-valued neural networks (QVNNs) with unbounded time-varying delays (UTVD) and two general classes of activation functions (AFs). The time delay is another important aspect that needs to be considered for the applications of neural networks (NNs). In fact, when it comes to the memory functions of neural network models of the human brain, the current state is often associated with the previous ones. Due to the finite switching speed and interneuron conduction distance of the amplifiers, time delays are invariably present in some real applications of artificial NNs. It should be noted that most researchers have focused on research by assuming bounded time

delays i.e., $\tau(t) \leq \tau$, where τ is a fix constant. However, answering the following questions is an interesting problem in both theory and application: What will happen if the delays are unbounded? Can we give a more detailed analysis of the dynamical behavior for the systems with unbounded delays? If so, how large can the delay be while the delayed systems still have stable EPs? In chapter paper, we have given confirmatory answers to all the above questions by introducing a modern concept of μ -stability, which is a generalization of power-stability discussed in [45]. We have pointed out that as the delay increases, the stability of the EPs will change from exponential-stable to power-stable, log-stable, and log-log-stable. Keeping all the existing EPs in mind, in this paper we have discussed multiple μ -stability concept with UTVD two general classes of AFs.

3.2 Model Description and Preliminaries

In this chapter, let us consider the following QVNNs with time-varying delays as

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^n h_{ij} g_j(x_j(t)) + \sum_{j=1}^n w_{ij} g_j(x_j(t - \tau_{ij}(t))) + k_i, \quad i = 1, 2, \dots, n, \quad t \geq 0, \quad (3.1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{H}^n$, $x_i(t)$ represents the state variable of i -th neuron at a time t ; $d_i \in \mathbb{R}$ with $d_i \geq 0$ is the self-feedback connection strength; $h_{ij} \in \mathbb{H}$ and $w_{ij} \in \mathbb{H}$ are, respectively, the instantaneous and delays synaptic connection strengths from j -th to i -th neurons of the network; $k_i \in \mathbb{H}$ is an external input to each neuron; $\tau_{ij}(t) \leq \tau(t)$, represents the varying delay with $\tau(t) \geq 0$; $g_j(\cdot) : \mathbb{H} \rightarrow \mathbb{H}$ denotes quaternion-valued activation function.

Let us use the following assumption throughout this chapter to explore the dynamics of the system (3.1).

Assumption 3.1. Let $x_j = x_j^{(0)} + x_j^{(1)}i + x_j^{(2)}j + x_j^{(3)}k \in \mathbb{H}$, be arbitrary. For all $j = 1, 2, \dots, n$, $g_j(x_j)$ can be expressed by its real and vector parts with

$$g_j(x_j) = g_j^{(0)}(x_j^{(0)}) + g_j^{(1)}(x_j^{(1)})i + g_j^{(2)}(x_j^{(2)})j + g_j^{(3)}(x_j^{(3)})k,$$

where $g_j^{(b)}(\cdot)$ with $b = 0, 1, 2, 3$, are real valued continuous functions on \mathbb{R} , which belong to the following two classes of functions.

Class \mathcal{A} : For $j = 1, 2, \dots, n$; $b = 0, 1, 2, 3$, $g_j^{(b)}(\cdot)$ are twice continuously differentiable functions, which agree with the following two properties:

(a) The following limits exist,

$$\lim_{\xi \rightarrow -\infty} g_i^{(b)}(\xi) = u_i^{(b)}, \quad \lim_{\xi \rightarrow +\infty} g_i^{(b)}(\xi) = v_i^{(b)}.$$

(b) There exist $\sigma_j^{(b)} \in \mathbb{R}$, such that

$$g_j^{(b)'}(\sigma_j^{(b)}) > g_j^{(b)'}(\xi) > 0, \quad (\xi - \sigma_j^{(b)})g_j^{(b)''}(\xi) < 0, \quad \text{for } \xi \in \mathbb{R} \setminus \sigma_j^{(b)},$$

where the superscripts ' and '' , respectively, indicate the first and second derivative.

Class \mathcal{B} : For $j = 1, 2, \dots, n$ and $b = 0, 1, 2, 3$, there exist $u_j^{(b)}, v_i^{(b)}, p_i^{(b)}, q_i^{(b)} \in \mathbb{R}$,

such that

$$g_j^{(b)}(\xi) = \begin{cases} u_j^{(b)}, & \text{if } z \in (-\infty, p_j^{(b)}), \\ \bar{g}_j^{(b)}(\xi), & \text{if } \xi \in [p_j^{(b)}, q_j^{(b)}], \\ v_j^{(b)}, & \text{if } \xi \in (q_j^{(b)}, +\infty), \end{cases}$$

where $\bar{g}_j^{(b)}(\cdot)$ are continuously differentiable and monotonically non-decreasing on $[p_j^{(b)}, q_j^{(b)}]$.

Remark 3.1. From Assumption 3.1, it is seen that Class \mathcal{A} consists of bounded sigmoid functions, for example $\tanh \xi$, $\arctan \xi$ and $\frac{1}{1+e^{-\xi}}$. Whereas Class \mathcal{B} consists of non-decreasing functions with saturation, for example, the piece-wise linear functions with two corner points at $p_i^{(b)}, q_i^{(b)}$ as

$$g_i^{(b)}(\xi) = u_i^{(b)} + \frac{v_i^{(b)} - u_i^{(b)}}{q_i^{(b)} - p_i^{(b)}}(\xi - p_i^{(b)}),$$

whose standard form is

$$g_i^{(b)}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|).$$

Remark 3.2. In [7], the multistability analysis of QVNNs with bounded time-varying delay was investigated, in which the AFs are chosen as piecewise linear belonging to Class \mathcal{B} . In this chapter, the multistability analysis of NNs is discussed with both classes of AFs with UTVD. Moreover, by applying some restrictions on delays, herein, the concept of μ -stability is discussed, which is a generalization of power stability, exponential stability, and log stability.

Based on the multiplication between two quaternions (see equation (1.13)) and by

the Assumption 3.1, the system (3.1) can be decomposed into following four real-valued systems as

$$\begin{aligned} \dot{x}_i^{(0)} = & -d_i x_i^{(0)} + \sum_{j=1}^n \left[h_{ij}^{(0)} g_j^{(0)}(x_j^{(0)}(t)) - h_{ij}^{(1)} g_j^{(1)}(x_j^{(1)}(t)) - \right. \\ & \left. h_{ij}^{(2)} g_j^{(2)}(x_j^{(2)}(t)) - h_{ij}^{(3)} g_j^{(3)}(x_j^{(3)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} g_j^{(0)}(x_j^{(0)}(t - \tau_{ij}(t))) - w_{ij}^{(1)} g_j^{(1)}(x_j^{(1)}(t - \tau_{ij}(t))) \right. \\ & \left. - w_{ij}^{(2)} g_j^{(2)}(x_j^{(2)}(t - \tau_{ij}(t))) - w_{ij}^{(3)} g_j^{(3)}(x_j^{(3)}(t - \tau_{ij}(t))) \right] + k_i^{(0)}, \end{aligned}$$

$$\begin{aligned} \dot{x}_i^{(1)} = & -d_i x_i^{(1)} + \sum_{j=1}^n \left[h_{ij}^{(0)} g_j^{(1)}(x_j^{(1)}(t)) + h_{ij}^{(1)} g_j^{(0)}(x_j^{(0)}(t)) + \right. \\ & \left. h_{ij}^{(2)} g_j^{(3)}(x_j^{(3)}(t)) - h_{ij}^{(3)} g_j^{(2)}(x_j^{(2)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} g_j^{(1)}(x_j^{(1)}(t - \tau_{ij}(t))) + w_{ij}^{(1)} g_j^{(0)}(x_j^{(0)}(t - \tau_{ij}(t))) \right. \\ & \left. + w_{ij}^{(2)} g_j^{(3)}(x_j^{(3)}(t - \tau_{ij}(t))) - w_{ij}^{(3)} g_j^{(2)}(x_j^{(2)}(t - \tau_{ij}(t))) \right] + k_i^{(1)}, \end{aligned}$$

$$\begin{aligned} \dot{x}_i^{(2)} = & -d_i x_i^{(2)} + \sum_{j=1}^n \left[h_{ij}^{(0)} g_j^{(2)}(x_j^{(2)}(t)) + h_{ij}^{(2)} g_j^{(0)}(x_j^{(0)}(t)) - \right. \\ & \left. h_{ij}^{(1)} g_j^{(3)}(x_j^{(3)}(t)) + h_{ij}^{(3)} g_j^{(1)}(x_j^{(1)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} g_j^{(2)}(x_j^{(2)}(t - \tau_{ij}(t))) + w_{ij}^{(2)} g_j^{(0)}(x_j^{(0)}(t - \tau_{ij}(t))) \right. \\ & \left. - w_{ij}^{(1)} g_j^{(3)}(x_j^{(3)}(t - \tau_{ij}(t))) + w_{ij}^{(3)} g_j^{(1)}(x_j^{(1)}(t - \tau_{ij}(t))) \right] + k_i^{(2)}, \end{aligned}$$

$$\begin{aligned} \dot{x}_i^{(3)} = & -d_i x_i^{(3)} + \sum_{j=1}^n \left[h_{ij}^{(0)} g_j^{(3)}(x_j^{(3)}(t)) + h_{ij}^{(3)} g_j^{(0)}(x_j^{(0)}(t)) + \right. \\ & \left. h_{ij}^{(1)} g_j^{(2)}(x_j^{(2)}(t)) - h_{ij}^{(2)} g_j^{(1)}(x_j^{(1)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} g_j^{(3)}(x_j^{(3)}(t - \tau_{ij}(t))) + w_{ij}^{(3)} g_j^{(0)}(x_j^{(0)}(t - \tau_{ij}(t))) \right. \\ & \left. + w_{ij}^{(1)} g_j^{(2)}(x_j^{(2)}(t - \tau_{ij}(t))) - w_{ij}^{(2)} g_j^{(1)}(x_j^{(1)}(t - \tau_{ij}(t))) \right] + k_1^{(3)}. \end{aligned}$$

The equivalent initial conditions of system (3.1) are given by

$$x_i^{(b)}(\theta) = \phi_i^{(b)}(\theta), \quad \theta \in (-\infty, 0],$$

where $\phi_i^{(b)}(\theta) \in \mathbf{C}((-\infty, 0], \mathbb{R})$, $b = 0, 1, 2, 3$, and $i = 1, 2, \dots, n$.

For $i = 1, 2, \dots, n$ and $g_i^{(b)}(\xi)$ in Class \mathcal{A} or Class \mathcal{B} , the following real-valued continuous functions can be defined as

$$\tilde{G}_i^{(b)}(\xi) = -d_i \xi + (h_{ii}^{(0)} + w_{ii}^{(0)}) g_i^{(b)}(\xi), \quad (3.2)$$

$$\hat{G}_i^{(b)}(\xi) = -d_i \xi + (h_{ii}^{(0)} + w_{ii}^{(0)}) g_i^{(b)}(\xi) + \hat{\eta}_i^{(b)}, \quad (3.3)$$

$$\check{G}_i^{(b)}(\xi) = -d_i \xi + (h_{ii}^{(0)} + w_{ii}^{(0)}) g_i^{(b)}(\xi) + \check{\eta}_i^{(b)}, \quad (3.4)$$

where $\xi \in \mathbb{R}$, $b = 0, 1, 2, 3$, and $\hat{\eta}_i^{(b)}$ and $\check{\eta}_i^{(b)}$ are real constants which are defined as

$$\begin{aligned} \hat{\eta}_i^{(0)} = & \sum_{j=1, j \neq i}^n \max \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)}) u_j^{(0)}, (h_{ij}^{(0)} + w_{ij}^{(0)}) v_j^{(0)} \right\} - \\ & \sum_{j=1}^n \left[\min \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)}) u_j^{(1)}, (h_{ij}^{(1)} + w_{ij}^{(1)}) v_j^{(1)} \right\} + \min \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)}) u_j^{(2)}, (h_{ij}^{(2)} + w_{ij}^{(2)}) v_j^{(2)} \right\} \right. \\ & \left. + \min \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)}) u_j^{(3)}, (h_{ij}^{(3)} + w_{ij}^{(3)}) v_j^{(3)} \right\} \right] + k_i^{(0)}, \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_i^{(0)} = & \sum_{j=1, j \neq i}^n \min \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(0)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(0)} \right\} - \\ & \sum_{j=1}^n \left[\max \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(1)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(1)} \right\} + \max \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(2)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(2)} \right\} \right. \\ & \left. + \max \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(3)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(3)} \right\} \right] + k_i^{(0)}, \end{aligned}$$

$$\begin{aligned} \hat{\eta}_i^{(1)} = & \sum_{j=1, j \neq i}^n \max \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(1)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(0)} \right\} + \\ & \sum_{j=1}^n \left[\max \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(0)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(0)} \right\} + \max \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(3)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(3)} \right\} \right. \\ & \left. - \min \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(2)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(2)} \right\} \right] + k_i^{(1)}, \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_i^{(1)} = & \sum_{j=1, j \neq i}^n \min \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(1)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(0)} \right\} + \\ & \sum_{j=1}^n \left[\min \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(0)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(0)} \right\} + \min \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(3)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(3)} \right\} \right. \\ & \left. - \max \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(2)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(2)} \right\} \right] + k_i^{(1)}, \end{aligned}$$

$$\begin{aligned} \hat{\eta}_i^{(2)} = & \sum_{j=1, j \neq i}^n \max \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(2)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(2)} \right\} + \\ & \sum_{j=1}^n \left[\max \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(0)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(0)} \right\} - \min \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(3)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(3)} \right\} \right. \\ & \left. + \max \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(1)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(1)} \right\} \right] + k_i^{(2)}, \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_i^{(2)} = & \sum_{j=1, j \neq i}^n \min \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(2)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(2)} \right\} + \\ & \sum_{j=1}^n \left[\min \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(0)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(0)} \right\} - \max \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(3)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(3)} \right\} \right. \\ & \left. + \min \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(1)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(1)} \right\} \right] + k_i^{(2)}, \end{aligned}$$

$$\begin{aligned} \hat{\eta}_i^{(3)} = & \sum_{j=1, j \neq i}^n \max \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(3)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(3)} \right\} + \\ & \sum_{j=1}^n \left[\max \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(0)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(0)} \right\} + \max \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(2)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(2)} \right\} \right. \\ & \left. - \min \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(1)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(1)} \right\} \right] + k_i^{(3)}, \end{aligned}$$

$$\begin{aligned} \check{\eta}_i^{(3)} = & \sum_{j=1, j \neq i}^n \min \left\{ (h_{ij}^{(0)} + w_{ij}^{(0)})u_j^{(3)}, (h_{ij}^{(0)} + w_{ij}^{(0)})v_j^{(3)} \right\} \\ & + \sum_{j=1}^n \left[\min \left\{ (h_{ij}^{(3)} + w_{ij}^{(3)})u_j^{(0)}, (h_{ij}^{(3)} + w_{ij}^{(3)})v_j^{(0)} \right\} + \min \left\{ (h_{ij}^{(1)} + w_{ij}^{(1)})u_j^{(2)}, (h_{ij}^{(1)} + w_{ij}^{(1)})v_j^{(2)} \right\} \right. \\ & \left. - \max \left\{ (h_{ij}^{(2)} + w_{ij}^{(2)})u_j^{(1)}, (h_{ij}^{(2)} + w_{ij}^{(2)})v_j^{(1)} \right\} \right] + k_i^{(3)}. \end{aligned}$$

The configurations for $\hat{G}_i^{(b)}(\xi)$ and $\check{G}_i^{(b)}(\xi)$ are depicted in Figure 3.1 and Figure 3.2.

Remark 3.3. From the definitions of $\hat{\eta}_i^{(b)}$ and $\check{\eta}_i^{(b)}$, it can be concluded that $\hat{G}_i^{(b)}(\xi) \geq \check{G}_i^{(b)}(\xi)$ for all $b = 0, 1, 2, 3$ and $i = 1, 2, \dots, n$. Moreover, we can say that $\hat{G}_i^{(b)}(\xi)$ and $\check{G}_i^{(b)}(\xi)$ are vertical shifts of $\tilde{G}_i^{(b)}(\xi)$. Since $c_i > 0$ and by virtue of Assumption 3.1, we can see that

$$\lim_{\xi \rightarrow -\infty} \check{G}_i^{(b)}(\xi) = +\infty, \quad \lim_{\xi \rightarrow +\infty} \hat{G}_i^{(b)}(\xi) = -\infty.$$

Thus, there exist $\check{p}_i^{(b)} < 0$ and $\hat{q}_i^{(b)} > 0$, in such a way that $\check{p}_i^{(b)} < p_i^{(b)}$, $q_i^{(b)} < \hat{q}_i^{(b)}$ and

$$\hat{G}_i^{(b)}(\check{p}_i^{(b)}) \geq \check{G}_i^{(b)}(\check{p}_i^{(b)}) > 0, \quad \check{G}_i^{(b)}(\hat{q}_i^{(b)}) \leq \hat{G}_i^{(b)}(\hat{q}_i^{(b)}) < 0.$$

3.3 Main Results

In this section, based on the geometrical behavior of AFs and by using Brouwer's fixed point theory, a few sufficient conditions are obtained to ensure the coexistence

of 81^n equilibria, out of which 16^n of those are locally μ -stable.

3.3.1 Coexistence of multiple equilibria of the system (3.1)

At first, let us propose some sufficient criteria to ensure the coexistence of multiple equilibrium points of the system (3.1). Here the first parametric condition for system (3.1) is proposed with AFs belonging to Class \mathcal{A} and Class \mathcal{B} , respectively as

$$(\mathbf{C}_1^{\mathcal{A}}) \quad \max_{\xi \in \mathbb{R}} g_i^{(b)'}(\xi) > \frac{d_i}{h_{ii}^{(0)} + w_{ii}^{(0)}} > 0, \quad i = 1, 2, \dots, n,$$

where $b = 0, 1, 2, 3$.

$$(\mathbf{C}_1^{\mathcal{B}}) \quad \max_{\xi \in \mathbb{R}} \bar{g}_i^{(b)'}(\xi) > \frac{d_i}{h_{ii}^{(0)} + w_{ii}^{(0)}} > 0, \quad i = 1, 2, \dots, n,$$

where $b = 0, 1, 2, 3$.

Lemma 3.1. For $i = 1, 2, \dots, n$, let us assume that $g_i^{(b)}(\cdot)$ be in Class \mathcal{A} (resp. Class \mathcal{B}) with $b = 0, 1, 2, 3$. If the parametric condition $(\mathbf{C}_1^{\mathcal{A}})$ (resp. $(\mathbf{C}_1^{\mathcal{B}})$) holds, then there exist real constants $\tilde{p}_i^{(b)}$ and $\tilde{q}_i^{(b)}$ such that

1. $\tilde{G}_i^{(b)}(\xi)$ is strictly decreasing on $(-\infty, \tilde{p}_i^{(b)}]$.
2. $\tilde{G}_i^{(b)}(\xi)$ is strictly increasing on $[\tilde{p}_i^{(b)}, \tilde{q}_i^{(b)}]$.
3. $\tilde{G}_i^{(b)}(\xi)$ is strictly decreasing on $[\tilde{q}_i^{(b)}, +\infty)$.

Proof. Here, the proof for the activation function $g_i^{(b)}$ in Class \mathcal{A} is derived only. The proof for the Class \mathcal{B} can be omitted because in a similar way we can prove it. Differentiating (3.2), we have $\tilde{G}_i^{(b)'}(\xi) = -d_i + (h_{ii}^{(0)} + w_{ii}^{(0)})g_i^{(b)'}(\xi)$. Since $\max_{\xi \in \mathbb{R}} g_i^{(b)'}(\xi) = g_i^{(b)'}(\sigma_i^{(b)})$, and from the condition $(\mathbf{C}_1^{\mathcal{A}})$, we get $\tilde{G}_i^{(b)'}(\sigma_i^{(b)}) =$

Now, for both classes of AFs, here the second parametric conditions are given which are used to derive the coexistence of multiple equilibrium points for the system (3.1):

(C₂) For each $i = 1, 2, \dots, n$,

$$\hat{G}_i^{(b)}(\hat{p}_i^{(b)}) < 0, \check{G}_i^{(b)}(\check{q}_i^{(b)}) > 0, b = 0, 1, 2, 3.$$

Lemma 3.2. For $i = 1, 2, \dots, n$, let $g_i^{(b)}$ in Class \mathcal{A} (resp. Class \mathcal{B}) with $b = 0, 1, 2, 3$. If both parametric conditions (C₁^A) (resp. (C₁^B)) and (C₂) hold, then there exist real constants $\hat{a}_i^{(b)}, \check{a}_i^{(b)}, \hat{b}_i^{(b)}, \check{b}_i^{(b)}, \hat{c}_i^{(b)}$ and $\check{c}_i^{(b)}$, with $\check{a}_i^{(b)} < \check{b}_i^{(b)} < \check{c}_i^{(b)}$ and $\hat{a}_i^{(b)} < \hat{b}_i^{(b)} < \hat{c}_i^{(b)}$ such that

$$\hat{G}_i^{(b)}(\hat{a}_i^{(b)}) = \check{G}_i^{(b)}(\check{a}_i^{(b)}) = \hat{G}_i^{(b)}(\hat{b}_i^{(b)}) = \check{G}_i^{(b)}(\check{b}_i^{(b)}) = \hat{G}_i^{(b)}(\hat{c}_i^{(b)}) = \check{G}_i^{(b)}(\check{c}_i^{(b)}) = 0.$$

Proof. Here, the proof for Class \mathcal{A} is only discussed. In a similar manner, the proof for Class \mathcal{B} can be derived. Since $\tilde{G}_i^{(b)}, \hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ have same monotonic intervals, therefore $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ are strictly decreasing on $(-\infty, \tilde{p}_i^{(b)}]$. Clearly, we have the following inequalities, $\check{G}_i^{(b)}(\tilde{p}_i^{(b)}) \leq \hat{G}_i^{(b)}(\tilde{p}_i^{(b)}) < 0$ and $\hat{G}_i^{(b)}(\tilde{p}_i^{(b)}) \geq \check{G}_i^{(b)}(\tilde{p}_i^{(b)}) > 0$. Then by using the intermediate value theorem on continuous functions, there exist $\check{a}_i^{(b)}$ and $\hat{a}_i^{(b)}$ such that $\hat{G}_i^{(b)}(\hat{a}_i^{(b)}) = 0$ and $\check{G}_i^{(b)}(\check{a}_i^{(b)}) = 0$. In a similar manner, the remaining constants can be obtained. This completes the proof. \square

Next, let us denote the intervals $I_i^{(b)l} = [\check{a}_i^{(b)}, \hat{a}_i^{(b)}]$, $I_i^{(b)m} = [\hat{b}_i^{(b)}, \check{b}_i^{(b)}]$, $I_i^{(b)r} = [\check{c}_i^{(b)}, \hat{c}_i^{(b)}]$, where $b = 0, 1, 2, 3$. Herein, the superscripts “ l ”, “ m ”, and “ r ”, respectively represent “left”, “middle”, and “right”. Then for each index $\alpha = (\alpha_1, \alpha_2, \dots$

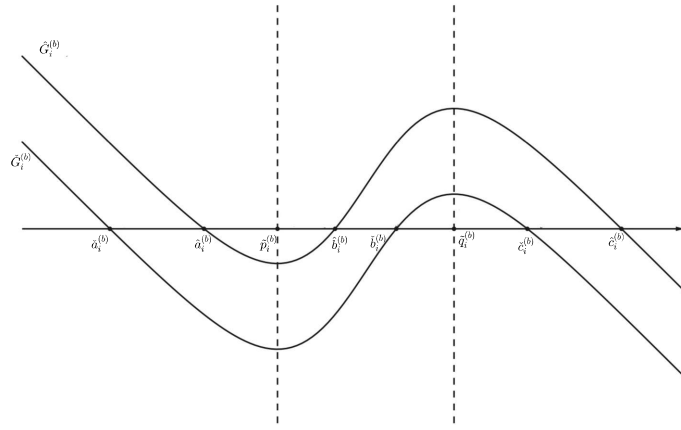


FIGURE 3.1: Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ for the function in Class \mathcal{A}

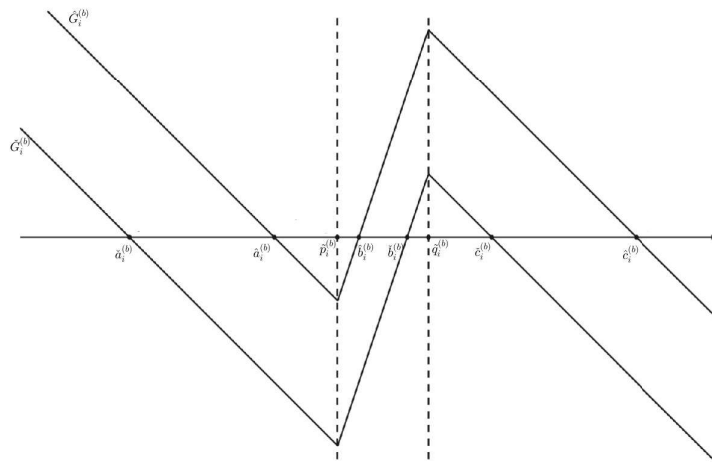


FIGURE 3.2: Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ for the function in Class \mathcal{B}

., α_{4n}), where α_i is “ l ”, “ m ” or “ r ”, we denote

$$\Omega^\alpha = \left\{ (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n \mid x_i^{(0)} \in I_i^{(0)\alpha_i}, x_i^{(1)} \in I_i^{(1)\alpha_{n+i}}, x_i^{(2)} \in I_i^{(2)\alpha_{2n+i}}, \right. \\ \left. x_i^{(3)} \in I_i^{(3)\alpha_{3n+i}}, \text{ for } i = 1, 2, \dots, n \right\},$$

and $\Omega = \left\{ \Omega^\alpha \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n}) \text{ with } \alpha_i \text{ is “}l\text{”, “}m\text{”, or “}r\text{”} \right\}$. It can be seen that, there are 3^{4n} elements of the type Ω^α in Ω , which are disjoint regions in \mathbb{H}^n . Now, let us introduce some more real-valued continuous functions to make the proof of the following lemmas in an easier way. For any given $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n$, let us define the functions $G_i^{(b)} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$G_i^{(b)}(\xi) = -d_i \xi + (h_{ii}^{(0)} + w_{ii}^{(0)})g_i^{(b)}(\xi) + \eta_i^{(b)}, \quad (3.5)$$

where $b = 0, 1, 2, 3$, $i = 1, 2, \dots, n$, and

$$\eta_i^{(0)} = \sum_{j=1, j \neq i}^n (h_{ii}^{(0)} + w_{ii}^{(0)})g_j^{(0)}(x_j^{(0)}) - \sum_{j=1}^n \left[(h_{ij}^{(1)} + w_{ij}^{(1)})g_j^{(1)}(x_j^{(1)}) + (h_{ij}^{(2)} + w_{ij}^{(2)})g_j^{(2)}(x_j^{(2)}) \right. \\ \left. + (h_{ij}^{(3)} + w_{ij}^{(3)})g_j^{(3)}(x_j^{(3)}) \right] + k_i^{(0)},$$

$$\eta_i^{(1)} = \sum_{j=1, j \neq i}^n (h_{ii}^{(0)} + w_{ii}^{(0)})g_j^{(1)}(x_j^{(1)}) + \sum_{j=1}^n \left[(h_{ij}^{(1)} + w_{ij}^{(1)})g_j^{(0)}(x_j^{(0)}) + (h_{ij}^{(2)} + w_{ij}^{(2)})g_j^{(3)}(x_j^{(3)}) \right. \\ \left. - (h_{ij}^{(3)} + w_{ij}^{(3)})g_j^{(2)}(x_j^{(2)}) \right] + k_i^{(1)},$$

$$\eta_i^{(2)} = \sum_{j=1, j \neq i}^n (h_{ii}^{(0)} + w_{ii}^{(0)})g_j^{(2)}(x_j^{(2)}) + \sum_{j=1}^n \left[(h_{ij}^{(2)} + w_{ij}^{(2)})g_j^{(0)}(x_j^{(0)}) - (h_{ij}^{(1)} + w_{ij}^{(1)})g_j^{(3)}(x_j^{(3)}) \right. \\ \left. + (h_{ij}^{(3)} + w_{ij}^{(3)})g_j^{(1)}(x_j^{(1)}) \right] + k_i^{(2)},$$

$$\eta_i^{(3)} = \sum_{j=1, j \neq i}^n (h_{ii}^{(0)} + w_{ii}^{(0)})g_j^{(3)}(x_j^{(3)}) + \sum_{j=1}^n \left[(h_{ij}^{(3)} + w_{ij}^{(3)})g_j^{(0)}(x_j^{(0)}) + (h_{ij}^{(1)} + w_{ij}^{(1)})g_j^{(2)}(x_j^{(2)}) - (h_{ij}^{(2)} + w_{ij}^{(2)})g_j^{(1)}(x_j^{(1)}) \right] + k_i^{(3)}.$$

From the definitions of $\hat{G}_i^{(b)}(\xi)$, $\check{G}_i^{(b)}(\xi)$ and $G_i^{(b)}(\xi)$, we can say that $\check{G}_i^{(b)}(\xi)$ and $\hat{G}_i^{(b)}(\xi)$ are also vertical shifts of $G_i^{(b)}(\xi)$. Moreover, we also have the following inequality

$$\check{G}_i^{(b)}(\xi) \leq G_i^{(b)}(\xi) \leq \hat{G}_i^{(b)}(\xi), \quad \forall \xi \in \mathbb{R}. \quad (3.6)$$

Lemma 3.3. Under the Assumption 3.1, if the parametric conditions (\mathbf{C}_1^A) (resp. (\mathbf{C}_1^B)) and (\mathbf{C}_2) hold, then there exist 3^{4n} equilibrium points for the network (3.1) with AFs of Class \mathcal{A} (resp. Class \mathcal{B}).

Proof. For any arbitrary $x = (x_1, x_2, \dots, x_n) \in \mathbb{H}^n$, we can define the functions $G_i^{(b)}(\cdot)$ on \mathbb{R} in the form of (3.5). Since $\check{G}_i^{(b)}(\xi)$ and $\hat{G}_i^{(b)}(\xi)$ are vertical shift of $G_i^{(b)}(\xi)$, then by using Remark 3.5 and Lemma 3.1, it is be easy to say that $G_i^{(b)}(\xi)$ is strictly decreasing on $(-\infty, \tilde{p}_i^{(b)}]$ and $[\hat{q}_i^{(b)}, +\infty)$, and strictly increasing on $[\tilde{p}_i^{(b)}, \tilde{q}_i^{(b)}]$. From the inequalities $\check{G}_i^{(b)}(\tilde{p}_i^{(b)}) > 0$ and given in (3.6), we have $G_i^{(b)}(\tilde{p}_i^{(b)}) > 0$. Again from the inequalities $\hat{G}_i^{(b)}(\tilde{p}_i^{(b)}) < 0$ and also in (3.6), we get $G_i^{(b)}(\tilde{p}_i^{(b)}) < 0$. Similarly by using the inequalities $\check{G}_i^{(b)}(\hat{q}_i^{(b)}) > 0$, $\hat{G}_i^{(b)}(\hat{q}_i^{(b)}) < 0$ and in (3.6), we obtain $G_i^{(b)}(\hat{q}_i^{(b)}) > 0$ and $G_i^{(b)}(\hat{q}_i^{(b)}) < 0$. So there exist exactly three points $\underline{x}_i^{(b)l} \in I_i^{(b)l}$, $\underline{x}_i^{(b)m} \in I_i^{(b)m}$ and $\underline{x}_i^{(b)r} \in I_i^{(b)r}$ such that $G_i^{(b)}(\underline{x}_i^{(b)l}) = G_i^{(b)}(\underline{x}_i^{(b)m}) = G_i^{(b)}(\underline{x}_i^{(b)r}) = 0$.

Now, for a given Ω^α in Ω , suppose the index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n})$ where, α_i is “ l ”, “ m ”, or “ r ”. By taking $x = (x_1, x_2, \dots, x_n) \in \Omega^\alpha$ arbitrarily and putting it into (3.5), we get the corresponding functions $G_i^{(b)}(\xi)$, and exactly one point $\underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)^T \in \Omega^\alpha$, where $\underline{x}_i = \underline{x}_i^{(0)\alpha_i} + \underline{x}_i^{(1)\alpha_{n+i}} + \underline{x}_i^{(2)\alpha_{2n+i}} + \underline{x}_i^{(3)\alpha_{3n+i}}$, such that $G_i^{(b)}(\underline{x}_i^{(b)\alpha_{bn+i}}) = 0$ with $b = 0, 1, 2, 3$ and $i = 1, 2, \dots, n$. Then we can define

a mapping $V : \Omega^\alpha \rightarrow \Omega^\alpha$ by $V(x) = \underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)$. It is easy to check that mapping V is a continuous function. By Brouwer's fixed point theorem, V has at least one fixed point $x = \underline{x} \in \Omega^\alpha$ which is also an equilibrium point of the system (3.1). Consequently, there exist 3^{4n} equilibrium points of the system (3.1) and each of those lies in one of the 3^{4n} regions Ω^α . \square

Next, for each index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n})$, where α_i is "l," or "r," we denote

$$\tilde{\Omega}^\alpha = \left\{ (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n \mid x_i^{(0)} \in \tilde{I}_i^{(0)\alpha_i}, x_i^{(1)} \in \tilde{I}_i^{(1)\alpha_{n+i}}, x_i^{(2)} \in \tilde{I}_i^{(2)\alpha_{2n+i}}, \right. \\ \left. x_i^{(3)} \in \tilde{I}_i^{(3)\alpha_{3n+i}}, \text{ for } i = 1, 2, \dots, n \right\},$$

and

$$\tilde{\Omega} = \left\{ \tilde{\Omega}^\alpha \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n}) \text{ with } \alpha_i \text{ is "l", or "r"} \right\}, \text{ where } \tilde{I}_i^{(b)l} = (-\infty, \hat{b}_i^{(b)}), \\ \tilde{I}_i^{(b)r} = (\hat{b}_i^{(b)}, +\infty) \text{ and } b = 0, 1, 2, 3.$$

Here, 2^{4n} elements of the type $\tilde{\Omega}^\alpha$ in $\tilde{\Omega}$ can be obtained, which are disjoint regions in \mathbb{H}^n .

Lemma 3.4. Under the Assumption 3.1, if the parametric conditions (\mathbf{C}_1^A) (resp. (\mathbf{C}_1^B)), (\mathbf{C}_2) and $w_{ii}^{(0)} \geq 0, i = 1, 2, \dots, n$ hold, then each region $\tilde{\Omega}^\alpha \in \tilde{\Omega}$ is positively invariant with respect to the state trajectory of the system (3.1) having AFs of Class \mathcal{A} (resp. Class \mathcal{B}).

Proof. Consider any one region $\tilde{\Omega}^\alpha \in \tilde{\Omega}$. Without loss of generality, assume the index $\alpha = (l, l, \dots, l)$. Then we may take the initial condition for the system (3.1) as $x_i(\theta) = \phi_i(\theta), \theta \in (-\infty, 0]$ with $x_i^{(b)}(\theta) = \phi_i^{(b)}(\theta) \in (-\infty, \hat{b}_i^{(b)})$, where $b = 0, 1, 2, 3$ and $i = 1, 2, \dots, n$. We claim that for all $t > 0$, the solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ with initial condition satisfies $-\infty < x_i^{(b)}(t) < \hat{b}_i^{(b)}$, where $b = 0, 1, 2, 3$ and $i = 1, 2, \dots, n$. If possible, let us assume that it is not true, then

there exists a time t_0 at which $x(t)$ first escapes from $\tilde{\Omega}^\alpha$. For convenience let us suppose that the real part of the i_0 -th component x_{i_0} of x is the first (or one of the first ones) that escapes from $(-\infty, \hat{b}_{i_0}^{(0)})$ at time t_0 . That is for all $i = 1, 2, \dots, n$ and $b = 0, 1, 2, 3$, there exist a $\epsilon_0 > 0$ such that $x_i^{(b)}(t) \leq \hat{b}_i^{(b)} - \epsilon_0, t \in (-\infty, t_0]$, $x_{i_0}^{(0)}(t_0) = \hat{b}_{i_0}^{(0)} - \epsilon_0$ and $\dot{x}_{i_0}^{(0)}(t_0) > 0$. Due to $w_{i_0 i_0}^{(0)} \geq 0$, the monotonicity of $g_{i_0}^{(0)}$ and the parametric condition (C_2) , we can obtain

$$\begin{aligned}
 \dot{x}_{i_0}^{(0)}(t_0) &= -d_{i_0} x_{i_0}^{(0)}(t_0) + k_{i_0}^{(0)} + \sum_{j=1}^n \left[h_{i_0 j}^{(0)} g_j^{(0)}(x_j^{(0)}(t_0)) - h_{i_0 j}^{(1)} g_j^{(1)}(x_j^{(1)}(t_0)) - \right. \\
 &\quad \left. h_{i_0 j}^{(2)} g_j^{(2)}(x_j^{(2)}(t_0)) - h_{i_0 j}^{(3)} g_j^{(3)}(x_j^{(3)}(t_0)) \right] \\
 &\quad + \sum_{j=1}^n \left[w_{i_0 j}^{(0)} g_j^{(0)}(x_j^{(0)}(t_0 - \tau_{i_0 j}(t_0))) - w_{i_0 j}^{(1)} g_j^{(1)}(x_j^{(1)}(t_0 - \tau_{i_0 j}(t_0))) - \right. \\
 &\quad \left. w_{i_0 j}^{(2)} g_j^{(2)}(x_j^{(2)}(t_0 - \tau_{i_0 j}(t_0))) - w_{i_0 j}^{(3)} g_j^{(3)}(x_j^{(3)}(t_0 - \tau_{i_0 j}(t_0))) \right] \\
 &\leq -d_{i_0} (\hat{b}_{i_0}^{(0)} - \epsilon_0) + (h_{i_0 i_0}^{(0)} + w_{i_0 i_0}^{(0)}) g_{i_0}^{(0)} (\hat{b}_{i_0}^{(0)} - \epsilon_0) + \sum_{j=1, j \neq i_0}^n \max \left\{ (h_{i_0 j}^{(0)} + w_{i_0 j}^{(0)}) u_j^{(0)}, \right. \\
 &\quad \left. (h_{i_0 j}^{(0)} + w_{i_0 j}^{(0)}) v_j^{(0)} \right\} - \sum_{j=1}^n \left[\min \left\{ (h_{i_0 j}^{(1)} + w_{i_0 j}^{(1)}) u_j^{(1)}, (h_{i_0 j}^{(1)} + w_{i_0 j}^{(1)}) v_j^{(1)} \right\} + \right. \\
 &\quad \min \left\{ (h_{i_0 j}^{(2)} + w_{i_0 j}^{(2)}) u_j^{(2)}, (h_{i_0 j}^{(2)} + w_{i_0 j}^{(2)}) v_j^{(2)} \right\} + \min \left\{ (h_{i_0 j}^{(3)} + w_{i_0 j}^{(3)}) u_j^{(3)}, \right. \\
 &\quad \left. \left. (h_{i_0 j}^{(3)} + w_{i_0 j}^{(3)}) v_j^{(3)} \right\} \right] + k_{i_0}^{(0)} \\
 &= \hat{G}_i^{(b)} (\hat{b}_{i_0}^{(b)} - \epsilon_0) < \hat{G}_{i_0}^{(0)} (\hat{b}_{i_0}^{(0)}) = 0,
 \end{aligned}$$

which is a contradiction. So the considered hypothesis is wrong. Hence, each $\tilde{\Omega}^\alpha$ is a positively invariant under the state trajectory of the system (3.1). This completes the proof. \square

Theorem 3.1. Under the Assumption 3.1, if the parametric conditions (C_1^A) , (C_2)

and $w_{ii}^{(0)} \geq 0$, $i = 1, 2, \dots, n$ hold, Furthermore, let us consider there exists a non-decreasing function $\mu(t) > 0$ with

$$\lim_{t \rightarrow +\infty} \mu(t) = +\infty, \quad \lim_{t \rightarrow +\infty} \frac{\dot{\mu}(t)}{\mu(t)} = \alpha_0, \quad \lim_{t \rightarrow +\infty} \frac{\mu(t)}{\mu(t - \tau(t))} = 1 + \beta_0, \quad (3.7)$$

where α_0, β_0 are non-negative constants. Moreover, if there exist positive constants $\xi_i^{(b)}$ for $b = 0, 1, 2, 3$ and $i = 1, 2, \dots, n$ such that

$$\begin{aligned} & (-d_i + \alpha_0)\xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(3)}| \xi_i^{(3)} + \\ & (1 + \beta_0) \left\{ \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(3)}| \xi_i^{(3)} \right\} < 0, \end{aligned} \quad (3.8)$$

$$\begin{aligned} & (-d_i + \alpha_0)\xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(3)}| \xi_i^{(2)} + \\ & (1 + \beta_0) \left\{ \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(3)}| \xi_i^{(2)} \right\} < 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} & (-d_i + \alpha_0)\xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(3)}| \xi_i^{(1)} + \\ & (1 + \beta_0) \left\{ \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(3)}| \xi_i^{(1)} \right\} < 0, \end{aligned} \quad (3.10)$$

$$\begin{aligned}
 & (-d_i + \alpha_0)\xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(2)}| \xi_i^{(1)} + \\
 & (1 + \beta_0) \left\{ \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(2)}| \xi_i^{(1)} \right\} < 0,
 \end{aligned} \tag{3.11}$$

where $\max\{g_j^{(b)'}(\hat{a}_j^{(b)}), g_j^{(b)'}(\check{c}_j^{(b)})\} < \rho_j^{(b)} < \min\{g_j^{(b)'}(\tilde{p}_j^{(b)}), g_j^{(b)'}(\tilde{q}_j^{(b)})\}$. Then the system (3.1) with the AFs of Class \mathcal{A} and UTVD has 16^n locally μ -stable equilibrium points located in $\bar{\Omega}^\alpha$.

Proof. For AFs $g_i^{(b)}$ in class \mathcal{A} ($b = 0, 1, 2, 3; i = 1, 2, \dots, n$), we introduce $\underline{s}_i^{(b)}, \bar{s}_i^{(b)}$ such that

$$\underline{s}_i^{(b)} = \min\{\xi | g_i^{(b)'}(\xi) = \rho_i^{(b)}\}, \bar{s}_i^{(b)} = \max\{\xi | g_i^{(b)'}(\xi) = \rho_i^{(b)}\},$$

where $\underline{s}_i^{(b)} \in (\hat{a}_i^{(b)}, \hat{b}_i^{(b)})$, $\bar{s}_i^{(b)} \in (\check{b}_i^{(b)}, \check{c}_i^{(b)})$. From Lemma 3.4 each $\bar{\Omega}^\alpha$ of the 16^n elements in $\bar{\Omega}$ is positive invariant set for the system (3.1), where

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n})$, α_i is “ l ”, or “ r ”,

$$\begin{aligned}
 \bar{\Omega}^\alpha = \left\{ (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n \mid x_i^{(0)} \in \bar{I}_i^{(0)\alpha_i}, x_i^{(1)} \in \bar{I}_i^{(1)\alpha_{n+i}}, x_i^{(2)} \in \bar{I}_i^{(2)\alpha_{2n+i}}, \right. \\
 \left. x_i^{(3)} \in \bar{I}_i^{(3)\alpha_{3n+i}}, \text{ for } i = 1, 2, \dots, n \right\},
 \end{aligned}$$

and

$\bar{\Omega} = \left\{ \bar{\Omega}^\alpha \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n}) \text{ with } \alpha_i \text{ is “}l\text{” or “}r\text{”} \right\}$, where $\bar{I}_i^{(b)l} = (-\infty, \underline{s}_i^{(b)}]$, $\bar{I}_i^{(b)r} = [\bar{s}_i^{(b)}, +\infty)$ and $b = 0, 1, 2, 3$. By using Lemma 3.3, the system (3.1) has unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ lying in $\bar{\Omega}^\alpha$. Now by using the translation $\bar{x}(t) = x(t) - x^*$, the system (3.1) can be transformed to

$$\dot{\bar{x}}_i(t) = -d_i \bar{x}_i(t) + \sum_{j=1}^n h_{ij} \bar{g}_j(\bar{x}_j(t)) + \sum_{j=1}^n w_{ij} \bar{g}_j(\bar{x}_j(t - \tau_{ij}(t))), \tag{3.12}$$

where $\bar{g}_i(\bar{x}_i(t)) = g_i(x_i(t)) - g_i(x_i^*)$. Again by using the multiplication rule between quaternions, the equation (3.12) can be decomposed into four real-valued systems of equations as

$$\begin{aligned} \dot{\bar{x}}_i^{(0)}(t) = & -d_i \bar{x}_i^{(0)} + \sum_{j=1}^n \left[h_{ij}^{(0)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t)) - h_{ij}^{(1)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t)) - \right. \\ & \left. h_{ij}^{(2)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t)) - h_{ij}^{(3)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t - \tau_{ij}(t))) - w_{ij}^{(1)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t - \tau_{ij}(t))) - \right. \\ & \left. w_{ij}^{(2)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t - \tau_{ij}(t))) - w_{ij}^{(3)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t - \tau_{ij}(t))) \right], \\ \dot{\bar{x}}_i^{(1)}(t) = & -d_i \bar{x}_i^{(1)} + \sum_{j=1}^n \left[h_{ij}^{(0)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t)) + h_{ij}^{(1)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t)) + \right. \\ & \left. h_{ij}^{(2)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t)) - h_{ij}^{(3)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t - \tau_{ij}(t))) + w_{ij}^{(1)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t - \tau_{ij}(t))) + \right. \\ & \left. w_{ij}^{(2)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t - \tau_{ij}(t))) - w_{ij}^{(3)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t - \tau_{ij}(t))) \right], \\ \dot{\bar{x}}_i^{(2)}(t) = & -d_i \bar{x}_i^{(2)} + \sum_{j=1}^n \left[h_{ij}^{(0)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t)) + h_{ij}^{(2)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t)) - \right. \\ & \left. h_{ij}^{(1)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t)) + h_{ij}^{(3)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t - \tau_{ij}(t))) + w_{ij}^{(2)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t - \tau_{ij}(t))) - \right. \\ & \left. w_{ij}^{(1)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t - \tau_{ij}(t))) + w_{ij}^{(3)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t - \tau_{ij}(t))) \right], \end{aligned}$$

$$\begin{aligned} \dot{\bar{x}}_i^{(3)}(t) = & -d_i \bar{x}_i^{(3)} + \sum_{j=1}^n \left[h_{ij}^{(0)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t)) + h_{ij}^{(3)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t)) + \right. \\ & \left. h_{ij}^{(1)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t)) - h_{ij}^{(2)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t)) \right] + \\ & \sum_{j=1}^n \left[w_{ij}^{(0)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t - \tau_{ij}(t))) + \bar{w}_{ij}^{(3)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t - \tau_{ij}(t))) + \right. \\ & \left. w_{ij}^{(1)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t - \tau_{ij}(t))) - w_{ij}^{(2)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t - \tau_{ij}(t))) \right], \end{aligned}$$

where $\bar{x}_i^{(b)}(t) = x_i^{(b)}(t) - x_i^{*(b)}$ and $\bar{g}_i^{(b)}(\bar{x}_i^{(b)}(t)) = g_i^{(b)}(x_i^{(b)}(t)) - g_i^{(b)}(x_i^{*(b)})$. By virtue of Mean Value Theorem, there exist $\gamma_j^{(b)}$ between $x_j^{(b)}(t)$ and $x_j^{*(b)}$, and also $\varsigma_j^{(b)}$ between $x_j^{(b)}(t - \tau_{ij}(t))$ and $x_j^{*(b)}$ such that

$$\begin{aligned} \bar{g}_j^{(b)}(\bar{x}_j^{(b)}(t)) &= g_j^{(b)'}(\gamma_j^{(b)}) \bar{x}_j^{(b)}(t), \\ \bar{g}_j^{(b)}(\bar{x}_j^{(b)}(t - \tau_{ij}(t))) &= g_j^{(b)'}(\varsigma_j^{(b)}) \bar{x}_j^{(b)}(t - \tau_{ij}(t)). \end{aligned}$$

Let $X_i^{(b)}(t) = \mu(t) \bar{x}_i^{(b)}(t)$ ($b = 0, 1, 2, 3$) and for our convenience we set

$$\begin{aligned} X^{(0)}(t) &= (X_1^{(0)}(t), X_2^{(0)}(t), \dots, X_n^{(0)}(t)), X^{(1)}(t) = (X_1^{(1)}(t), X_2^{(1)}(t), \dots, X_n^{(1)}(t)), X^{(2)}(t) = \\ &= (X_1^{(2)}(t), X_2^{(2)}(t), \dots, X_n^{(2)}(t)) \text{ and } X^{(3)}(t) = (X_1^{(3)}(t), X_2^{(3)}(t), \dots, X_n^{(3)}(t)), \\ X(t) &= (X^{(0)}(t)^T, X^{(1)}(t)^T, X^{(2)}(t)^T, X^{(3)}(t)^T). \end{aligned}$$

From the equation (3.7) and inequalities (3.8)-(3.11), we can find a sufficiently large time $\mathbf{T} > 0$, such that for all $t > \mathbf{T}$, $t - \tau_{ij}(t) \geq t - \tau(t) > 0$ and the limiting expressions of (3.7) attain their limiting values $\forall t > \mathbf{T}$.

Next, we define

$$Z(t) = \sup_{s \leq t} \|X(s)\|_{\xi}, \quad t \geq \mathbf{T}, \quad (3.13)$$

where $\|X(s)\|_{\xi} = \max\{\|X^{(0)}(t)\|_{\xi^{(0)}}, \|X^{(1)}(t)\|_{\xi^{(1)}}, \|X^{(2)}(t)\|_{\xi^{(2)}}, \|X^{(3)}(t)\|_{\xi^{(3)}}\}$
 with $\|X^{(b)}(t)\|_{\xi^{(b)}} = \max_{1 \leq i \leq n} \{(\xi_i^{(b)})^{-1} |X_i^{(b)}(t)|\}$, $b = 0, 1, 2, 3$.

It is claimed that $Z(t)$ is bounded, which implies $\|X(t)\|_\xi$ is bounded too. From eqn. (3.13), definitely we have $\|X(s)\|_\xi \leq Z(t), \forall t \geq \mathbf{T}$. If possible there exists a special time t such that $\|X(t)\|_\xi = Z(t)$, then there are following four possible cases:

Case 3.1. An index $i^{(0)} = i^{(0)}(t)$ depending on t exists such that

$$\|X(t)\|_\xi = \|X^{(0)}(t)\|_{\xi^{(0)}} = (\xi_{i^{(0)}}^{(0)})^{-1} |X_{i^{(0)}}^{(0)}(t)|. \quad (3.14)$$

Case 3.2. An index $i^{(1)} = i^{(1)}(t)$ depending on t exists such that

$$\|X(t)\|_\xi = \|X^{(1)}(t)\|_{\xi^{(1)}} = (\xi_{i^{(1)}}^{(1)})^{-1} |X_{i^{(1)}}^{(1)}(t)|. \quad (3.15)$$

Case 3.3. An index $i^{(2)} = i^{(2)}(t)$ depending on t exists such that

$$\|X(t)\|_\xi = \|X^{(2)}(t)\|_{\xi^{(2)}} = (\xi_{i^{(2)}}^{(2)})^{-1} |X_{i^{(2)}}^{(2)}(t)|. \quad (3.16)$$

Case 3.4. An index $i^{(3)} = i^{(3)}(t)$ depending on t exists such that

$$\|X(t)\|_\xi = \|X^{(3)}(t)\|_{\xi^{(3)}} = (\xi_{i^{(3)}}^{(3)})^{-1} |X_{i^{(3)}}^{(3)}(t)|. \quad (3.17)$$

Here we discuss only Case 3.1, similarly, we can prove the remaining cases.

From (3.14), we have $\xi_{i^{(0)}}^{(0)} \|X(t)\|_{\xi} = |X_{i^{(0)}}^{(0)}(t)|$. Now

$$\begin{aligned}
 \frac{d|X_{i^{(0)}}^{(0)}(t)|}{dt} &= \text{sign}\{X_{i^{(0)}}^{(0)}(t)\} \dot{X}_{i^{(0)}}^{(0)}(t) \\
 &= \text{sign}\{\bar{x}_{i^{(0)}}^{(0)}(t)\} \dot{\mu}(t) \bar{x}_{i^{(0)}}^{(0)}(t) + \text{sign}\{\bar{x}_{i^{(0)}}^{(0)}(t)\} \mu(t) \dot{\bar{x}}_{i^{(0)}}^{(0)}(t) \\
 &= \text{sign}\{\bar{x}_{i^{(0)}}^{(0)}(t)\} \dot{\mu}(t) \bar{x}_{i^{(0)}}^{(0)}(t) + \text{sign}\{\bar{x}_{i^{(0)}}^{(0)}(t)\} \mu(t) \left\{ -d_{i^{(0)}} \bar{x}_{i^{(0)}}^{(0)}(t) + \sum_{j=1}^n \left[h_{i^{(0)j}}^{(0)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t)) \right. \right. \\
 &\quad \left. \left. - h_{i^{(0)j}}^{(1)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t)) - h_{i^{(0)j}}^{(2)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t)) - h_{i^{(0)j}}^{(3)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t)) \right] + \right. \\
 &\quad \left. \sum_{j=1}^n \left[w_{i^{(0)j}}^{(0)} \bar{g}_j^{(0)}(\bar{x}_j^{(0)}(t - \tau_{i^{(0)j}}(t))) - w_{i^{(0)j}}^{(1)} \bar{g}_j^{(1)}(\bar{x}_j^{(1)}(t - \tau_{i^{(0)j}}(t))) - \right. \right. \\
 &\quad \left. \left. w_{i^{(0)j}}^{(2)} \bar{g}_j^{(2)}(\bar{x}_j^{(2)}(t - \tau_{i^{(0)j}}(t))) - w_{i^{(0)j}}^{(3)} \bar{g}_j^{(3)}(\bar{x}_j^{(3)}(t - \tau_{i^{(0)j}}(t))) \right] \right\} \\
 &\leq \dot{\mu}(t) |\bar{x}_{i^{(0)}}^{(0)}(t)| + \mu(t) \left\{ -d_{i^{(0)}} |\bar{x}_{i^{(0)}}^{(0)}(t)| + \sum_{j=1}^n \left[|h_{i^{(0)j}}^{(0)}| |g_j^{(0)'(\gamma_j^{(0)})}| |\bar{x}_j^{(0)}(t)| + |h_{i^{(0)j}}^{(1)}| |g_j^{(1)'(\gamma_j^{(1)})}| \right. \right. \\
 &\quad \times |\bar{x}_j^{(1)}(t)| + |h_{i^{(0)j}}^{(2)}| |g_j^{(2)'(\gamma_j^{(2)})}| |\bar{x}_j^{(2)}(t)| + |h_{i^{(0)j}}^{(3)}| |g_j^{(3)'(\gamma_j^{(3)})}| |\bar{x}_j^{(3)}(t)| \left. \right] + \sum_{j=1}^n \left[|w_{i^{(0)j}}^{(0)}| \right. \\
 &\quad \times |g_j^{(0)'(\varsigma_j^{(0)})}| |\bar{x}_j^{(0)}(t - \tau_{i^{(0)j}}(t))| + |w_{i^{(0)j}}^{(1)}| |g_j^{(1)'(\varsigma_j^{(1)})}| |\bar{x}_j^{(1)}(t - \tau_{i^{(0)j}}(t))| + |w_{i^{(0)j}}^{(2)}| \\
 &\quad \times |g_j^{(2)'(\varsigma_j^{(2)})}| |\bar{x}_j^{(2)}(t - \tau_{i^{(0)j}}(t))| + |w_{i^{(0)j}}^{(3)}| |g_j^{(3)'(\varsigma_j^{(3)})}| |\bar{x}_j^{(3)}(t - \tau_{i^{(0)j}}(t))| \left. \right] \right\} \\
 &\leq \dot{\mu}(t) |\bar{x}_{i^{(0)}}^{(0)}(t)| + \mu(t) \left\{ -d_{i^{(0)}} |\bar{x}_{i^{(0)}}^{(0)}(t)| + \sum_{j=1}^n \left[|h_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| |\bar{x}_j^{(0)}(t)| + |h_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| \times |\bar{x}_j^{(1)}(t)| + \right. \right. \\
 &\quad |h_{i^{(0)j}}^{(2)}| |\rho_j^{(2)}| |\bar{x}_j^{(2)}(t)| + |h_{i^{(0)j}}^{(3)}| |\rho_j^{(3)}| |\bar{x}_j^{(3)}(t)| \left. \right] + \sum_{j=1}^n \left[|w_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| |\bar{x}_j^{(0)}(t - \tau_{i^{(0)j}}(t))| + \right. \\
 &\quad |w_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| |\bar{x}_j^{(1)}(t - \tau_{i^{(0)j}}(t))| + |w_{i^{(0)j}}^{(2)}| |\rho_j^{(2)}| |\bar{x}_j^{(2)}(t - \tau_{i^{(0)j}}(t))| + \\
 &\quad \left. \left. |w_{i^{(0)j}}^{(3)}| |\rho_j^{(3)}| |\bar{x}_j^{(3)}(t - \tau_{i^{(0)j}}(t))| \right] \right\} \\
 &\leq (-d_{i^{(0)}} + \frac{\dot{\mu}(t)}{\mu(t)}) |X_{i^{(0)}}^{(0)}(t)| + \sum_{j=1}^n \left[|h_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| |X_j^{(0)}(t)| + |h_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| |X_j^{(1)}(t)| + \right. \\
 &\quad |h_{i^{(0)j}}^{(2)}| |\rho_j^{(2)}| |X_j^{(2)}(t)| + |h_{i^{(0)j}}^{(3)}| |\rho_j^{(3)}| |X_j^{(3)}(t)| \left. \right] + \sum_{j=1}^n \left[|w_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| |X_j^{(0)}(t - \tau_{i^{(0)j}}(t))| + \right. \\
 &\quad |w_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| |X_j^{(1)}(t - \tau_{i^{(0)j}}(t))| + |w_{i^{(0)j}}^{(2)}| |\rho_j^{(2)}| |X_j^{(2)}(t - \tau_{i^{(0)j}}(t))| + \\
 &\quad \left. \left. |w_{i^{(0)j}}^{(3)}| |\rho_j^{(3)}| |X_j^{(3)}(t - \tau_{i^{(0)j}}(t))| \right] \frac{\mu(t)}{\mu(t - \tau_{i^{(0)j}}(t))} \right. \\
 &\leq (-d_{i^{(0)}} + \frac{\dot{\mu}(t)}{\mu(t)}) \xi_{i^{(0)}}^{(0)} \|X^{(0)}(t)\|_{\xi^{(0)}} + \sum_{j=1}^n \left[|h_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| \xi_j^{(0)} \|X^{(0)}(t)\|_{\xi^{(0)}} + |h_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| \xi_j^{(1)} \right. \\
 &\quad \times \|X^{(1)}(t)\|_{\xi^{(1)}} + |h_{i^{(0)j}}^{(2)}| |\rho_j^{(2)}| \xi_j^{(2)} \|X^{(2)}(t)\|_{\xi^{(2)}} + |h_{i^{(0)j}}^{(3)}| |\rho_j^{(3)}| \xi_j^{(3)} \|X^{(3)}(t)\|_{\xi^{(3)}} \left. \right] + \\
 &\quad \sum_{j=1}^n \left[|w_{i^{(0)j}}^{(0)}| |\rho_j^{(0)}| \xi_j^{(0)} \|X_j^{(0)}(t - \tau_{i^{(0)j}}(t))\|_{\xi^{(0)}} + |w_{i^{(0)j}}^{(1)}| |\rho_j^{(1)}| \xi_j^{(1)} \|X_j^{(1)}(t - \tau_{i^{(0)j}}(t))\|_{\xi^{(1)}} + \right.
 \end{aligned}$$

$$\begin{aligned}
& \left[|w_{i(0)j}^{(2)}| \rho_j^{(2)} \xi_j^{(2)} \|X_j^{(2)}(t - \tau_{i(2)j}(t))\|_{\xi^{(2)}} + |w_{i(0)j}^{(3)}| \rho_j^{(3)} \xi_j^{(3)} \|X_j^{(3)}(t - \tau_{i(3)j}(t))\|_{\xi^{(3)}} \right] \frac{\mu(t)}{\mu(t - \tau_{i(0)j}(t))} \\
\leq & \left\{ (-d_{i(0)} + \frac{\dot{\mu}(t)}{\mu(t)}) \xi_{i(0)}^{(0)} + \sum_{j=1}^n \left[|h_{i(0)j}^{(0)}| \rho_j^{(0)} \xi_j^{(0)} + |h_{i(0)j}^{(1)}| \rho_j^{(1)} \xi_j^{(1)} + |h_{i(0)j}^{(2)}| \rho_j^{(2)} \xi_j^{(2)} + |h_{i(0)j}^{(3)}| \rho_j^{(3)} \xi_j^{(3)} \right] + \right. \\
& \left. \sum_{j=1}^n \left[|w_{i(0)j}^{(0)}| \rho_j^{(0)} \xi_j^{(0)} + |w_{i(0)j}^{(1)}| \rho_j^{(1)} \xi_j^{(1)} + |w_{i(0)j}^{(2)}| \rho_j^{(2)} \xi_j^{(2)} + |w_{i(0)j}^{(3)}| \rho_j^{(3)} \xi_j^{(3)} \right] \frac{\mu(t)}{\mu(t - \tau_{i(0)j}(t))} \right\} Z(t) \\
\leq & \left\{ (-d_{i(0)} + \alpha_0) \xi_{i(0)}^{(0)} + \sum_{j=1}^n \left[|h_{i(0)j}^{(0)}| \rho_j^{(0)} \xi_j^{(0)} + |h_{i(0)j}^{(1)}| \rho_j^{(1)} \xi_j^{(1)} + |h_{i(0)j}^{(2)}| \rho_j^{(2)} \xi_j^{(2)} + |h_{i(0)j}^{(3)}| \rho_j^{(3)} \xi_j^{(3)} \right] + \right. \\
& \left. (1 + \beta_0) \sum_{j=1}^n \left[|w_{i(0)j}^{(0)}| \rho_j^{(0)} \xi_j^{(0)} + |w_{i(0)j}^{(1)}| \rho_j^{(1)} \xi_j^{(1)} + |w_{i(0)j}^{(2)}| \rho_j^{(2)} \xi_j^{(2)} + |w_{i(0)j}^{(3)}| \rho_j^{(3)} \xi_j^{(3)} \right] \right\} Z(t) \\
& < 0.
\end{aligned}$$

Hence, it can be concluded that $\|X(t)\|_{\xi}$ will be non-increasing and hence $Z(t)$ and $\|X(t)\|_{\xi}$ are bounded. Consequently, for $b = 0, 1, 2, 3$, each $\|X^{(b)}(t)\|_{\xi^{(b)}}$ is bounded. Therefore, for $i = 1, 2, \dots, n$ and $b = 0, 1, 2, 3$, each $|X_i^{(b)}(t)|$ is bounded, hence there exist real constants $\lambda_i^{(b)} > 0$ such that $|X_i^{(b)}(t)| \leq \lambda_i^{(b)}, \forall t \geq \mathbf{T}$. Hence, we must have $|\mu(t)\bar{x}_i^{(b)}(t)| \leq \lambda_i^{(b)}, \forall t \geq \mathbf{T} \implies |\bar{x}_i^{(b)}(t)| \leq \frac{\lambda_i^{(b)}}{\mu(t)}, \forall t \geq \mathbf{T} \implies |x_i^{(b)}(t) - x_i^{*(b)}| \leq \frac{\lambda_i^{(b)}}{\mu(t)}, \forall t \geq \mathbf{T}$. Therefore, x^* is locally μ -stable in $\bar{\Omega}^\alpha$. Since $\bar{\Omega}^\alpha$ is arbitrary, therefore the system (3.1) with AFs of Class \mathcal{A} and UTVD can have 81^n equilibrium points, out of which 16^n are locally μ -stable. This completes the proof. \square

Theorem 3.2. Under the Assumption 3.1, if the parametric conditions (\mathbf{C}_1^B) , (\mathbf{C}_2) and $w_{ii}^{(0)} \geq 0, i = 1, 2, \dots, n$ hold. Furthermore, let us consider there exists a non-decreasing function $\mu(t) > 0$ such that the conditions in equation (3.7) hold. Moreover, if there exist positive constants $\xi_i^{(b)}$ for $b = 0, 1, 2, 3, i = 1, 2, \dots, n$ such that inequalities (3.8)-(3.11) hold. Then there exist 16^n locally μ -stable equilibrium points located in $\check{\Omega}^\alpha$ for the system (3.1) having UTVD and the AFs given in Class \mathcal{B} .

Proof. For AFs $g_i^{(b)}(\cdot)$ in Class \mathcal{B} , let us define $\check{\Omega}$ and $\check{\Omega}^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n})$, α_i is “ l ,” or “ r ” as

$$\check{\Omega}^\alpha = \left\{ (x_1, x_2, \dots, x_n)^T \in \mathbb{H}^n \mid x_i^{(0)} \in \check{I}_i^{(0)\alpha_i}, x_i^{(1)} \in \check{I}_i^{(1)\alpha_{n+i}}, x_i^{(2)} \in \check{I}_i^{(2)\alpha_{2n+i}}, \right. \\ \left. x_i^{(3)} \in \check{I}_i^{(3)\alpha_{3n+i}}, \text{ for } i = 1, 2, \dots, n \right\},$$

and

$\check{\Omega} = \left\{ \check{\Omega}^\alpha \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{4n}) \text{ with } \alpha_i \text{ is “}l\text{” , or “}r\text{”} \right\}$, where $\check{I}_i^{(b)l} = (-\infty, \check{p}_i^{(b)}]$, $\check{I}_i^{(b)r} = [\check{q}_i^{(b)}, +\infty)$ and $b = 0, 1, 2, 3$. From Lemma 3.4, any region $\check{\Omega}^\alpha$ is positively invariant for QVNNs (3.1). By Lemma 3.3, the system (3.1) has an unique equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ lying in $\check{\Omega}^\alpha$. Now by using the translation $\bar{x}(t) = x(t) - x^*$, system (3.1) can be transformed to (3.12). Next we decompose (3.12) into four real-valued system of equations $\dot{\bar{x}}_i^{(b)}$, $b = 0, 1, 2, 3$, which are discussed in the Theorem 3.1. For any arbitrary given region $\check{\Omega}^\alpha$ and for the AFs in Class \mathcal{B} , we can derive $\rho_j^{(b)} = 0$ ($j = 1, 2, \dots, n; b = 0, 1, 2, 3$). From (3.8)-(3.11), it follows that

$$(-d_i + \alpha_0) < 0 \tag{3.18}$$

and

$$\bar{g}_i^{(b)}(\bar{x}_i^{(b)}(t)) = g_i^{(b)}(x_i^{(b)}(t)) - g_i^{(b)}(x_i^{*(b)}) = 0, \\ \bar{g}_i^{(b)}(\bar{x}_i^{(b)}(t - \tau_{ij}(t))) = g_i^{(b)}(x_i^{(b)}(t - \tau_{ij}(t))) - g_i^{(b)}(x_i^{*(b)}) = 0. \tag{3.19}$$

Here, we also consider the function $Z(t)$ which is defined in (3.13) in the proof of Theorem 3.1. Using the similar way as in the proof of Theorem 3.1, we can easily prove that $Z(t)$ is bounded and hence $\|X(t)\|$ is bounded too. From eqn. (3.13), definitely we have $\|X(s)\|_\xi \leq Z(t)$, $\forall t \geq \mathbf{T}$. If possible there exist a special time t such that $\|X(t)\|_\xi = Z(t)$, then there are four possible cases, viz., Case 3.1 - Case

3.4 which are already discussed in the proof of Theorem 3.1. Here we discuss only Case 3.1, the rest of the cases will be similar.

From (3.14), we have $\xi_{i(0)}^{(0)} \|X(t)\|_{\xi} = |X_{i(0)}^{(0)}(t)|$.

Now,

$$\begin{aligned}
 \frac{d|X_{i(0)}^{(0)}(t)|}{dt} &= \text{sign}\{X_{i(0)}^{(0)}(t)\} \dot{X}_{i(0)}^{(0)}(t) \\
 &= \text{sign}\{\bar{x}_{i(0)}^{(0)}(t)\} \dot{\mu}(t) \bar{x}_{i(0)}^{(0)}(t) + \text{sign}\{\bar{x}_{i(0)}^{(0)}(t)\} \mu(t) \dot{\bar{x}}_{i(0)}^{(0)}(t) \\
 &= \text{sign}\{\bar{x}_{i(0)}^{(0)}(t)\} \dot{\mu}(t) \bar{x}_{i(0)}^{(0)}(t) - \text{sign}\{\bar{x}_{i(0)}^{(0)}(t)\} \mu(t) d_{i(0)} \bar{x}_{i(0)}^{(0)}(t) \\
 &= \frac{\dot{\mu}(t)}{\mu(t)} |X_{i(0)}^{(0)}(t)| - d_{i(0)} |X_{i(0)}^{(0)}(t)| \\
 &= (-d_{i(0)} + \frac{\dot{\mu}(t)}{\mu(t)}) \xi_{i(0)}^{(0)} \|X^{(0)}(t)\|_{\xi^{(0)}} \\
 &\leq (-d_{i(0)} + \frac{\dot{\mu}(t)}{\mu(t)}) \xi_{i(0)}^{(0)} Z(t) \\
 &< 0.
 \end{aligned}$$

Using the similar logic given in Theorem 3.1, we can say that the QVNNs (3.1) with Class **B** AFs and UTVD have 16^n equilibrium points which are locally μ -stable located in $\check{\Omega}^\alpha$. This completes the proof. \square

Corollary 3.1. (Multiple Exponential Stability) : Under the Assumption 3.1, if the parametric conditions (\mathbf{C}_1^A) , (\mathbf{C}_2) and $w_{ii}^{(0)} \geq 0$, $i = 1, 2, \dots, n$ hold, Moreover, if there exist positive constants $\xi_i^{(b)}$ for $b = 0, 1, 2, 3$, $i = 1, 2, \dots, n$ such that

$$\begin{aligned}
 &-d_i \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(3)}| \xi_i^{(3)} + \\
 &\left\{ \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(3)}| \xi_i^{(3)} \right\} < 0,
 \end{aligned} \tag{3.20}$$

$$\begin{aligned}
 & -d_i \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(3)}| \xi_i^{(2)} + \\
 & \left\{ \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(3)}| \xi_i^{(2)} \right\} < 0,
 \end{aligned} \tag{3.21}$$

$$\begin{aligned}
 & -d_i \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(3)}| \xi_i^{(1)} + \\
 & \left\{ \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(3)}| \xi_i^{(1)} \right\} < 0,
 \end{aligned} \tag{3.22}$$

$$\begin{aligned}
 & -d_i \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(2)}| \xi_i^{(1)} + \\
 & \left\{ \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(2)}| \xi_i^{(1)} \right\} < 0,
 \end{aligned} \tag{3.23}$$

where $\max\{g_j^{(b)'}(\hat{a}_j^{(b)}), g_j^{(b)'}(\hat{c}_j^{(b)})\} < \rho_j^{(b)} < \min\{g_j^{(b)'}(\tilde{p}_j^{(b)}), g_j^{(b)'}(\tilde{q}_j^{(b)})\}$. Then the QVNNs (3.1) with time-varying delays $\tau_{ij}(t) \leq \tau$, have 16^n locally exponential stable equilibrium points located in $\bar{\Omega}^\alpha$.

Proof. Now from the inequalities (3.20)-(3.23), we can pick a small real constant $\epsilon > 0$ such that

$$\begin{aligned}
 & (-d_i + \epsilon) \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(3)}| \xi_i^{(3)} + \\
 & e^{\epsilon\tau} \left\{ \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(3)}| \xi_i^{(3)} \right\} < 0,
 \end{aligned}$$

$$(-d_i + \epsilon)\xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(3)}| \xi_i^{(2)} + e^{\epsilon\tau} \left\{ \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(0)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(1)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(2)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(3)}| \xi_i^{(2)} \right\} < 0,$$

$$(-d_i + \epsilon)\xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(3)}| \xi_i^{(1)} + e^{\epsilon\tau} \left\{ \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(0)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(2)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(1)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(3)}| \xi_i^{(1)} \right\} < 0,$$

$$(-d_i + \epsilon)\xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(2)}| \xi_i^{(1)} + e^{\epsilon\tau} \left\{ \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(0)}| \xi_i^{(3)} + \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(3)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(1)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(2)}| \xi_i^{(1)} \right\} < 0,$$

hold. Let $\mu(t) = e^{\epsilon t}$, then

$$\lim_{t \rightarrow +\infty} \frac{\dot{\mu}(t)}{\mu(t)} = \epsilon, \quad \lim_{t \rightarrow +\infty} \frac{\mu(t)}{\mu(t - \tau(t))} = e^{\epsilon\tau}.$$

Corollary 3.1 is a direct consequence of Theorem 3.1. □

Corollary 3.2. (Multiple Power Stability) : Under the Assumption 3.1, if the conditions (\mathbf{C}_1^A) , (\mathbf{C}_2) , (3.20)-(3.23) and $w_{ii}^{(0)} \geq 0$, $i = 1, 2, \dots, n$ hold. Then, QVNNs (3.1) with $\tau_{ij}(t) \leq \delta t$ ($0 < \delta < 1$) have 81^n equilibria, out of which 16^n are locally power stable located in $\overline{\Omega}^\alpha$.

Proof. The proof of this corollary is very similar to the proof of Corollary 3.1. Because of space limitations, we focused only on the inequality (3.20). Pick a constant

$\epsilon > 0$ sufficiently small such that

$$-d_i \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(0)} |h_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |h_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |h_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |h_{ij}^{(3)}| \xi_i^{(3)} + (1 - \delta)^{-\epsilon} \left\{ \sum_{j=1}^n \rho_j^{(0)} |w_{ij}^{(0)}| \xi_i^{(0)} + \sum_{j=1}^n \rho_j^{(1)} |w_{ij}^{(1)}| \xi_i^{(1)} + \sum_{j=1}^n \rho_j^{(2)} |w_{ij}^{(2)}| \xi_i^{(2)} + \sum_{j=1}^n \rho_j^{(3)} |w_{ij}^{(3)}| \xi_i^{(3)} \right\} < 0.$$

It is noted that here the remaining three inequalities have been skipped to write due to page limitations. Considering $\mu(t) = t^\epsilon$, we obtain

$$\lim_{t \rightarrow +\infty} \frac{\dot{\mu}(t)}{\mu(t)} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\mu(t)}{\mu(t - \tau(t))} = (1 - \delta)^{-\epsilon}.$$

Thus the corollary 3.2 is a direct consequence of Theorem 3.1. □

Corollary 3.3. (Multiple log Stability) : Under the Assumption 3.1, if the conditions (\mathbf{C}_1^A) , (\mathbf{C}_2) , (3.20)-(3.23) and $w_{ii}^{(0)} \geq 0$, $i = 1, 2, \dots, n$ hold. Then QVNNs (3.1) with $\tau_{ij}(t) \leq t - \frac{t}{\ln t}$, have 81^n equilibria, out of which 16^n are locally log stable.

Here, we can pick $\mu(t) = \ln(1 + t)$. Then we can derive $\alpha_0 = 0$ and $\beta_0 = 0$.

Remark 3.6. Here we analyze several verities of stability in the μ -stability framework. The stability becomes exponential if $\tau_{ij} \leq \tau$. It is noted that by increasing the delays we can observe that the equilibrium points are still stable but the convergence becomes slower. More precisely, the equilibrium points become power stable, if $\tau_{ij} \leq \delta t$ ($0 < \delta < 1$), and becomes log-stability, if $\tau_{ij} \leq t - \frac{t}{\ln t}$.

Remark 3.7. The main goal in this chapter is to increase the storage capacity, which can be done by increasing the number of stable equilibrium points. Through this chapter, it is tried to show that QVNNs will have a more stable number of equilibria as compared CVNNs. In [46], the authors have showed that real-valued NNs can have 2^n stable equilibrium points while in [47], the authors showed that CVNNs can have 4^n stable equilibrium points, but μ -stability is not discussed in both of these two articles. Extending the results of these two articles, we have shown that

QVNNs (3.1) can have 2^{4n} μ -stable equilibrium points, which are much more than the stable points shown in [46, 47]. Hence, QVNNs (3.1) have more storage capacity than CVNNs. Apart from this, through this chapter, the different types of stability are mentioned, which are based on delays.

3.4 Numerical Examples

In this section, the numerical examples demonstrate the usefulness and superiority of our theoretical results. In Example 3.1, the dynamical behavior of QVNNs (3.1) is discussed with two different types of delays, the first one is constant delays and another is time-varying unbounded delays. It should be noted that to avoid lengthy calculations, Example 3.1 is presented with the activation function in Class \mathcal{B} .

Example 3.1. For $n = 2$, let us take the parameters of QVNNs (3.1) as

$$\begin{aligned}
 d_1 &= 1, & d_2 &= 3, \\
 h_{11} &= 3 - 0.4i + 0.4j + k, & h_{12} &= 1 + 0.5i - 0.8j - 0.2k, \\
 h_{21} &= 1 + 0.5i + 0.7j + 0.5k, & h_{22} &= 4 + 0.4i - 0.8j - 0.5k, \\
 w_{11} &= 4 - 0.3i - j - 0.2k, & w_{12} &= 0.5 - 0.4i + 0.5j - 0.8k, \\
 w_{21} &= -0.6 + 0.2i - j - k, & w_{22} &= 4 + 0.4i + 0.6j + 0.5k, \\
 k_1 &= 0, & k_2 &= 0,
 \end{aligned}$$

and the AFs $g_j^{(b)}(\xi)$ which belong to Class \mathcal{B} as

$$g_j^{(b)}(\xi) = \frac{|\xi + 1| - |\xi - 1|}{2}, \text{ clearly } g_j^{(b)}(\xi) = \begin{cases} -1 & , \xi \in (-\infty, -1) \\ \xi & , \xi \in [-1, 1] \\ 1 & , \xi \in (1, +\infty) \end{cases}, \quad (3.24)$$

where $b = 0, 1, 2, 3$ and $j = 1, 2$. Obviously, we have $u_j^{(b)} = -1, v_j^{(b)} = 1, p_j^{(b)} = -1$ and $q_j^{(b)} = 1$.

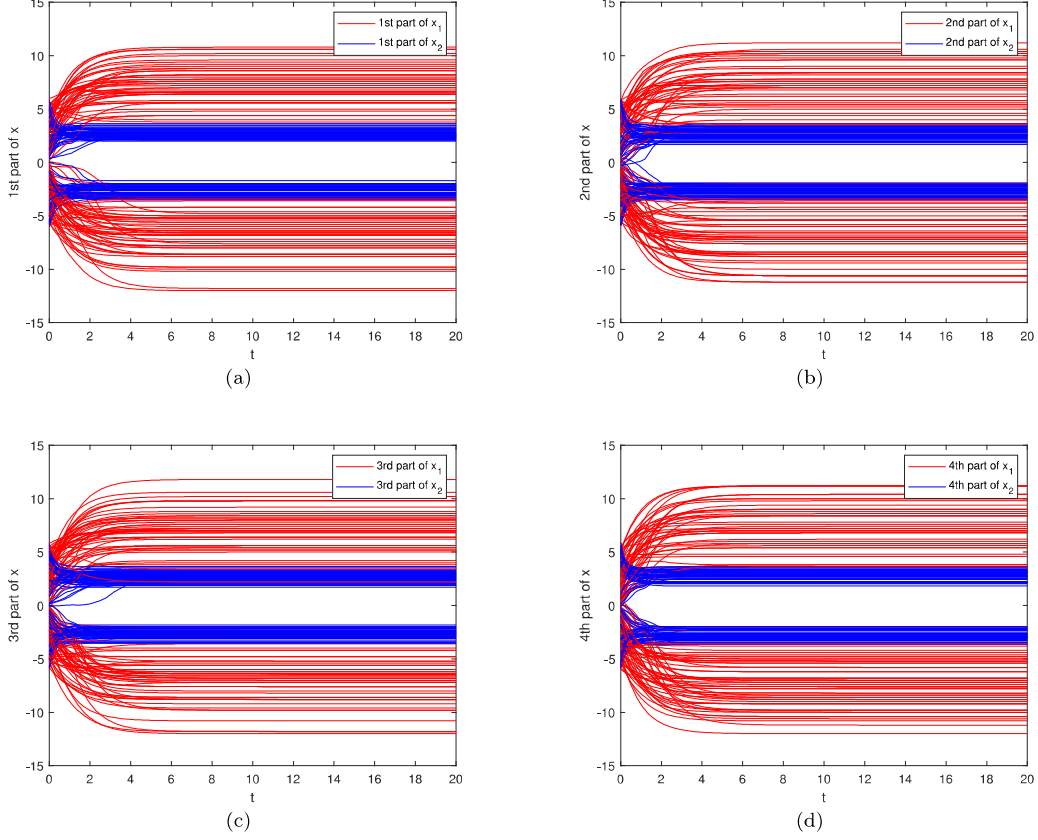


FIGURE 3.3: Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (3.1) with $\tau_{ij} = 1$ ($i, j = 1, 2$), respectively.

For all $i = 1, 2$ and $b = 0, 1, 2, 3$, the condition $(\mathbf{C}_1^{\mathcal{B}})$ holds. Also we have $w_{ii} \geq 0$.

Next for $b = 0, 1, 2, 3$, we have

$$\begin{aligned} \hat{G}_1^{(b)}(\xi) &= -\xi + 7g_1^{(b)}(\xi) + \hat{\eta}_1^{(b)}, & \check{G}_1^{(b)}(\xi) &= -\xi + 7g_1^{(b)}(\xi) + \check{\eta}_1^{(b)}, \\ \hat{G}_2^{(b)}(\xi) &= -3\xi + 8g_2^{(b)}(\xi) + \hat{\eta}_2^{(b)}, & \check{G}_2^{(b)}(\xi) &= -3\xi + 8g_2^{(b)}(\xi) + \check{\eta}_2^{(b)}, \end{aligned}$$

where $\hat{\eta}_1^{(b)} = 5, \check{\eta}_1^{(b)} = -5, \hat{\eta}_2^{(b)} = 2.9$ and $\check{\eta}_2^{(b)} = -2.9$.

Now $\hat{G}_1^{(b)}(p_1^{(b)}) = -1 < 0, \hat{G}_2^{(b)}(p_2^{(b)}) = -2.1 < 0,$

$$\check{G}_1^{(b)}(q_1^{(b)}) = 1 > 0, \quad \check{G}_2^{(b)}(q_2^{(b)}) = 2.1 > 0,$$

which clearly shows that the condition (C_2) is satisfied. For the multiple exponential stability we take $\tau_{ij} = 1$ ($i, j = 1, 2$). Next, for the AFs in Class \mathcal{B} , we must have $\rho_j^{(b)} = 0$ ($j = 1, 2; b = 0, 1, 2, 3$). Consequently, all the conditions (3.20)-(3.23) are satisfied. Hence from corollary 3.1, the system (3.1) has 81^2 equilibria, out of which 16^2 are locally exponentially stable. Now, we perform numerical simulation of the QVNNs (3.1) by using the Quaternion Toolbox for Matlab. By tracking twenty thousand random initial conditions, we get twenty thousand numerical solutions and these solutions converge exponentially to the 256 stable equilibrium points. We may approximate the values of these 256 stable equilibrium points, three of which are listed below:

$$\begin{pmatrix} -9.8000 + 7.6000i - 2.0000j + 6.8000k \\ -2.6333 - 3.2333i + 2.5667j - 2.3000k \end{pmatrix},$$

$$\begin{pmatrix} -6.2000 + 8.2000i - 3.6000j - 4.2000k \\ -2.9667 - 3.1667i + 2.5000j + 2.7667k \end{pmatrix},$$

$$\begin{pmatrix} 4.4000 - 6.6000i - 6.2000j - 6.8000k \\ -2.2333 - 2.8333i + 3.1667j - 3.1000k \end{pmatrix}.$$

Due to page limitation, here only three equilibrium points have been listed. Figures 3.3(a)-3.3(d) demonstrate the dynamical behavior of the system (3.1) with constant delays $\tau_{ij} = 1$ ($i, j = 1, 2$), where 100 random constant quaternion valued-vectors are chosen as initial conditions. From Figures 3.3(a)-3.3(d), it is observed that each state neuron converges to a stable state.

In the same example, multiple power stability of QVNNs is discussed for activation function in Class \mathcal{B} . Here we choose the time-varying delays $\tau_{ij} = 0.8t$ ($i, j = 1, 2$). Clearly, all the sufficient conditions for Corollary 3.2 are satisfied. Therefore the

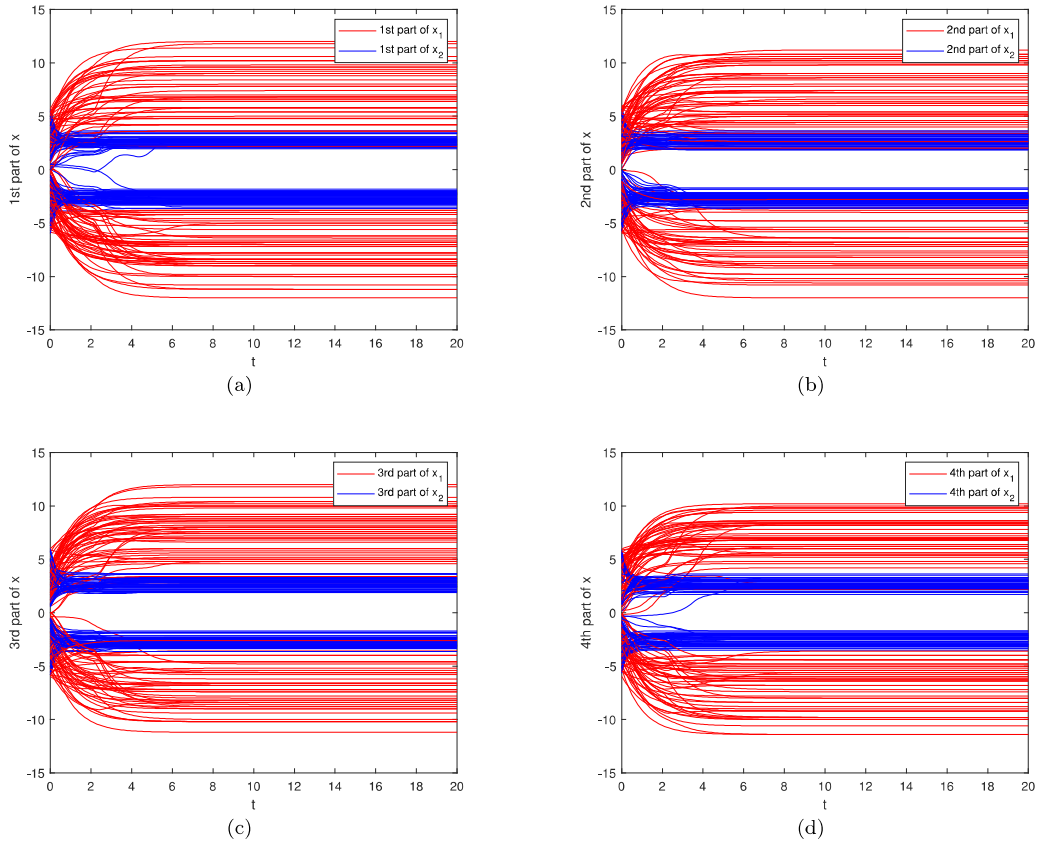


FIGURE 3.4: Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (3.1) with $\tau_{ij} = 0.8t$ ($i, j = 1, 2$), respectively.

system (3.1) has 81^n equilibrium points, out of which 16^n are locally power stable. Clearly Figures 3.4(a)-3.4(d) demonstrate the dynamical behavior of the system (3.1) with time-varying delays $\tau_{ij} = 0.8t$ ($i, j = 1, 2$), where one hundred random initial conditions are considered.

Through Figures 3.3(a)-3.3(d) and Figures 3.4(a)-3.4(d), it is seen that on increasing the delay, the equilibrium points are still stable, but the convergence speed is decreased. Hence, we can say that exponential stability is faster than power stability.

3.5 Conclusions

In this chapter, the multiple μ -stability problem of QVNNs given in equation (3.1) with UTVD has been investigated with two general classes of AFs. For avoiding the non-commutativity of quaternions, the QVNNs are decomposed into four equivalent real-valued systems, and then the state space \mathbb{H}^n is decomposed into 3^{4n} disjoint regions. Based on the geometrical behaviors of the AFs and by the inequality technique, several sufficient criteria have been achieved to guarantee the coexistence of 3^{4n} equilibria of the QVNNs, out of which 2^{4n} are locally μ -stable. Next, by imposing appropriate restrictions on the delay term $\tau_{ij}(t)$ and selecting appropriate functions of $\mu(t)$, multiple power stability, multiple exponential stability, and multiple log stability of the QVNN (3.1) have been obtained. The effectiveness of our proposed theory is validated through an example.
