Chapter 2

Theory of α-Fractal Function on SG

In this chapter, we deal with FIF and the box dimension of its graph corresponding to a continuous function defined on SG. This chapter also explores the so-called fractal operator, which is associated with the α-fractal function. Furthermore, we also provide some results on constrained approximation by fractal polynomials and study best approximation properties of fractal polynomials defined on SG. We also try to estimate bounds for fractal dimensions of the graph of a FIF using the method of oscillation of functions.

2.1 Fractal Interpolation Function on SG

Let $f \in \mathcal{C}(SG)$. We construct an IFS whose attractor is the graph of a continuous function on SG such that $f^{\alpha}|_{V_1} = f|_{V_1}$. Let $K = SG \times \mathbb{R}$. Define maps $W_i : K \to K$ by

$$W_i(t, x) = \Big(L_i(t), F_i(t, x)\Big), \quad i = 1, 2, 3,$$

where $F_i(t,x) : SG \times \mathbb{R} \to \mathbb{R}$ are required to satisfy the following conditions:

$$|F_i(.,x) - F_i(.,x')| \le c|x - x'|$$

with $c < 1$, that is, F_i is a contraction map with respect to the second variable and $F_i(p_j, f(p_j)) = f(L_i(p_j))$. In particular, we take

$$F_i(t,x) = \alpha_i(t)x + f(L_i(t)) - \alpha_i(t)b(t),$$

where $b : SG \to \mathbb{R}$ is a continuous function such that $b(p_j) = f(p_j),\ j = 1,2,3$ and for each $i = 1,2,3$, $\alpha_i : SG \to \mathbb{R}$ is a continuous function with $\|\alpha_i\|_\infty < 1$. We get an IFS $\{K; W_i, i = 1,2,3\}$.

Theorem 2.1.1. *Let $f : SG \to \mathbb{R}$ be a given function. Then the IFS $\{K; W_i, i = 1,2,3\}$ defined above has a unique attractor $Gr(f^\alpha)$.*

$$Gr(f^\alpha) = \Big\{(t, f^\alpha(t)) : f^\alpha \text{ is a continuous function and } f^\alpha|_{V_1} = f|_{V_1} \forall\ t \in SG\Big\}.$$

Furthermore, we have the following functional equation

$$f^\alpha(t) = f(t) + \alpha_i(L_i^{-1}(t))(f^\alpha - b)\big(L_i^{-1}(t)\big) \quad \forall\ \ t \in L_i(SG),\ \ i \in \{1,2,3\}. \tag{2.1.1}$$

Proof. Let $\mathcal{C}_f(SG) = \{g \in \mathcal{C}(SG) : g(p_j) = f(p_j),\ \forall\ j \in \{1,2,3\}\}$. We observe that the set $\mathcal{C}_f(SG)$ is a closed set of $\mathcal{C}(SG)$. Since $(\mathcal{C}(SG), \|.\|_\infty)$ is a Banach space, we get that $\mathcal{C}_f(SG)$ is a complete metric space. We define a map $T : \mathcal{C}_f(SG) \to \mathcal{C}_f(SG)$ by

$$(Tg)(t) = f(t) + \alpha_i(L_i^{-1}(t))\ (g - b)(L_i^{-1}(t))$$

for all $t \in L_i(SG)$, where $i \in \{1,2,3\}$. It is obvious that T is well-defined. Let $g, h \in \mathcal{C}_f(SG)$. Then

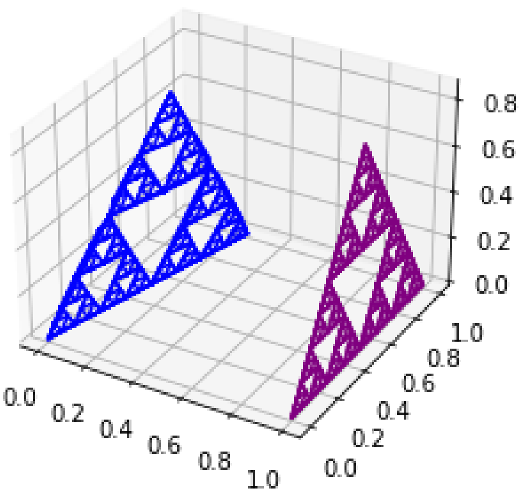
$$\begin{aligned}
\big|(Tg)(t) - (Th)(t)\big| &= \Big|\alpha_i(L_i^{-1}(t))\ (g-b)(L_i^{-1}(t)) - \alpha_i(L_i^{-1}(t))\ (h-b)(L_i^{-1}(t))\Big| \\
&= |\alpha_i(L_i^{-1}(t))\ (g-h)(L_i^{-1}(t))| \\
&= |\alpha_i(L_i^{-1}(t))|\ |(g-h)(L_i^{-1}(t))| \\
&\leq \|\alpha_i\|_\infty \|g-h\|_\infty \\
&\leq \|\alpha\|_\infty \|g-h\|_\infty,
\end{aligned}$$

which is true for all $t \in L_i(SG)$ and for all $i \in \{1,2,3\}$. Therefore, we obtain $\|Tg - Th\|_\infty \leq \|\alpha\|_\infty \|g-h\|_\infty$. Using $\|\alpha\|_\infty = \max_{i=1,2,3} \|\alpha_i\|_\infty < 1$, we get that T is a contraction map on $\mathcal{C}_f(SG)$. With the help of Banach contraction principle, we get a unique fixed point of T, namely $f^\alpha \in \mathcal{C}_f(SG)$. Finally, we check that $Gr(f^\alpha)$ is an attractor of the IFS. We have $T(f^\alpha) = f^\alpha$, that is, $f^\alpha(t) = F_i\big(L_i^{-1}(t), f^\alpha(L_i^{-1}(t))\big)\ \forall\ t \in L_i(SG), i = 1,2,3$. Furthermore, we get $f^\alpha(L_i(t)) = F_i(t, f^\alpha(t))$ for $i = 1,2,3$.

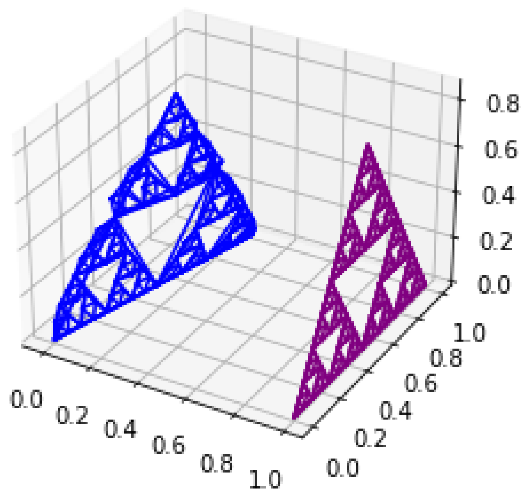
$$\begin{aligned}
\cup_{i=1}^3 W_i(Gr(f^\alpha)) &= \cup_{i=1}^3 \big\{W_i(t, f^\alpha(t)) : t \in SG\big\} \\
&= \cup_{i=1}^3 \Big\{\big(L_i(t), F_i(t, f^\alpha(t))\big) : t \in SG\Big\} \\
&= \cup_{i=1}^3 \Big\{\big(L_i(t), f^\alpha(L_i(t))\big) : t \in SG\Big\} \\
&= \cup_{i=1}^3 \big\{(t, f^\alpha(t)) : t \in L_i(SG)\big\} \\
&= Gr(f^\alpha),
\end{aligned}$$

which proves the result. □

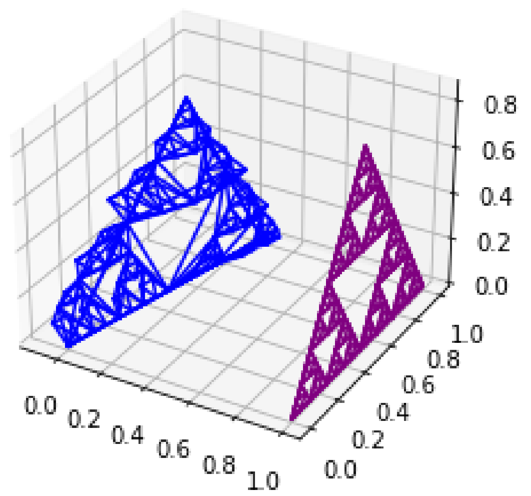
Next, we plot the graphs of f^α for different values of the parameters, that is, original function $f(x,y)$, base function $b(x,y)$ and scale vector α. One can easily identify the variation in these graphs by changing the values of the parameters. Hence, $Gr(f^{\alpha})$ depends on these parameters, see Figures [2.1.1-2.1.4].



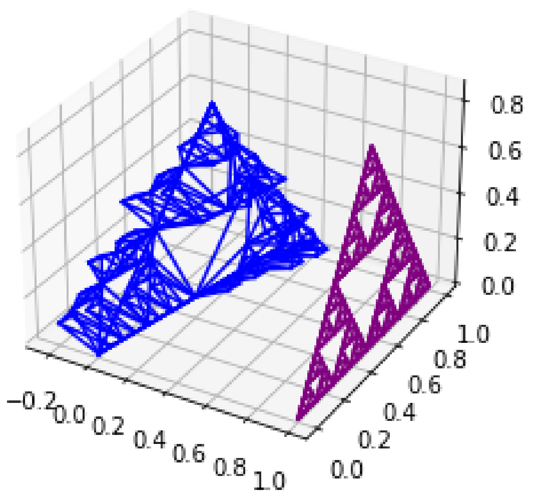
$Gr(f^{\alpha})$ at $\alpha = 0.1$



$Gr(f^{\alpha})$ at $\alpha = 0.3$

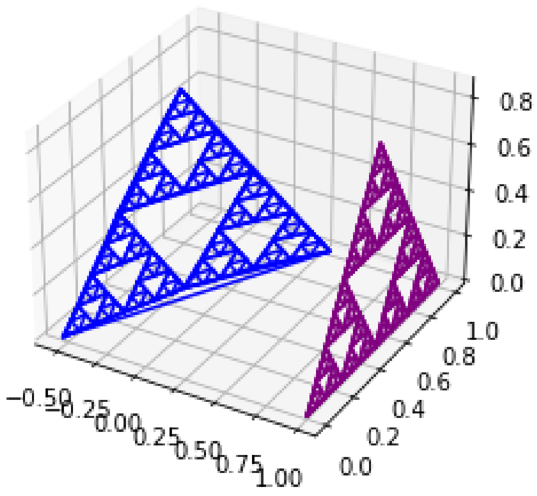


$Gr(f^{\alpha})$ at $\alpha = 0.6$

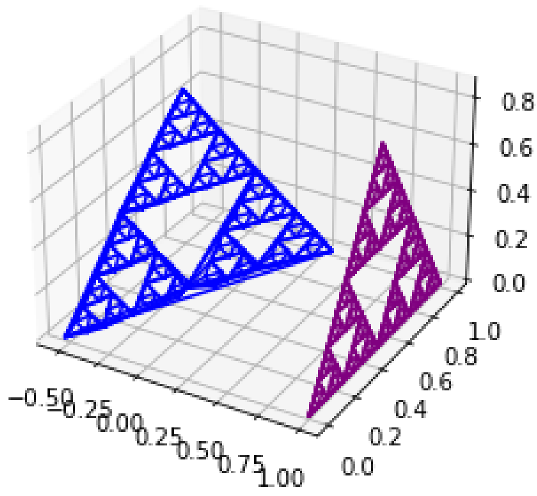


$Gr(f^{\alpha})$ at $\alpha = 0.9$

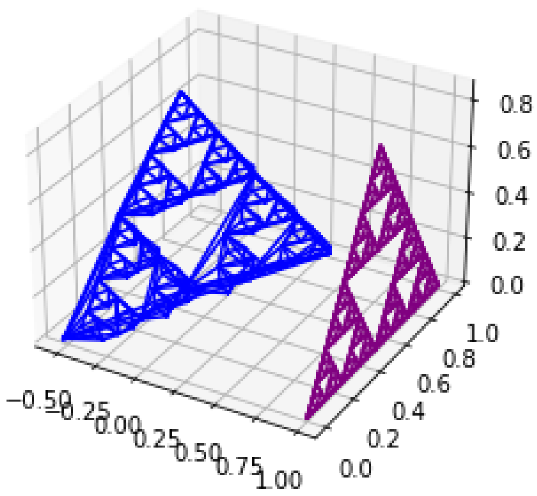
FIGURE 2.1.1: $Gr(f^{\alpha})$ for the various values of the α, where $b(x,y) = \frac{x}{4} + \frac{y}{9} - 1.3y(x-0.5)$, $f(x,y) = \frac{x}{4} + \frac{y}{9}$.



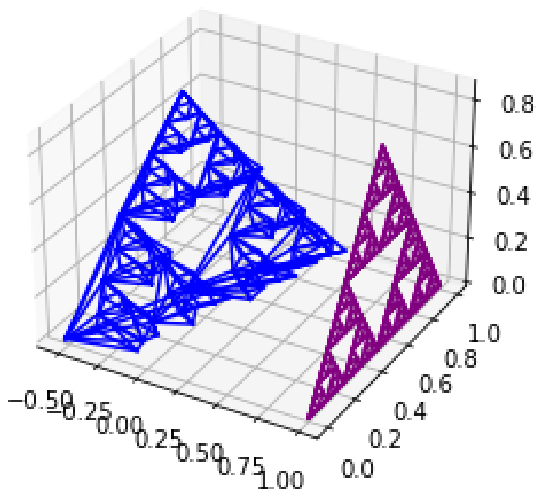
$Gr(f^{\alpha})$ at $\alpha = 0.1$



$Gr(f^{\alpha})$ at $\alpha = 0.3$

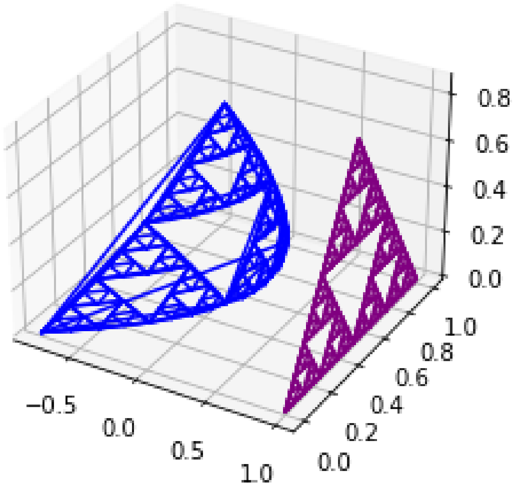


$Gr(f^{\alpha})$ at $\alpha = 0.6$

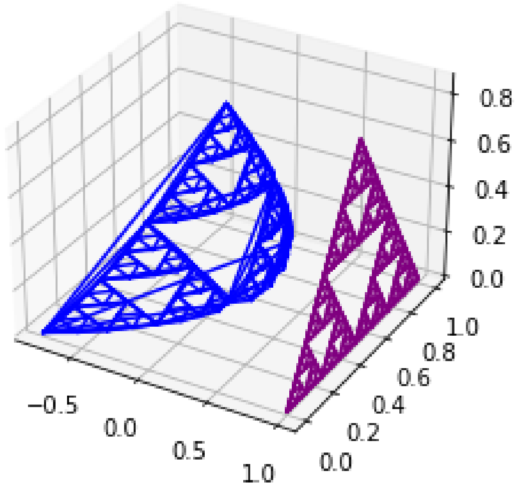


$Gr(f^{\alpha})$ at $\alpha = 0.9$

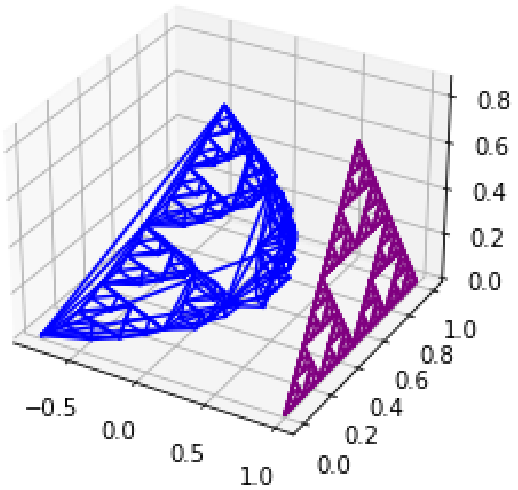
FIGURE 2.1.2: $Gr(f^{\alpha})$ for the various values of the α, where $b(x, y) = sin(x + 3.7) + 1.3x - x^2y + 0.866x^2 + xy - 0.866x$, $f(x, y) = sin(x + 3.7) + 1.3x$.



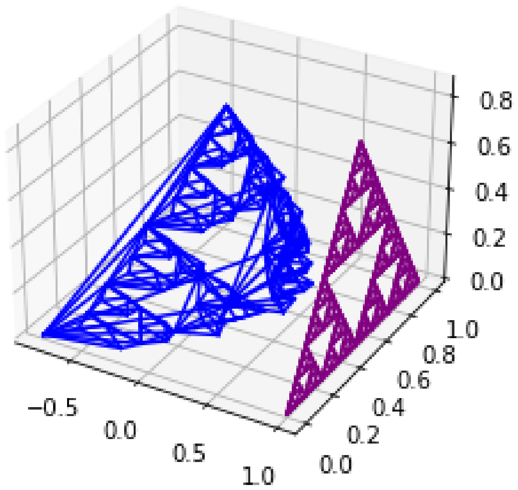
$Gr(f^{\alpha})$ at $\alpha = 0.1$



$Gr(f^{\alpha})$ at $\alpha = 0.3$

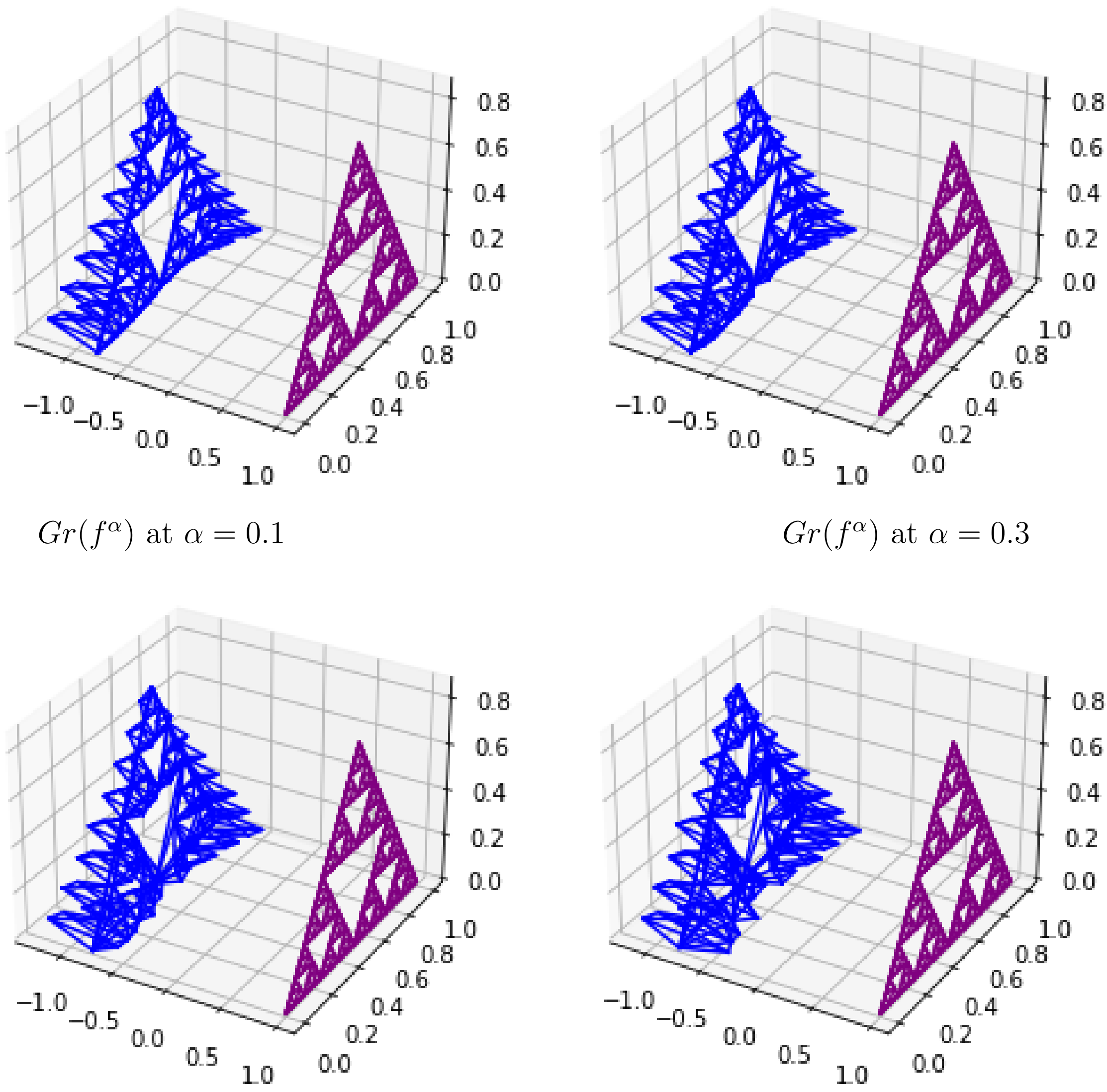


$Gr(f^{\alpha})$ at $\alpha = 0.6$



$Gr(f^{\alpha})$ at $\alpha = 0.9$

FIGURE 2.1.3: $Gr(f^{\alpha})$ for the various values of the α, where $b(x, y) = cos(2x + 5) + sin(x + 2.7) - 1.5 + 1.3x - x^2y + 0.866x^2 + xy - 0.866x$, $f(x, y) = cos(2x + 5) + sin(x + 2.7) - 1.5 + 1.3x$.



$Gr(f^\alpha)$ at $\alpha = 0.1$ $Gr(f^\alpha)$ at $\alpha = 0.3$

$Gr(f^\alpha)$ at $\alpha = 0.6$ $Gr(f^\alpha)$ at $\alpha = 0.9$

FIGURE 2.1.4: $Gr(f^\alpha)$ for the various values of the α, where $b(x,y) = cos(100x + 5) + sin(x+2.7) - 1.5 + 1.3x - x^2y + 0.866x^2 + xy - 0.866x$, $f(x,y) = cos(100x + 5) + sin(x+2.7) - 1.5 + 1.3x$.

Theorem 2.1.2. *Let* f, α_i $(i = 1,2,3)$ *and* $b \in \mathcal{H}^\sigma(SG)$ *such that* $b(p_j) = f(p_j)$ $\forall\, j \in \{1,2,3\}$. *If* $2^\sigma 3\|\alpha\| < 1$, *then* $f^\alpha \in \mathcal{H}^\sigma(SG)$, *where* $\|\alpha\| = \|\alpha\|_\infty + [\alpha]_\sigma$.

Proof. Let $\mathcal{H}^\sigma_f(SG) = \{g \in \mathcal{H}^\sigma(SG) : g(p_j) = f(p_j),\ \forall\, j \in \{1,2,3\}\}$. Note that the set $\mathcal{H}^\sigma_f(SG)$ is a closed subset of $\mathcal{H}^\sigma(SG)$. It follows that $\mathcal{H}^\sigma_f(SG)$ is complete. Let us define a map $T : \mathcal{H}^{\sigma}_{f}(SG) \to \mathcal{H}^{\sigma}_{f}(SG)$ by

$$(Tg)(t) = f(t) + \alpha_i(L_i^{-1}(t))\ (g-b)(L_i^{-1}(t))$$

for all $t \in L_i(SG)$ and $i \in \{1,2,3\}$. Next, we find that T is well defined. Here

$$\begin{aligned}
[Tg]_\sigma \le 3 \max_{i\in\{1,2,3\}} \sup_{t\neq t', t,t'\in L_i(SG)} & \frac{|Tg(t)-Tg(t')|}{\|t-t'\|_2^\sigma} \\
\le & 3 \max_{i\in\{1,2,3\}} \Bigg[\sup_{t\neq t', t,t'\in L_i(SG)} \frac{|f(t)-f(t')|}{\|t-t'\|_2^\sigma} \\
+ & 3 \sup_{t\neq t', t,t'\in L_i(SG)} \frac{|\alpha_i(L_i^{-1}(t))|\Big|(g-b)(L_i^{-1}(t)) - (g-b)(L_i^{-1}(t'))\Big|}{\|t-t'\|_2^\sigma} \\
+ & 3 \sup_{t\neq t', t,t'\in L_i(SG)} \frac{|(g-b)(L_i^{-1}(t'))|\Big|\alpha(L_i^{-1}(t)) - \alpha(L_i^{-1}(t'))\Big|}{\|t-t'\|_2^\sigma} \Bigg] \\
\le & \ 3[f]_\sigma + 2^\sigma 3\|\alpha\|_\infty\big([g]_\sigma + [b]_\sigma\big) + 2^\sigma 3 \|g-b\|_\infty [\alpha]_\sigma,
\end{aligned}$$

where $\|\alpha\|_\infty = \max_{i=1,2,3} \|\alpha_i\|_\infty$. For $g, h \in \mathcal{H}^{\sigma}_{f}(SG)$

$$\begin{aligned}
\|Tg - Th\| &= \|Tg-Th\|_\infty + [Tg-Th]_\sigma \\
&\le 3\|\alpha\|_\infty \|g-h\|_\infty + 2^\sigma 3\|\alpha\|_\infty [g-h]_\sigma + 2^\sigma 3 \|g-h\|_\infty [\alpha]_\sigma \\
&\le 2^\sigma 3 \|\alpha\| \|g-h\|,
\end{aligned}$$

whence, T is a contraction mapping on $\mathcal{H}^{\sigma}_{f}(SG)$. T has a unique fixed point in view of the Banach contraction principle, say $f^\alpha \in \mathcal{H}^{\sigma}_{f}(SG)$. □

Remark 2.1.1. *We recall Equation (2.1.1):*

$$f^\alpha(t) = f(t) + \alpha_i(L_i^{-1}(t))(f^\alpha - b)\big(L_i^{-1}(t)\big) \quad \forall\ \ t \in L_i(SG), \quad i \in \{1,2,3\}.$$

Now for every $t \in L_i(SG)$ *and* $i \in \{1,2,3\}$*, we have*

$$\begin{aligned}|f^\alpha(t) - f(t)| &= |\alpha_i(L_i^{-1}(t))(f^\alpha - b)\big(L_i^{-1}(t)\big)| \\ &= |\alpha_i(L_i^{-1}(t))|\ |(f^\alpha - b)\big(L_i^{-1}(t)\big)| \\ &\leq \|\alpha_i\|_\infty \|f^\alpha - b\|_\infty \\ &\leq \|\alpha\|_\infty \|f^\alpha - b\|_\infty.\end{aligned}$$

The above inequality implies that

$$\|f^\alpha - f\|_\infty \leq \|\alpha\|_\infty \|f^\alpha - b\|_\infty. \tag{2.1.2}$$

Using triangle inequality, we obtain $\|f^\alpha - f\|_\infty \leq \|\alpha\|_\infty \|f^\alpha - f\|_\infty + \|\alpha\|_\infty \|f - b\|_\infty$*. Finally, we have* $\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - b\|_\infty$*, and*

$$\|f^\alpha\|_\infty - \|f\|_\infty \leq \|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - b\|_\infty.$$

Therefore, we get $\|f^\alpha\|_\infty \leq \|f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - b\|_\infty$*. The above theorem coveys that Hölder constant of the map* f^α *depends only on the generating function* f *and the parameter maps* b *and* α_i*.*

With a bounded linear operator $L : \mathcal{C}(SG) \to \mathcal{C}(SG)$ satisfying $(Lg)(p_1) = g(p_1)$, $(Lg)(p_2) = g(p_2)$, and $(Lg)(p_3) = g(p_3)$, let us define a fractal operator $\mathcal{F}^\alpha : \mathcal{C}(SG) \to \mathcal{C}(SG)$ by $\mathcal{F}^\alpha(f) = f^\alpha$, where f^α is the fractal perturbation of f corresponding to the base function $b = Lf$ and scale vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{C}(SG))^3$. The upcoming theorem contains some elementary properties of the fractal operator $\mathcal{F}^\alpha$. Fractal operator associated with univariate and bivariate function can be found in [10, 43]. The proof of the next theorem is similar to that in the previous literature, hence, we omit.

Theorem 2.1.3. *Let Id be the identity operator on $\mathcal{C}(SG)$. Denote $\|\alpha\|_\infty = \max\{\|\alpha_i\|_\infty : i = 1, 2, 3\}$. Then the following statements hold:*

1. *Let $f \in \mathcal{C}(SG)$ be arbitrary. Then the perturbation error is of the form:*

$$\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - Lf\|_\infty.$$

Note that, for $\|\alpha\|_\infty = 0$ or $L = Id$, we have $\mathcal{F}^\alpha = Id$.

2. *Under the uniform norm on $\mathcal{C}(SG)$, the fractal operator $\mathcal{F}^\alpha$ is a bounded linear operator. Moreover, the operator norm holds, i.e.,*

$$\|\mathcal{F}^\alpha\| \leq 1 + \frac{\|\alpha\|_\infty \ \ \|Id - L\|}{1 - \|\alpha\|_\infty}.$$

3. *For $\|\alpha\|_\infty < \|L\|^{-1}$, $\mathcal{F}^\alpha$ is bounded below. In particular, $\mathcal{F}^\alpha$ is injective.*

4. *For $\|\alpha\|_\infty < (1 + \|Id - L\|)^{-1}$, the inverse of $\mathcal{F}^\alpha$ exists, which is also bounded and therefore a topological isomorphism. Furthermore,*

$$\|(\mathcal{F}^\alpha)^{-1}\| \leq \frac{1 + \|\alpha\|_\infty}{1 - \|\alpha\|_\infty \|L\|}.$$

5. *For $\|\alpha\|_\infty \neq 0$, the fixed points of L and $\mathcal{F}^\alpha$ are the same.*

6. *If the point spectrum of L contains 1, then $1 \leq \|\mathcal{F}^\alpha\|$.*

7. *The fractal operator $\mathcal{F}^\alpha$ is not the compact operator in the case of $\|\alpha\|_\infty < \|L\|^{-1}$.*

8. *If $\|\alpha\|_\infty < (1 + \|Id - L\|)^{-1}$, then $\mathcal{F}^\alpha$ is Fredholm and its index is 0.*

Theorem 2.1.4. *For the suitable continuous functions b, α_i on SG, let us consider the germ function f such that*

$$\begin{aligned} &|f(t) - f(t')| \leq k_f \|t - t'\|_2^\sigma, \\ &|b(t) - b(t')| \leq k_b \|t - t'\|_2^\sigma, \\ &|\alpha_i(t) - \alpha_i(t')| \leq k_\alpha \|t - t'\|_2^\sigma \end{aligned} \tag{2.1.3}$$

for every $t, t' \in SG, i \in \{1, 2, 3\}$, $0 < \sigma \leq 1$, and $k_f, k_b, k_\alpha > 0$. If $\|\alpha\| < \frac{1}{2^\sigma 3}$, then $\overline{\dim}_B\big(Gr(f^\alpha)\big) \leq 1 + \frac{\log 3}{\log 2} - \sigma$.

Proof. In the light of Theorem 2.1.2 and $\|\alpha\| < \frac{1}{2^\sigma 3}$, we get that f^α is Hölder continuous with the same exponent σ. From the previous result, we get

$$|f^\alpha(t) - f^\alpha(t')| \leq k_{f^\alpha} \|t - t'\|_2^\sigma$$

for some $k_{f^\alpha} > 0$. For $Gr(f^\alpha)$, we obtain the upper bound for the upper box dimension as follows: For $\delta = \frac{1}{2^n}$, let $N_\delta(Gr(f^\alpha))$ denote the number of $\delta-$triangular cubes intersecting $Gr(f^\alpha)$. Then

$$\begin{aligned} N_\delta(Gr(f^\alpha)) &\leq 3^n 2 + 2^n \sum_{w \in \{1,2,3\}^n} \mathrm{OSC}_f[L_w(SG)] \\ &\leq 3^n 2 + 2^n \sum_{w \in \{1,2,3\}^n} \frac{k_{f^\alpha}}{2^{n\sigma}} \\ &= 3^n 2 + k_{f^\alpha} 3^n 2^{n(1-\sigma)}. \end{aligned} \tag{2.1.4}$$

Finally, we have

$$\overline{\dim}_B\big(Gr(f^\alpha)\big) \leq 1 + \frac{\log 3}{\log 2} - \sigma.$$

□

Theorem 2.1.5. *Let $f, \alpha_i (i = 1, 2, 3)$ and $b \in \mathcal{H}^\sigma(SG)$ such that $b(p_j) = f(p_j)$ $\forall j \in \{1, 2, 3\}$. If $2^\sigma 3\|\alpha\| < 1$, where $\|\alpha\| := \|\alpha\|_\infty + [\alpha]_\sigma$, then*

$$\frac{\log 3}{\log 2} \leq \dim_H(Gr(f^\alpha)) \leq 1 + \frac{\log 3}{\log 2} - \sigma.$$

Proof. Given a map $T : Gr(f^\alpha) \to SG$ defined by $T((t, f^\alpha(t))) = t$, we get

$$\|T((t, f^\alpha(t))) - T((t', f^\alpha(t')))\|_2 = \|t - t'\|_2 \leq \|(t, f^\alpha(t)) - (t', f^\alpha(t'))\|_2.$$

Therefore, it can be concluded that T is a Lipschitz map. Considering the properties of Hausdorff dimension, we have $\dim_H(T(Gr(f^\alpha))) \leq \dim_H(Gr(f^\alpha))$. We can easily verify that T is a bijective map. Hence, we have $\dim_H(Gr(f^\alpha)) \geq \dim_H(SG) = \frac{\log 3}{\log 2}$. We recall a well-known result that relates the Hausdorff dimension and box dimension in the following sense:

$$\dim_H(C) \leq \underline{\dim}_B(C) \leq \overline{\dim}_B(C)$$

for any bounded set $C \subset \mathbb{R}^n$. Theorem 2.1.2 and the first part of Theorem 2.1.4 yields the required upper bound of the Hausdorff dimension of graph of f^α. □

In this chapter, we consider $\boldsymbol{\omega} = \omega^1 \dots \omega^N$, where $\omega^i \in I = \{1, 2, 3\}$ and $N \in \mathbb{N}$. For $\boldsymbol{\omega} \in I^N$, $L_{\boldsymbol{\omega}}$ is defined by

$$L_{\boldsymbol{\omega}} = L_{\omega^1} \circ \cdots \circ L_{\omega^N}.$$

Consider $V_* = \bigcup_{N=1}^\infty V_N$. In this chapter α- fractal function defined on SG is same as previous chapter.

The Weierstrass Approximation Theorem shows that the continuous real-valued functions on a compact interval can be uniformly approximated by polynomials. Whereas the polynomials defined on SG, which are obtained from the class of multi-harmonic functions, do not follow the Weierstrass Approximation Theorem.

2.2 Associated Fractal Operator on $\mathcal{C}(SG)$

Navascués and her fellow researchers explored the properties of fractal operators on certain function spaces in [68, 69]. The fractal operator on $\mathcal{C}(SG)$ is discussed in detail in [70].

Let $f \in \mathcal{C}(SG)$ be arbitrarily chosen. If we choose $b = Lf$, where $L : \mathcal{C}(SG) \to \mathcal{C}(SG)$ is a bounded linear operator satisfying $(Lf)|_{V_0} = f|_{V_0}$, while generating the α-fractal function f^α. Then the following self-referential equation is satisfied by the corresponding fractal function f^α:

$$f^\alpha(t) = f(t) + \alpha_{\boldsymbol{\omega}}(L_{\boldsymbol{\omega}}^{-1}(t))(f^\alpha - Lf)\big(L_{\boldsymbol{\omega}}^{-1}(t)\big), \quad \forall\ t \in L_{\boldsymbol{\omega}}(SG),\ \ \boldsymbol{\omega} \in I^N.$$

Definition 2.2.1. An α-fractal operator $\mathcal{F}^\alpha : \mathcal{C}(SG) \to \mathcal{C}(SG)$ is defined by

$$\mathcal{F}^\alpha(f) = f^\alpha.$$

Remark 2.2.1. *We can see that f^α and the operator $\mathcal{F}^\alpha$ satisfying*

$$\mathcal{F}^\alpha(f)(x) = f(x), \quad \forall\ x \in V_N.$$

Definition 2.2.2. Corresponding to the aforementioned class $\mathcal{H}_n(SG)$ and the bounded linear operator $\mathcal{F}^\alpha : \mathcal{C}(SG) \to \mathcal{C}(SG)$. The space $\mathcal{F}^\alpha\left(\mathcal{H}_n(SG)\right) = \mathcal{H}_n^\alpha(SG)$

represents the space of all the fractal polynomials on SG, i.e., a continuous function $h^\alpha : SG \to \mathbb{R}$ is a fractal polynomial on SG if $h^\alpha = \mathcal{F}^\alpha(h)$ for $h \in \mathcal{H}_n(SG)$.

Definition 2.2.3. Corresponding to the aforementioned class $\mathcal{S}(H_0, V_m, \mathbb{R})$. The space $\mathcal{F}^\alpha\big(\mathcal{S}(H_0, V_m, \mathbb{R})\big)$ represents the class of all (real-valued) fractal piecewise harmonic functions, i.e., a continuous function $h^\alpha : SG \to \mathbb{R}$ is a fractal piecewise harmonic function if $h^\alpha = \mathcal{F}^\alpha(h)$ for $h \in \mathcal{S}(H_0, V_m, \mathbb{R})$.

2.3 Constrained Approximation Aspects of the Constructed Class of Fractal Polynomials and Best Approximation Property

The following work in this section is motivated by the results provided by Viswanathan et al. [26] and Chand et al. [64].

It is shown in this section that under the suitable parameter settings, the fractal polynomial possesses certain properties as the original function f. We also discuss some approximation results of the fractal polynomials under certain restrictions. Furthermore, we prove some results related to one-sided approximation. Let us now present the following notations:

$$m_* = \min\{b(t) : t \in SG\}, \quad m_{\boldsymbol{\omega}} = \min\left\{f(t) : t \in L_{\boldsymbol{\omega}}(SG), \ \ \boldsymbol{\omega} \in I^N\right\}, \tag{2.3.1}$$

$$M^* = \max\{b(t) : t \in SG\}, \quad M_{\boldsymbol{\omega}} = \max\left\{f(t) : t \in L_{\boldsymbol{\omega}}(SG), \ \ \boldsymbol{\omega} \in I^N\right\}. \tag{2.3.2}$$

Theorem 2.3.1. *Consider a function $f \in \mathcal{C}(SG)$ such that $0 \leq f(t) \leq \tilde{M}$ for all $t \in SG$. If $\alpha_{\boldsymbol{\omega}} \in \mathcal{C}(SG)$ is chosen such that $\|\alpha_{\boldsymbol{\omega}}\|_\infty < 1$ and for all $\boldsymbol{\omega} \in I^N$, then*

$$\max\left\{\frac{-m_{\boldsymbol{\omega}}}{\tilde{M} - m_*}, \frac{M_{\boldsymbol{\omega}} - \tilde{M}}{M^*}\right\} \leq \alpha_{\boldsymbol{\omega}}(t) \leq \min\left\{\frac{m_{\boldsymbol{\omega}}}{M^*}, \frac{\tilde{M} - M_{\boldsymbol{\omega}}}{\tilde{M} - m_*}\right\},$$

corresponding $f^\alpha \in \mathcal{C}(SG)$ and $0 \leq f^\alpha(t) \leq \tilde{M}\ \forall\ t \in SG$ hold.

Proof. Note that the self-referential equation is satisfied by f^α, that is,

$$f^\alpha(t) = F_{\boldsymbol{\omega}}(L_{\boldsymbol{\omega}}^{-1}(t), f^\alpha(L_{\boldsymbol{\omega}}^{-1}(t)) \quad \forall t \in L_{\boldsymbol{\omega}}(SG), \quad \boldsymbol{\omega} \in I^N, \tag{2.3.3}$$

where $F_{\boldsymbol{\omega}}(t,x) = \alpha_{\boldsymbol{\omega}}(t)x + f(L_w(t)) - \alpha_{\boldsymbol{\omega}}(t)b(t)$ and it satisfies the interpolatory condition $f^\alpha|_{V_N} = f|_{V_N}$. We need to show $0 \leq f^\alpha(t) \leq \tilde{M}$ for all $t \in SG$. Since $SG = \bigcup_{\boldsymbol{\omega} \in \{1,2,3\}^N} L_{\boldsymbol{\omega}}(SG)$, a piecewise harmonic function f^α is constructed iteratively through the functional equation given in (2.3.3) and $f^\alpha|_{V_N} = f|_{V_N}$, this is sufficient to prove that $\forall\ \boldsymbol{\omega}$, and $(t,x) \in SG \times [0, \tilde{M}]$

$$0 \leq F_{\boldsymbol{\omega}}(t,x) \leq \tilde{M}.$$

Let $(t,x) \in SG \times [0, \tilde{M}]$ and

$$d_{\boldsymbol{\omega}}(t) = f(L_{\boldsymbol{\omega}}(t)) - \alpha_{\boldsymbol{\omega}}(t)b(t).$$

First, we consider $0 \leq \alpha_{\boldsymbol{\omega}}(t) < 1$. This assumption with $0 \leq x \leq \tilde{M}$ yields,

$$\begin{aligned} d_{\boldsymbol{\omega}}(t) \leq F_{\boldsymbol{\omega}}(t,x) &= \alpha_{\boldsymbol{\omega}}(t)x + d_{\boldsymbol{\omega}}(t) \\ &\leq \alpha_{\boldsymbol{\omega}}(t)\tilde{M} + d_{\boldsymbol{\omega}}(t). \end{aligned} \tag{2.3.4}$$

Therefore, $0 \le F_{\omega}(t,x) \le \tilde{M}$ holds if

$$0 \le d_{\omega}(t) \le \tilde{M}(1-\alpha_{\omega}(t)). \tag{2.3.5}$$

Keeping in mind the fact that $f(L_{\omega}(t)) \ge m_{\omega}$ and $b(t) \le M^*$ for all $t \in SG$, it can be readily verified that $\alpha_{\omega}(t) \le \frac{m_{\omega}}{M^*}$ ensures the first inequality in (2.3.5). Recall that if M^* is zero, then no further conditions on $\alpha_{\omega}(t)$ are required to ensure $d_{\omega}(t) \ge 0$. Similarly, using $f(L_{\omega}(t)) \le M_{\omega}$ and $b(t) \ge m^*$ $\forall\, t \in SG$, we assert that $\alpha_{\omega}(t) \le \frac{\tilde{M}-M_{\omega}}{\tilde{M}-m_*}$ guarantees the second inequality in (2.3.5). We thus choose α_{ω} as shown below to ensure that (2.3.5) holds,

$$\alpha_{\omega}(t) \le \min\left\{\frac{m_{\omega}}{M^*}, \frac{\tilde{M}-M_{\omega}}{\tilde{M}-m_*}\right\}. \tag{2.3.6}$$

Now we assume $-1 < \alpha_{\omega}(t) < 0$. Let $0 \le x \le \tilde{M}$, then by using simple steps, one gets

$$\alpha_{\omega}(t)\tilde{M} + d_{\omega}(t) \le \alpha_{\omega}(t)x + d_{\omega}(t) \le d_{\omega}(t), \tag{2.3.7}$$

consequently, for $0 \le F_{\omega}(t,x) \le \tilde{M}$, this is sufficient to show that

$$-\alpha_{\omega}(t)M \le d_{\omega}(t) \le \tilde{M}. \tag{2.3.8}$$

The similar calculations as for (2.3.5) indicate that (2.3.8) is satisfied if

$$\alpha_{\omega}(t) \ge \max\left\{\frac{-m_{\omega}}{\tilde{M}-m_*}, \frac{M_{\omega}-\tilde{M}}{M^*}\right\}.$$

Hence, we have established the assertion. □

Remark 2.3.1. *If $f \in \mathcal{C}(SG)$ with $f(t) \le 0$ for all $t \in SG$, then its fractal counterpart f^{α} can be constructed and for all $t \in SG$ that holds $f^{\alpha}(t) \le 0$. To* *obtain this, Theorem 2.3.1 is applied for the positive function $\hat{f} = -f$ and associated function $\hat{s} = -s$. We take $\alpha_{\boldsymbol{\omega}} \in SG$ satisfying $\|\alpha_{\boldsymbol{\omega}}\|_\infty < 1$ and*

$$\max\left\{\frac{-M_{\boldsymbol{\omega}}}{m - M^*}, \frac{m_{\boldsymbol{\omega}} - m}{m_*}\right\} \leq \alpha_{\boldsymbol{\omega}}(t) \leq \min\left\{\frac{M_{\boldsymbol{\omega}}}{m_*}, \frac{m - m_{\boldsymbol{\omega}}}{m - M^*}\right\},$$

guarantees $m \leq f^\alpha(t) \leq 0$ for all $t \in SG$.

The subsequent theorems are motivated by the work discussed in [58] and [71].

Theorem 2.3.2. *Let $f \in \mathcal{C}(SG)$ and f^α be an α-fractal function corresponding to f. Then $f^\alpha(t) \leq f(t)$ for every $t \in SG$ provided $\alpha_{\boldsymbol{\omega}}(t) \geq 0$ and $b(t) \geq f(t)$ for every $t \in SG$.*

Proof. We know that f^α satisfies the following self-referential equation

$$f^\alpha(L_{\boldsymbol{\omega}}(t)) = f(L_{\boldsymbol{\omega}}(t)) + \alpha_{\boldsymbol{\omega}}(t)(f^\alpha - b)(t),$$

for every $t \in SG$ and $\boldsymbol{\omega} \in I^N$. The previous theorem showed that in establishing $(f^\alpha - f)(t) \leq 0$ for all $t \in SG$. It is sufficient to demonstrate that if $(f^\alpha - f)(t) \leq 0$ for any level on SG points, then its next level points ensure $(f^\alpha - f)(L_{\boldsymbol{\omega}}(t)) \leq 0$. We can find that the above condition is equivalent to $(f^\alpha - f)(L_{\boldsymbol{\omega}}(t)) \leq 0$ for every $\boldsymbol{\omega} \in I^N$ whenever $(f^\alpha - f)(t) \leq 0$. Therefore, using the above, we have

$$\begin{aligned}(f^\alpha - f)(L_{\boldsymbol{\omega}}(t)) &= \alpha_{\boldsymbol{\omega}}(t)(f^\alpha - b)(t) \\ &= \alpha_{\boldsymbol{\omega}}(t)(f^\alpha - f)(t) + \alpha_{\boldsymbol{\omega}}(t)(f - b)(t).\end{aligned} \tag{2.3.9}$$

If we set up $\alpha_{\boldsymbol{\omega}}$ and b as above mentioned theorem then we can determine $(f^\alpha - f)(L_{\boldsymbol{\omega}}(t)) \leq 0$ by using the assumption $(f^\alpha - f)(t) \leq 0$, which completes the proof. □

Remark 2.3.2. *Using the same steps, one can verify that if $\alpha_{\boldsymbol{\omega}}(t) \geq 0$ and $b(t) \leq f(t)$, then $f^{\alpha}(t) \geq f(t)$ for every $t \in SG$.*

Theorem 2.3.3. *Suppose $f \in \mathcal{C}(SG)$ satisfying $f(t) \geq 0$ for all $t \in SG$. Let $\epsilon > 0$, then we can choose a fractal piecewise harmonic function h^{α} such that $h^{\alpha} \geq 0$ for every $t \in SG$ and $\|f - h^{\alpha}\|_{\infty} < \epsilon$.*

Proof. Let $\epsilon > 0$ and $f \in \mathcal{C}(SG)$ with $f(t) \geq 0$. Using Theorem 1.9.5, one can choose a piecewise harmonic function g such that

$$\|f - g\|_{\infty} < \frac{\epsilon}{4}.$$

For $t \in SG$, define $h(t) = g(t) + \frac{\epsilon}{4}$. Then

$$h(t) = g(t) - f(t) + f(t) + \frac{\epsilon}{4} \geq -\|f - g\|_{\infty} + f(t) + \frac{\epsilon}{4} > f(t) \geq 0.$$

Furthermore,

$$\|f - h\|_{\infty} \leq \|f - g\|_{\infty} + \|g - h\|_{\infty} < \frac{\epsilon}{2}.$$

Therefore, we obtain a piecewise harmonic function h with $h(t) \geq 0$ and $\|f - h\|_{\infty} < \frac{\epsilon}{2}$. Theorem 2.3.1 yields to get h^{α}, a positive fractal perturbation of h, now we choose $\alpha_{\boldsymbol{\omega}} \in \mathcal{C}(SG)$ such that $\|\alpha\|_{\infty} \leq \frac{\epsilon}{\epsilon + 2\|h - b\|_{\infty}}$, we get

$$\begin{aligned}
\|f - h^{\alpha}\|_{\infty} &\leq \|f - h\|_{\infty} + \|h - h^{\alpha}\|_{\infty} \\
&\leq \|f - h\|_{\infty} + \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|h - b\|_{\infty} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned} \tag{2.3.10}$$

implying the desired conclusion. □

Theorem 2.3.4. *Let $f \in \mathcal{C}(SG)$ and $\epsilon > 0$. Then a fractal piecewise harmonic function h^α can be obtained, which always lies above f, i.e., $h^\alpha(t) \geq f(t)$, $\forall\, t \in SG$ and $\|f - h^\alpha\|_\infty < \epsilon$.*

Proof. For a given $f \in \mathcal{C}(SG)$ and $\epsilon > 0$, it follows from Theorem 1.9.5 that there exists a piecewise harmonic function h with $h(t) \geq f(t)$ for every $t \in SG$ and

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

Now we choose $b(t) \leq h(t)$ and $\alpha_{\boldsymbol{\omega}} \geq 0$ for every $t \in SG$ with $\|\alpha\|_\infty \leq \frac{\epsilon}{\epsilon + 2\|h-b\|_\infty}$. In light of Theorem 2.3.2, we obtain $h^\alpha(t) \geq h(t)$ for every $t \in SG$. Moreover,

$$\|f - h^\alpha\|_\infty < \epsilon.$$

We, therefore, have a fractal polynomial h^α satisfying the required conditions. □

We now give some basics of best one-sided approximation [66]. Consider $\beta \in \left[\frac{\log 3}{\log 2}, \frac{\log 3}{\log 2} + 1\right]$ and define

$$\mathcal{C}_\beta(SG) := \{f \in \mathcal{C}(SG) : \overline{\dim}_B(Gr(f)) \leq \beta\}.$$

In the light of [72, Proposition 3.4], note that $\mathcal{C}_\beta(SG)$ is a normed linear space (NLS). Let $\{g_1, \dots, g_n\}$ be a linearly independent subset of $\mathcal{C}_\beta(SG)$. Further, for a bounded below and Lebesgue integrable function $f : SG \to \mathbb{R}$, we define

$$\mathcal{Y}_f^\beta := \Big\{h \in span\{g_1, \dots, g_n\} : h(t) \leq f(t) \text{ for all } t \in SG\Big\}.$$

In view of [70], the above set is non-empty. A function $h_f \in \mathcal{Y}_f^\beta$ is said to be a best one-sided approximation from below to f on SG if

$$\int_{SG} h_f \, d\mu_p = \sup\left\{\int_{SG} h \, d\mu_p : h \in \mathcal{Y}_f^\beta\right\}.$$

Similarly, best one-sided approximation from above can be defined, see, for instance, [66].

Theorem 2.3.5. *Let $f \in \mathcal{B}(SG)$. Then a best one-sided approximant in $\mathcal{Y}_f^\alpha$ from below to f on SG exists.*

Proof. Define $A = \sup\left\{\int_{SG} h \, d\mu_p : h \in \mathcal{Y}_f^\beta\right\}$. Using a basic result of real analysis [73], there exists a sequence $(h_m) \in \mathcal{Y}_f^\beta$ such that

$$\int_{SG} h_m \, d\mu_p \to A \text{ as } m \to \infty. \tag{2.3.11}$$

Now for some positive number M_{SG}, we obtain

$$\begin{aligned}\int_{SG} |h_m| \, d\mu_p &= \int_{SG} |h_m - A + A| \, d\mu_p \\ &\leq \int_{SG} |h_m - A| \, d\mu_p + \int_{SG} A \, d\mu_p \leq M_{SG}.\end{aligned} \tag{2.3.12}$$

Recall that any $A \subset X$ in finite dimensional NLS is closed and bounded if and only if A is compact. Since $\mathcal{Y}_f^\beta$ is compact, we have a subsequence $\{h_{m_k}\}$ such that $\{h_{m_k}\}$ converges to h in $\mathcal{L}^1(SG)$. Recall that on a finite dimensional linear space, all norms are equivalent. Since $\mathcal{H}_n^\alpha(SG)\big(\supseteq \mathcal{Y}_f^\beta\big)$ is finite dimensional, it follows that the subsequence $\{h_{m_k}\}$ also converges to h uniformly. Since $h_m(x) \leq f(x) \ \forall \ x \in SG$ and $h_{m_k} \to h$ uniformly, we get $h(x) \leq f(x), \forall x \in SG$. Thus $h \in \mathcal{Y}_f^\beta$. Using (2.3.11), we have

$$\int_{SG} h \, d\mu_p = \lim_{k\to\infty} \int_{SG} h_{m_k} \, d\mu_p = A.$$

This completes the proof.

□

Theorem 2.3.6. *Let $f \in \mathcal{B}(SG)$. Then we can choose fractal piecewise harmonic function of best one-sided approximant from above to f on SG.*

Proof. The proof is omitted as it is similar to the previous proof. □

Definition 2.3.7. The set valued mapping $P_S : Y \to \mathcal{P}(Y)$ is defined by

$$P_S(y) = \{s \in S : d(y, s) = d(y, S)\},$$

where the distance function $d(\cdot, S) : Y \to [0, \infty)$ is defined by

$$d(y, S) = \inf\{d(y, t) : t \in S\}.$$

Here the set P_S is called the metric projection of y contained in S. Let us denote the cardinality of $P_S(y)$ by $|P_S(y)|$. Then S is said to be proximinal set if $|P_S(y)| \geq 1 \; \forall \; y \in Y$ and Chebyshev set if $|P_S(y)| = 1 \; \forall \; y \in Y$. If $t \in P_S(y)$, then the element t is called a best approximation to y from S.

There are many notions and techniques useful in functional analysis that arise from the study of best approximation. A substantial amount of research on best approximation in Banach spaces has been done since 1970, since [74] was published. Next, we collect some examples showing that proximinal set does not always exist.

Example 2.3.1. *If $X = \mathbb{R}^2$ with the l_1-norm $\|(x, y)\| = |x| + |y|$ and $K = \{(x, y) : y = \pm 2x\}$. Then for $x = (1, 1) \in \mathbb{R}^2$, we get $P_K(x) = \left\{\left(\frac{1}{2}, 1\right)\right\}$, and for $x = (0, 1) \in \mathbb{R}^2$, we get $P_K(x) = \left\{\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right)\right\}$. Hence, K is not Chebyshev.*

Example 2.3.2. *Let $M = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. If $c \in \mathbb{R}^2 \backslash M$, then the nearest point in M to c is the unique foot of the perpendicular that is drawn from c to the y-axis. Thus M is a Chebyshev set.*

Example 2.3.3. *Assume that $M = \mathbb{R}^2 \backslash B[0, 1], \mathbb{R}^2$ endowed with the Euclidean norm. Since M is open, no point $x \in B[0, 1] = \mathbb{R}^2 \backslash M$ has a nearest point. Therefore, M is not proximinal.*

Example 2.3.4. *Let $M = \mathbb{R}^2 \backslash B(0, 1) \subseteq \mathbb{R}^2$ endowed with the Euclidean norm. It is straightforward to verify that for each $c \in B(0, 1) \backslash \{0\}$,*

$$P_M(c) = \{c/\|c\|\}$$

and $P_M(0) = \{t \in \mathbb{R}^2 : \|t\| = 1\}$. Therefore, M is proximinal but not Chebyshev set.

The following theorem discusses several essential aspects of the approximation class $\mathcal{H}_n^\alpha(SG)$. We refer the reader to the book [75] for a detailed description of various definitions that are used in Theorem 2.3.9.

Proposition 2.3.8. *Let us note the following:*

1. *([75], P. 440). Let Y be a NLS and W be a non-empty approximately compact subset of Y, then the metric projection $P_W : Y \to W$ is upper semicontinuous (u.s.c), with compact values.*

2. *([75], P. 434). Let Z, W be topological spaces and W be Hausdorff. If the set valued map $S : Z \to W$ is u.s.c with compact values, then S is closed, that*

is, for each net z_λ in Z, $z_\lambda \to z_0$, $w_\lambda \in S(z_\lambda)$, and $w_\lambda \to w_0$. It follows that $w_0 \in S(z_0)$.

Theorem 2.3.9. *Let $\mathcal{H}_n^\alpha(SG)$ be the space of all the fractal polynomials. Then*

(i) $\mathcal{H}_n^\alpha(SG)$ is a proximinal subset of $\mathcal{C}(SG)$. In fact, $\mathcal{H}_n^\alpha(SG)$ is strongly proximinal.

(ii) For each $f \in \mathcal{C}(SG)$, the set of best approximants $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is convex and closed. In particular, $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is weakly closed.

(iii) The metric projection $P_{\mathcal{H}_n^\alpha(SG)} : \mathcal{C}(SG) \to \mathcal{H}_n^\alpha(SG)$ is upper semicontinuous, closed, and locally bounded.

(iv) $P_{\mathcal{H}_n^\alpha(SG)}$ has a first Baire class selector $\mathcal{J} : \mathcal{C}(SG) \to \mathcal{H}_n^\alpha(SG)$, whose set of points of norm discontinuity is an F_σ-set of the first category in SG.

Proof. Firstly from $\mathcal{H}_n^\alpha(SG) = \mathcal{F}^\alpha(\mathcal{H}_n(SG))$ and $\mathcal{F}^\alpha$ is a linear map. It is clear that $\mathcal{H}_n^\alpha(SG)$ is finite dimensional. Further every subspace whose dimension is finite of a NLS are proximinal (see, for instance, [75]). This leads to the conclusion that $\mathcal{H}_n^\alpha(SG)$ is proximinal. It is possible to demonstrate that every subspace of NLS, which is of finite dimension, is strongly proximinal by applying the notion of compactness given in [76], and thus $\mathcal{H}_n^\alpha(SG)$ is strongly proximinal.

Note that $\mathcal{H}_n^\alpha(SG)$ is finite dimensional and proximinal, therefore $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is closed. Since $\mathcal{H}_n^\alpha(SG)$ is convex, it follows that $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is convex for each f. Using the fact that $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is closed and convex subset of a NLS, we get $P_{\mathcal{H}_n^\alpha(SG)}(f)$ is weakly closed (see [75]). Proceeding to the next step, we show that $\mathcal{H}_n^\alpha(SG)$ is approximately compact. Let $f \in \mathcal{C}(SG)$ and $h_m^\alpha \in \mathcal{H}_n^\alpha(SG)$, we shall call the sequence $\{h_m^\alpha\}$ a minimizing sequence for f, that is, $\|h_m^\alpha - f\|_\infty \to d(f, \mathcal{H}_n^\alpha(SG))$ as $m \to \infty$. Then one gets $M \in \mathbb{N}$ such that for $m \geq M$,

$$\|h_m^\alpha\|_\infty \leq \|h_m^\alpha - f\|_\infty + \|f\|_\infty \leq 1 + d(f, \mathcal{H}_n^\alpha(SG)) + \|f\|_\infty := A_1.$$

Since $\mathcal{H}_n^\alpha(SG)$ is finite-dimensional, from the above-mentioned inequality, it follows that h_m^α is the member of compact set defined by the inequality $\|g\|_\infty \leq A_2 := \max\left\{\|h_1^\alpha\|_\infty, \|h_2^\alpha\|_\infty, \ldots, \left\|h_{M-1}^\alpha\right\|_\infty, A_1\right\}$. Hence, with the help of compact argument, one can choose a subsequence $\{h_{m_k}^\alpha\}$ such that $h_{m_k}^\alpha \to h^\alpha$ for some $h^\alpha \in \mathcal{H}_n^\alpha(SG)$ and hence $\mathcal{H}_n^\alpha(SG)$ approximately compact. Therefore, item 1 of Proposition 2.3.8 allows us to deduce that the metric projection $P_{\mathcal{H}_n^\alpha(SG)}$ is $u.s.c$ with $P_{\mathcal{H}_n^\alpha(SG)}(f)$ compact for each f. Since $\mathcal{H}_n^\alpha(SG)$ is Hausdorff, item 2 of Proposition 2.3.8 yields that the multifunction $P_{\mathcal{H}_n^\alpha(SG)} : \mathcal{C}(SG) \to \mathcal{H}_n^\alpha(SG)$ is closed. It is simple to verify that a metric projection is locally bounded. For the completeness, we shall give some explanation. Let $f^* \in \mathcal{C}(SG)$. Consider the ball $B(f^*, r)$ centred at f^*, where $f \in B(f^*, r)$ and the radius $r > 0$, and $h^\alpha \in P_{\mathcal{H}_n^\alpha(SG)}(f)$. Then

$$\begin{aligned}
\|h^\alpha\|_\infty &\leq \|h^\alpha - f\|_\infty + \|f - f^*\| + \|f^*\|_\infty \\
&= d(f, \mathcal{H}_n^\alpha(SG)) + \|f - f^*\| + \|f^*\|_\infty \\
&\leq d(f^*, \mathcal{H}_n^\alpha(SG)) + 2\|f - f^*\| + \|f^*\|_\infty \\
&\leq d(f^*, \mathcal{H}_n^\alpha(SG)) + 2r + \|f^*\|_\infty \\
&:= A_3,
\end{aligned} \tag{2.3.13}$$

demonstrating the local boundedness of $P_{\mathcal{H}_n^\alpha(SG)}$.

Note that every NLS of finite dimension is reflexive, and the Radon-Nikodym property holds for every reflexive space. Furthermore, by a result of Jayne-Rogers ([77], Theorem 7), it implies that $P_{\mathcal{H}_n^\alpha(SG)}$ has a first Baire class selector $\mathcal{J}$. In fact, its set of discontinuities is an F_σ- set of first category in $\mathcal{C}(SG)$. □

2.4 Some Remarks on the Class of Polynomials on SG

In this section, we will point out that several distinct properties exist between the polynomials defined on any interval of the real line and polynomials defined on SG.

Let $p_i : D \subset \mathbb{R} \to \mathbb{R}$ be the polynomials of degree $\leq n_i$. Then $\dim(p_i) = \dim(D)$.

In $\big($ [33], Theorem 3.5$\big)$, the following theorem contradicts the above result on the space of the polynomials defined on SG also known as multi-harmonic space.

Theorem 2.4.1. *If $f \in \mathcal{C}(SG)$ and $E(f) < \infty$, then*

$$\frac{\log 3}{\log 2} \leq \dim_H(Gr(f)) \leq \overline{\dim}_B(Gr(f) \leq \frac{\log(108/5)}{2\log 2}.$$

The polynomial of degree n defined on the interval has at most n zeros. In the multi-harmonic spaces, also known as polynomials defined on SG may have infinite zeros. Every polynomial defined on the compact interval is of bounded variation. But this property does not hold in the case of polynomials defined on SG. This follows from the two interesting notions of bounded variation. One is demonstrated by Verma et al. in $\big($[33], Remark 4.13$\big)$, and the other one is proposed by Ruiz et al. in $\big($[78], Theorem 5.2$\big)$. Recall that on SG any non-constant piecewise harmonic function is not of bounded variation.

Let $p, q : [a, b] \to \mathbb{R}$ be two polynomials with degree $\leq n$, defined on the compact interval $[a, b]$, then the degree of pq is $\leq n^2$, whereas, every non-constant polynomial p defined on SG, i.e., p is a harmonic function in the domain of Δ, then p^2 is not in the domain of Δ, see, for instance, $\big($[79], Corollary 3.3$\big)$. Regardless of this significant disadvantage, we can overcome it by defining a distinct Laplacian with a distinct measure. Kusuoka [80, 81] defined a measure for which the domain of its Laplacian is closed under multiplication. However, the Kusuoka measure is not a self-similar measure and that makes it significantly more difficult to study.

2.5 Dimensional Results

In [5, 25, 38, 55, 82], there are various approaches available for estimating and computing the fractal dimension of a set. In this section, we provide some bounds for the dimension of the graph of an α-fractal function. Bounds on box dimension of graph of univariate α-fractal function is estimated by Navascués and her collaborators in [25, 82]. Their research inspired us for the following results presented in this chapter. .

In this chapter, we use method of variation/oscillation to estimate some bounds on fractal dimension of graph of α- fractal function at N-th level on SG with α as variable scaling vector. Therefore, we denote

$$\begin{aligned}
D_{\boldsymbol{\omega}} &= L_{\boldsymbol{\omega}}(SG), \text{ where } \boldsymbol{\omega} = \omega^1 \dots \omega^N \text{ and } \omega^i \in I,\\
D_{\omega_1\omega_2\dots\omega_m} &= L_{\omega_1\omega_2\dots\omega_m}(SG), \text{ where } \omega_k = \omega_k^1 \dots \omega_k^N \text{ and } \omega_k^i \in I,\\
L_{\omega_1\omega_2\dots\omega_m} &= L_{\omega_1} \circ \dots \circ L_{\omega_m}.
\end{aligned}$$

Now we define

$$\begin{aligned}
G_{\boldsymbol{\omega}} &= \{(t, f^{\alpha}(t) : t \in L_{\boldsymbol{\omega}}(SG) \text{ and } \boldsymbol{\omega} \in I^N\} \text{ and}\\
\bar{\alpha}_{\boldsymbol{\omega}} &= \max\{\|\alpha_{\boldsymbol{\omega}}\|_\infty : t \in L_{\boldsymbol{\omega}}(SG) \text{ and } \boldsymbol{\omega} \in I^N\},\\
\alpha_{\max} &= \max\{\|\alpha_{\boldsymbol{\omega}}\|_\infty : \boldsymbol{\omega} \in I^N\},\\
\alpha_{\min} &= \min\{\|\alpha_{\boldsymbol{\omega}}\|_\infty : \boldsymbol{\omega} \in I^N\}.
\end{aligned}$$

For $k \in \mathbb{N}$, we define $\bar{\alpha}_{\omega_k} = \max\{\|\alpha_{\boldsymbol{\omega_k}}\|_\infty : t \in L_{\omega_k}(SG) \text{ and } \omega_k \in I^N\}$.

Remark 2.5.1. *Let $\bar{\gamma} = \sum_{\boldsymbol{\omega} \in I^N} \bar{\alpha}_{\boldsymbol{\omega}}$. For $m \in \mathbb{N}$, one gets*

$$\bar{\gamma}^m = \sum_{\omega_k \in I^N} \bar{\alpha}_{\omega_m} \dots \bar{\alpha}_{\omega_1}.$$

Lemma 2.5.1. *Let f^α be the FIF constructed using the functions f, $\alpha_{\boldsymbol{\omega}}$, b as above. If the functions f, $\alpha_{\boldsymbol{\omega}}$, b are Hölder continuous with exponents $\sigma_f, \sigma_\alpha, \sigma_b$ and Hölder constants K_f, K_α, K_b respectively, then*

$$\begin{aligned} OSC_{F_{\omega_m}}\left(G_{\omega_1\omega_2\dots\omega_{m-1}}\right) \leq & \left\|\alpha_{\omega_m}\right\|_\infty \; OSC_{f^\alpha}\left(L_{\omega_1\omega_2\dots\omega_{m-1}}(SG)\right) \\ & + \frac{K_b\left\|\alpha_{\omega_m}\right\|_\infty + K_\alpha\;\left(\left\|b\right\|_\infty + \left\|f^\alpha\right\|_\infty\right)}{2^{N\sigma(m-1)}} + \frac{K_f}{2^{N\sigma}}, \end{aligned} \tag{2.5.1}$$

where $\sigma = \min\{\sigma_f, \sigma_\alpha, \sigma_b\}$.

Proof. Note that

$$F_{\boldsymbol{\omega}_1}(t, x) = \alpha_{\boldsymbol{\omega}_1}(t)x + f(L_{\boldsymbol{\omega}_1}(t)) - \alpha_{\boldsymbol{\omega}_1}(t)b(t).$$

Taking $t, u \in SG$, one obtains

$$\begin{aligned} |F_{\omega_1}(t, f^\alpha(t)) - F_{\omega_1}(u, f^\alpha(u))| = & \big|\alpha_{\omega_1}(t)f^\alpha(t) + f(L_{\omega_1}(t)) - \alpha_{\omega_1}(t)b(t) \\ & - \alpha_{\omega_1}(u)f^\alpha(u) - f(L_{\omega_1}(u)) + \alpha_{\omega_1}(u)b(u)\big| \\ \leq & \left|\alpha_{\omega_1}(t)\right| \left|f^\alpha(t) - f^\alpha(u)\right| + \left|\alpha_{\omega_1}(t)\right| \left|b(t) - b(u)\right| \\ & + \left(\left|b(u)\right| + \left|f^\alpha(u)\right|\right) \left|\left(\alpha_{\omega_1}(t) - \alpha_{\omega_1}(u)\right)\right| \\ & + \left|f(L_{\omega_1}(t)) - f(L_{\omega_1}(u))\right| \\ \leq & \left\|\alpha_{\omega_1}\right\|_\infty \left|f^\alpha(t) - f^\alpha(u)\right| + K_b\left\|\alpha_{\omega_1}\right\|_\infty \left\|t - u\right\|_2^{\sigma_b} \\ & + K_f\left\|L_{\omega_1}(t) - L_{\omega_1}(u)\right\|. \end{aligned} \tag{2.5.2}$$

Therefore, we have

$$|F_{\omega_1}(t, f^\alpha(t)) - F_{\omega_1}(u, f^\alpha(u))| \leq \left\|\alpha_{\omega_1}\right\|_\infty \mathrm{OSC}_{f^\alpha}(SG) + \left\|\alpha_{\omega_1}\right\|_\infty K_b + K_\alpha \left(\left\|b\right\|_\infty + \left\|f^\alpha\right\|_\infty\right) + \frac{K_f}{2^{N\sigma}}. \tag{2.5.3}$$

Performing iteration, we have

$$\mathrm{OSC}_{F_{\omega_m}}\left(G_{\omega_1\omega_2\ldots\omega_{m-1}}\right) \leq \left\|\alpha_{\omega_m}\right\|_\infty \mathrm{OSC}_{f^\alpha}\left(L_{\omega_1\omega_2\ldots\omega_{m-1}}(SG)\right) + \frac{K_b\left\|\alpha_{\omega_m}\right\|_\infty + K_\alpha \left(\left\|b\right\|_\infty + \left\|f^\alpha\right\|_\infty\right)}{2^{N\sigma(m-1)}} + \frac{K_f}{2^{N\sigma}}. \tag{2.5.4}$$

This completes the proof. □

Theorem 2.5.2. *The following properties hold for f^α:*

(1). If $\alpha_{\min} > \frac{1}{2^{N\sigma}}$, then $\frac{\log 3}{\log 2} \leq \dim_H(Gr(f^\alpha)) \leq \underline{\dim}_B(Gr(f^\alpha)) \leq \overline{\dim}_B(Gr(f^\alpha)) \leq 1 + \frac{\log(\bar{\gamma})}{N \log 2}$.

(2). If $\frac{1}{2^{N\sigma}} < \alpha_{max} < 1$, then $\frac{\log 3}{\log 2} \leq \dim_H(Gr(f^\alpha)) \leq \underline{\dim}_B(Gr(f^\alpha)) \leq \overline{\dim}_B(Gr(f^\alpha)) \leq 1 + \frac{\log 3}{\log 2} + \frac{\log(\alpha_{\max})}{N \log 2}$.

Proof. Note that f^α satisfies the following equation:

$$f^\alpha(t) = F_{\boldsymbol{\omega}}\left(L_{\boldsymbol{\omega}}^{-1}(t), f^\alpha(t)\right) \ \ \forall\, t \in L_{\boldsymbol{\omega}}(SG) \ \ \textit{and}\ \boldsymbol{\omega} \in I^N.$$

Let $t, u \in L_{\omega_1}(SG)$. Then we have

$$\begin{aligned} |f^\alpha(t) - f^\alpha(u)| &= |F_{\omega_1}\big(L_{\omega_1}^{-1}(t), f^\alpha(t)\big) - F_{\omega_1}\big(L_{\omega_1}^{-1}(u), f^\alpha(u)\big)| \\ &\leq \sup_{\tilde{t},\tilde{u} \in L_{\omega_1}(SG)} |F_{\omega_1}\big(L_{\omega_1}^{-1}(\tilde{t}), f^\alpha(\tilde{t})\big) - F_{\omega_1}\big(L_{\omega_1}^{-1}(\tilde{u}), f^\alpha(\tilde{u})\big)| \qquad (2.5.5) \\ &= \mathrm{OSC}_{F_{\omega_1}}\big[Gr(f^\alpha)\big]. \end{aligned}$$

Hence, for any $t, u \in L_{\omega_1}(SG)$,

$$\mathrm{OSC}_{f^\alpha}\big[L_{\omega_1\omega_2\ldots\omega_m}\big(SG)\big] \leq \mathrm{OSC}_{F_{\omega_m}}\big[G_{\omega_1\omega_2\ldots\omega_{m-1}}\big].$$

Using the previous lemma, one obtains

$$\begin{aligned} &\mathrm{OSC}_{f^\alpha}\big[L_{\omega_1\omega_2\cdots\omega_m}\big(SG)\big] \\ &\quad \leq \|\alpha_{\omega_m}\|_\infty \|\alpha_{\omega_{m-1}}\|_\infty \ldots \|\alpha_{\omega_1}\|_\infty \mathrm{OSC}_{f^\alpha}[SG] \\ &\qquad + K_b\bigg[\frac{\|\alpha_{\omega_m}\|_\infty}{2^{(m-1)N\sigma}} + \frac{\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty}{2^{(m-2)N\sigma}} + \cdots + \|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty \ldots \|\alpha_{\omega_1}\|_\infty\bigg] \\ &\qquad + K_f\bigg[\frac{1}{2^{mN\sigma}} + \frac{\|\alpha_{\omega_m}\|_\infty}{2^{(m-1)N\sigma}} + \frac{\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty}{2^{(m-2)N\sigma}} \\ &\qquad + \cdots + \frac{\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty \ldots \|\alpha_{\omega_2}\|_\infty}{2^{N\sigma}}\bigg] \\ &\qquad + K_\alpha\big(\|b\|_\infty + \|f^\alpha\|_\infty\big)\bigg[\frac{1}{2^{(m-1)N\sigma}} + \frac{\|\alpha_{\omega_m}\|_\infty}{2^{(m-2)N\sigma}} + \frac{\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty}{2^{(m-3)N\sigma}} \\ &\qquad + \cdots + \big(\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty \ldots \|\alpha_{\omega_2}\|_\infty\big)\bigg]. \end{aligned} \tag{2.5.6}$$

Now we are well-equipped to estimate the fractal dimension of $Gr(f^\alpha)$.

(1). If $\alpha_{min} > \frac{1}{2^{N\sigma}}$, then we get

$$
\begin{aligned}
&\mathrm{OSC}_{f^\alpha}\big[L_{\omega_1\omega_2\ldots\omega_m}(SG)\big] \\
&\quad \le \|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_1}\|_\infty \mathrm{OSC}_{f^\alpha}[SG] \\
&\qquad + K_b\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_1}\|_\infty\left[1+\frac{1}{\alpha_{\min}2^{N\sigma}}+\cdots+\frac{1}{\alpha_{\min}^m 2^{mN\sigma}}\right] \\
&\qquad + \frac{K_f\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_2}\|_\infty}{2^{N\sigma}}\left[1+\frac{1}{\alpha_{\min}2^{N\sigma}}+\cdots+\frac{1}{\alpha_{\min}^{m-1} 2^{(m-1)N\sigma}}\right] \\
&\qquad + K_\alpha\big(\|b\|_\infty+\|f^\alpha\|_\infty\big)\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_2}\|_\infty\Bigg[1+\frac{1}{\alpha_{\min}2^{N\sigma}}+\ldots \\
&\qquad \cdots+\frac{1}{\alpha_{\min}^{m-1} 2^{(m-1)N\sigma}}\Bigg] \\
&\quad \le \|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_1}\|_\infty\Bigg[\mathrm{OSC}_{f^\alpha}[SG] \\
&\qquad + \left(K_b+\frac{K_f}{\alpha_{\min}2^{N\sigma}}+\frac{K_\alpha\big(\|b\|_\infty+\|f^\alpha\|_\infty\big)}{\alpha_{\min}}\right)\left(\frac{1}{1-\frac{1}{\alpha_{\min}2^{N\sigma}}}\right)\Bigg].
\end{aligned}
$$

Finally, we have

$$
\mathrm{OSC}_{f^\alpha}\big[L_{\omega_1\omega_2\ldots\omega_m}(SG)\big] \le R_1\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_1}\|_\infty,
$$

where $R_1 = \mathrm{OSC}_{f^\alpha}[SG]+\left(K_b+\frac{K_f}{\alpha_{\min}2^{N\sigma}}+\frac{K_\alpha\left(\|b\|_\infty+\|f^\alpha\|_\infty\right)}{\alpha_{\min}}\right)\left(\frac{1}{1-\frac{1}{\alpha_{\min}2^{N\sigma}}}\right)$. Now Lemma 1.6.2 yields

$$
\begin{aligned}
N_\delta(Gr(f^\alpha)) &\le 2.3^{Nm}+2^{Nm}\sum_{\omega\in I^N}\mathrm{OSC}_{f^\alpha}\big[L_{\omega_1\omega_2\ldots\omega_m}(SG)\big] \\
&\le 2.3^{Nm}+2^{Nm}\sum_{\omega\in I^N}R_1\|\alpha_{\omega_m}\|_\infty\|\alpha_{\omega_{m-1}}\|_\infty\ldots\|\alpha_{\omega_1}\|_\infty \qquad (2.5.7) \\
&\le 2.3^{Nm}+2^{Nm}R_1\bar{\gamma}^m,
\end{aligned}
$$

where $\bar{\gamma} = \sum_{\boldsymbol{\omega} \in I^N} \bar{\alpha}_{\boldsymbol{\omega}}$. Furthermore, we derive that

$$\begin{aligned}\overline{\lim}_{\delta \to 0} \frac{\log(N_\delta(Gr(f^\alpha)))}{-\log(\delta)} &\leq \overline{\lim}_{m \to \infty} \frac{\log(2.3^{Nm} + 2^{Nm} R_1 \bar{\gamma}^m)}{Nm \log(2)} \\ &= \overline{\lim}_{m \to \infty} \frac{\log(2^{Nm} R_1 \bar{\gamma}^m) + \log\left(1 + \frac{2.3^{Nm}}{2^{Nm} R_1 \bar{\gamma}^m}\right)}{Nm \log 2}.\end{aligned} \tag{2.5.8}$$

Using $\bar{\gamma} = \sum_{\boldsymbol{\omega} \in I^N} \bar{\alpha}_{\boldsymbol{\omega}}$, we get $\bar{\gamma} \geq 3^N \alpha_{\min}$. Combining inequality $\bar{\gamma} \geq 3^N \alpha_{\min}$ and $N > N\sigma$ with the above inequality (i.e. $\alpha_{min} > \frac{1}{2^{N\sigma}}$). It gives $\frac{3^N}{\bar{\gamma} 2^N} < 1$. Therefore,

$$\overline{\lim}_{\delta \to 0} \frac{\log(N_\delta(Gr(f^\alpha)))}{-\log(\delta)} \leq 1 + \frac{\log(\bar{\gamma})}{N \log 2}.$$

(2). If $\frac{1}{2^{N\sigma}} < \alpha_{\max} < 1$, then we get

$$\begin{aligned}&\text{OSC}_{f^\alpha}\left[L_{\omega_1\omega_2\ldots\omega_m}(SG)\right] \\ &\quad \leq \alpha_{\max}^m \text{OSC}_{f^\alpha}[SG] \\ &\qquad + K_b \alpha_{\max}^m \left[1 + \frac{1}{\alpha_{\max} 2^{N\sigma}} + \cdots + \frac{1}{\alpha_{\max}^m 2^{mN\sigma}}\right] \\ &\qquad + \frac{K_f \alpha_{\max}^{m-1}}{2^{N\sigma}} \left[1 + \frac{1}{\alpha_{\max} 2^{N\sigma}} + \cdots + \frac{1}{\alpha_{\max}^{m-1} 2^{(m-1)N\sigma}}\right] \\ &\qquad + K_\alpha \big(\|b\|_\infty + \|f^\alpha\|_\infty\big) \alpha_{\max}^{m-1} \left[1 + \frac{1}{\alpha_{\max} 2^{N\sigma}} + \cdots + \frac{1}{\alpha_{\max}^{m-1} 2^{(m-1)N\sigma}}\right] \\ &\quad \leq \alpha_{max}^m \left[\text{OSC}_{f^\alpha}[SG] + \left(K_b + \frac{K_f}{\alpha_{\max} 2^{N\sigma}} + \frac{K_\alpha\big(\|b\|_\infty + \|f^\alpha\|_\infty\big)}{\alpha_{\max}}\right)\left(\frac{1}{1 - \frac{1}{\alpha_{\max} 2^{N\sigma}}}\right)\right].\end{aligned}$$

Hence,

$$\text{OSC}_{f^\alpha}\left[L_{\omega_1\omega_2\ldots\omega_m}(SG)\right] \leq R_2 \alpha_{max}^m,$$

where $R_2 = \mathrm{OSC}_{f^\alpha}[SG] + \left(K_b + \frac{K_f}{\alpha_{\max}2^{N\sigma}} + \frac{K_\alpha\left(\|b\|_\infty + \|f^\alpha\|_\infty\right)}{\alpha_{\max}}\right)\left(\frac{1}{1-\frac{1}{\alpha_{\max}2^{N\sigma}}}\right)$.
Further, Lemma 1.6.2 yields

$$\begin{aligned} N_\delta(Gr(f^\alpha)) &\leq 2.3^n + 2^n \sum_{\omega \in I^n} \mathrm{OSC}_{f^\alpha}\left[L_{\omega_1\omega_2\ldots\omega_m}(SG)\right] \\ &\leq 2.3^{Nm} + 2^{Nm}.3^{Nm} R_2 \alpha_{\max}^m. \end{aligned} \tag{2.5.9}$$

Now it follows that

$$\begin{aligned} \overline{\lim}_{\delta\to 0} \frac{\log(N_\delta(Gr(f^\alpha)))}{-\log(\delta)} &\leq \overline{\lim}_{m\to\infty} \frac{\log(2.3^{Nm} + 2^{Nm}.3^{Nm} R_2 \alpha_{\max}^m)}{Nm\log(2)} \\ &= \overline{\lim}_{m\to\infty} \frac{\log(2^{Nm}.3^{Nm} R_2\alpha_{\max}^m) + \log\left(1 + \frac{2.3^{Nm}}{2^{Nm}.3^{Nm}R_2\alpha_{\max}^m}\right)}{Nm\log 2}. \end{aligned} \tag{2.5.10}$$

Plugging inequality $2^N > 2^{N\sigma}$ with the aforementioned inequality (i.e. $\frac{1}{2^{N\sigma}} < \alpha_{\max} < 1$)), we get

$$\overline{\lim}_{\delta\to 0} \frac{\log(N_\delta(Gr(f^\alpha)))}{-\log(\delta)} \leq 1 + \frac{\log 3}{\log 2} + \frac{\log(\alpha_{\max})}{N\log 2}.$$

Hence, we complete the proof. □

Remark 2.5.2. *Theorem 2.5.2 can be compared with Theorem 4.3 in the cited article [83]. In the aforementioned Theorem 2.5.2, we consider α as a variable scaling vector and provide a proof for the Nth level of SG. However, in the following theorem of [83], α is a constant scaling vector and is proven for the 1st level of SG.*

Theorem 2.5.3 ([83], Theorem 4.3)**.** *Let f and b be Hölder continuous functions on SG with exponent σ_f and σ_b, respectively, and the interpolation points are not coplanar. Let f^α be the α-fractal function corresponding to f. Let $\phi = \sum_{i=1}^{i=3}|\alpha_i|$ and $\tilde{\sigma} = \min\{\sigma_f, \sigma_b\}$. Then the box dimension of $Gr(f^\alpha)$ has following bounds:*

(1) If $\frac{\phi 2^{\tilde{\sigma}}}{3} \leq 1$, then $\frac{\log 3}{\log 2} \leq \dim_B(Gr(f^\alpha)) \leq 1 - \tilde{\sigma} + \frac{log3}{log2}$.

(2) If $\frac{\phi 2^{\tilde{\sigma}}}{3} > 1$*, then* $\frac{\log 3}{\log 2} \leq \dim_B(Gr(f^\alpha)) \leq 1 + \frac{log\phi}{log2}$.

2.6 Conclusions

We have constructed α-fractal function $f^\alpha \in \mathcal{C}(SG)$ corresponding to $f \in \mathcal{C}(SG)$ through suitable IFS. Some properties of the fractal operator $\mathcal{F}^\alpha : \mathcal{C}(SG) \to \mathcal{C}(SG)$ that maps a continuous function f to its fractal analogue f^α are also highlighted. Furthermore, graphs of f^α are plotted for different values of parameters to observe the continuous dependence of the graph of f^α on parameters. This chapter also provides estimates of the box dimension of the graph of α-fractal functions on SG by considering appropriate parameters. We have also demonstrated that α-fractal functions preserve certain properties of the original function. Furthermore, we have proven the existence of a fractal harmonic function, the best one-sided approximant from below to the original function on SG.

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