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Finite-time stabilization of stochastic spin-$\frac{1}{2}$ system via strict Lyapunov control

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ABSTRACT

Finite-time stabilization of open quantum systems is essential for advancing time-critical applications in quantum computing, communication, and metrology. This paper addresses the problem of stabilizing a two-level open quantum system — modeled by the quantum stochastic master equation (SME) — to a target eigenstate within a finite time. We develop a continuous-time measurement-based feedback control strategy rooted in strict Lyapunov theory, which ensures practical finite-time convergence of the system state in the mean-square sense. The control law is explicitly constructed as a nonlinear state-feedback function of the density matrix, and we derive sufficient conditions for finite-time convergence using Lyapunov bounds. Furthermore, we show that when these conditions are relaxed, the same control law yields exponential stability. The proposed method also provides an explicit settling-time function that quantifies the time to convergence in terms of initial conditions and system parameters. Numerical simulations validate the theoretical results and demonstrate the effectiveness of the proposed approach compared to exponential stabilization method.

1. Introduction

The control of quantum systems [1–3], subject to continuous measurement, is a critical area of research in quantum technologies. These systems are often described by the quantum stochastic master equation [4–6]. This equation models the evolution of quantum states under continuous-time measurements, yielding solutions known as quantum trajectories. The behavior of these trajectories under various conditions has been the subject of extensive research [7–10], crucial for applications ranging from quantum computation to metrology.

The quantum stochastic master equation (SME) naturally decomposes into three fundamental components: (i) the unitary evolution generated by the system's free Hamiltonian, (ii) the deterministic dissipative dynamics described by Lindblad operators [11,12], and (iii) the stochastic terms representing measurement back-action. While the Hamiltonian governs the coherent dynamics of the isolated system, the Lindblad operators model the average effect of environmental interactions, and the stochastic terms capture the random fluctuations induced by continuous measurement. This tripartite structure presents unique challenges for quantum feedback control [13], requiring strategies that simultaneously: (i) account for the system's intrinsic Hamiltonian evolution, (ii) compensate for dissipative effects, and (iii) mitigate measurement-induced disturbances. Only through such comprehensive control can quantum systems be effectively stabilized to the desired target state.

Finite-time stabilization [14–17], where the goal is to bring the system to a target state within a fixed time horizon, is especially relevant for practical quantum applications like quantum computing [18] and real-time quantum communication [19]. Traditional control methods focus on asymptotic stabilization [20–26], which is insufficient for tasks requiring timely state preparation. Therefore, novel control techniques are necessary to ensure finite-time convergence despite the challenges posed by environmental interactions and measurement back-action. Continuous-time measurement-based control provides a promising approach for finite-time stabilization. By continuously monitoring the system and applying real-time feedback, the system can be dynamically guided towards the target state within a fixed time frame.

In the context of feedback control, various methods have been developed in [6] to design feedback controllers that stabilize quantum systems. For example, a quantum feedback controller has been proposed to globally stabilize a quantum spin-$\frac{1}{2}$ system even in the presence of imperfect measurements. This controller employs a global Lyapunov function to ensure stability. Further advancements include the construction of switching feedback controllers for N-level quantum angular momentum systems using stochastic Lyapunov techniques

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and the development of continuous feedback controllers that ensure robust stabilization by driving the system away from undesirable eigenstates [7]. In [27], a filter based bang–bang like control is proposed for target state stabilization of time delay quantum systems.

Recent studies have demonstrated exponential stability for specific target subspaces in N-level quantum systems under open-loop control and perfect measurement conditions [28]. It also demonstrates that incorporating measurements can enhance the convergence rate. Proportional output feedback has also been shown to achieve stochastic exponential stability in two-level open quantum systems in [29]. Recently, in [30], exponential feedback stabilization of a two-level system is developed, providing a lower bound of convergence rate. In [31], the same controller is modified for improved rapid convergence. These existing works primarily concentrate on asymptotic stabilization, addressing how the system behaves as time progresses towards infinity. However, these studies typically do not provide explicit bounds or estimates on the time required for convergence. Compared to asymptotic or exponential stabilization, where the system only approaches the desired state as time tends to infinity, finite-time stabilization offers a significantly stronger and more practical guarantee: the system reaches a target neighborhood of the desired state within a finite, explicitly computable *settling time*. This property is particularly critical in quantum technologies, where operations must be completed within strict coherence windows before decoherence dominates the system dynamics. As such, finite-time control not only enhances convergence speed and robustness, but also enables predictable timing essential for the reliable execution of quantum tasks, such as state preparation, measurement, and error correction.

A recent study by [32] discusses the finite-time control of a closed two-level quantum system. The derived methodology, however, is not applicable to open quantum systems. Additionally, the expression for the settling time provided is often not valid for all initial conditions. Most recently, a similar notion of switching based finite-time stabilization is proposed for Markovian open quantum system considering Markovian master equation in [33], whereas our methodology is proposed for SME. These gaps in the analysis can be significant for applications where understanding and optimizing the time of convergence is crucial for performance and control efficiency.

This paper aims to advance the existing research by addressing the finite-time stabilization control of open quantum systems. We propose a novel control strategy that guarantees rapid convergence to specified quantum states within a fixed time horizon. Given that weak measurement-based controlled quantum systems can be categorized as stochastic systems, this study aims to bridge this gap by developing finite-time stabilization techniques tailored for quantum systems. The proposed approach extends the classical finite-time stabilization theory into the quantum domain. Since the target eigenstate in our case lies on the boundary of the Bloch sphere, we have utilized the idea of practical finite-time stability [34,35] to avoid switching at the boundary. Our contributions are threefold: Firstly, we establish the theoretical foundations for finite-time stabilization using continuous-time measurement-based control. Secondly, we design and analyze a feedback controller that achieves the desired state within a finite time, providing explicit bounds on the convergence time. Lastly, we explore the practical implications of our findings through numerical simulations using the Kraus operator, demonstrating the effectiveness of our proposed methods. Following are the main contributions:

1. Developed Lyapunov-based sufficient conditions that ensure practical finite-time stability of weakly measured quantum systems. These conditions rigorously bound the system's behavior within a small neighborhood of the target state in finite time.
2. Designed and implemented a control law capable of steering a spin-$\frac{1}{2}$ system from an arbitrary initial state to a desired target eigenstate in finite time. The controller exploits the system's intrinsic structure and ensures rapid state convergence.
3. Derived an explicit settling time function, which provides a closed-form estimate for the finite convergence time.
4. Verified the proposed control methodology with numerical simulations, demonstrating its effectiveness. Moreover, compared our finite-time strategy against established exponential stabilization method, highlighting significant improvements in convergence speed and the settling-time function bound.

Table 1
List of symbols and notation.

Symbol	Description
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of positive real numbers
i	Imaginary unit ($i^2 = -1$)
$\mathrm{tr}(A)$	Trace of a square matrix A
$[X, Y]$	Commutator of operators X and Y, defined as $XY - YX$
$X^\dagger$	Hermitian adjoint (conjugate transpose) of matrix X
$\mathrm{sign}(x)$	Signum function: $\frac{x}{\lvert x\rvert}$ for $x \neq 0$, and 0 for $x = 0$
$\mathbb{P}\{\cdot\}$	Probability of an event (probability measure)
$\mathbb{E}[\cdot]$	Expectation of a random variable
$\lVert \cdot \rVert$	Euclidean norm of a vector
sup	Supremum (least upper bound) over a set
∇	Gradient operator

The remainder of this paper is organized as follows. Section 2 provides an overview of the mathematical preliminaries, including the quantum stochastic master equation and problem formulation. Section 3 introduces the proposed finite-time control strategies and presents the theoretical guarantees of their performance as well as the exponential stability derivation from the same control with relaxed conditions. Section 4 offers a detailed analysis of the control through numerical simulations. Finally, Section 5 concludes the paper with discussions on future research directions.

Notations. The primary mathematical notations used throughout the paper are summarized in Table 1.

2. Preliminaries

2.1. Stability notions for stochastic system

Here, we provide relevant definitions and notions of stochastic stability needed for our control formulation.

Definition 1 (*Convex $\mathcal{K}_\infty$ Function*)**.** A function $\gamma : [0,\infty) \to [0,\infty)$ belongs to the class of convex $\mathcal{K}_\infty$ functions if:

(i) γ is continuous, strictly increasing, and unbounded,
(ii) $\gamma(0) = 0$,
(iii) γ is convex on $[0,\infty)$.

Consider the system described as

$$dq = G(q)dt + H(q)dW \tag{1}$$

where $q \in \mathbb{R}^n$ is the system state, W is an independent standard Wiener process of dimension r, $G : \mathbb{R}^n \to \mathbb{R}^n$ and $H : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are continuous such that $G(0) = 0$ and $H(0) = 0$. The solution of (1) is denoted as $q(t, q_0)$ where q_0 is the initial state at $t = t_0$. The equilibrium $q = 0$ of system (1) is said to be

(i) *Stable in Probability [36]*: The trivial solution $q(t) \equiv 0$ is said to be stable in probability if for all $\epsilon > 0$,

$$\lim_{\lVert q_0\rVert \to 0} \mathbb{P}\left\{ \sup_{t\geq 0} \lVert q(t; q_0)\rVert \geq \epsilon \right\} = 0.$$

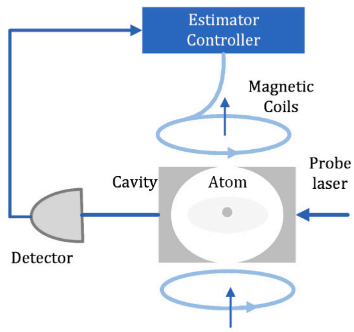

Fig. 1. Spin system under continuous measurement.

(ii) *Exponential stability [36]*: if there exist constants $k > 0$ and $r > 0$, such that

$$\mathbb{E}[\|q\| \mid q_0] \leq k\|q_0\|e^{-rt}$$

where $\mathbb{E}[\|q\| \mid q_0]$ denotes the expectation of $\|q\|$ for the given initial condition q_0.

(iii) *Practical Finite-Time Stable in Mean Square [34]*: if for all $q(t_0) = q_0$ there exist a constant $\epsilon > 0$ and a settling time $\bar{T}(\epsilon, q_0)$ such that $\mathbb{E}[\|q(t, q_0)\|^2] < \epsilon$ for all $t > t_0 + \bar{T}$.

To provide sufficient conditions for practical finite-time stability of the stochastic system, following Lemma is needed.

Lemma 1 (*[34]*). *For the system $\dot{q}(t) = G(q(t), u)$, if there exist a twice continuously differentiable function $\zeta(q(t)) \in C^2$; positive constants k, δ, $\sigma \in (0, 1)$; and $\mathcal{K}_\infty$-functions γ_1 and γ_2 such that*

$$\begin{cases} \gamma_1(\|q(t)\|) \leq \zeta(q(t)) \leq \gamma_2(\|q(t)\|) \\ \zeta(q(t)) - \zeta(q(s)) \leq -k\int_s^t \zeta^\sigma(q(v))dv + \delta(t-s) \end{cases} \tag{2}$$

with $0 \leq s \leq t$. Then, $\|q(t)\| < \epsilon$, $\forall\ t > \Upsilon$, where $\Upsilon = \frac{1}{(1-\sigma)\beta k}\left[\zeta^{1-\sigma}(q(0)) - \left(\frac{\delta}{(1-\beta)k}\right)^{(1-\sigma)/\sigma}\right]$, $\epsilon = \gamma_1^{-1}[(\frac{\delta}{(1-\beta)k})^{\frac{1}{\sigma}}]$ and $\beta \in (0, 1)$.

Theorem 1 (*[36]*). *Consider a function $V(q) \in C^2$, with respect to q. Assume there exist strictly positive constants l_1, l_2 and l_3, such that*

1. $\mathcal{L}V \leq -l_3\|q\|$,
2. $l_1\|q\| \leq V(q) \leq l_2\|q\|$.

where $\mathcal{L}$ denotes the infinitesimal generator, then the solution $q = 0$ of system (1) is exponentially stable.

2.2. System description

In this paper, we consider a finite-dimensional, two-level open quantum system subject to continuous measurement as shown in Fig. 1. The qubit system is fixed inside a cavity and interacts with the confined electromagnetic field. The outgoing field is continuously measured using the homodyne measurement process. The detailed experimental set-up can be studied in [37]. The overall dynamics of the quantum system leads to the following stochastic master equation

$$\begin{aligned} d\rho &= \left(-\frac{i}{\hbar}[\mathcal{H}, \rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right)dt \\ &+ \sqrt{\eta}\left(L\rho + \rho L^\dagger - \mathrm{tr}(L\rho + L^\dagger\rho)\rho\right)dW \\ &=: g(\rho)dt + h(\rho)dW \end{aligned} \tag{3}$$

where $\mathcal{H} = \frac{1}{2}w_{12}\sigma_z + \frac{u}{2}\sigma_y$ is the Hamiltonian operator on the Hilbert space $\mathscr{H}$. u denotes the feedback control such that $u := u(\rho)$. L is the only Lindblad operator on $\mathscr{H}$ given as $L = \sqrt{\Gamma}\sigma_z/2$ where $\Gamma > 0$ is the strength with which light and atoms interact. $\eta = (0, 1]$ denotes the efficiency of detector. W denotes the standard Wiener process. The given SME exhibits a unique solution within the space of density matrices

$$\mathcal{D} = \{\rho \mid \rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{tr}(\rho) = 1\}$$

Now, each measurement signal follows the following output relationship

$$dY = dW + \sqrt{\eta}\,\mathrm{tr}(L\rho + L^\dagger\rho)dt \tag{4}$$

Each density operator ρ can be expressed as Bloch sphere coordinates using the relation

$$\rho = \frac{1}{2}(\mathbf{I} + x\sigma_x + y\sigma_y + z\sigma_z) \tag{5}$$

where $\sigma_x, \sigma_y, \sigma_z$ denotes the Pauli matrices along x, y and z axis respectively and $\mathbf{I}$ is a 2×2 identity matrix. We can further write (3) using (5) as

$$dx = \left(-w_{12}y - \frac{\Gamma x}{2} + uz\right)dt - \sqrt{\eta\Gamma}xzdW \tag{6a}$$

$$dy = \left(w_{12}x - \frac{\Gamma y}{2}\right)dt - \sqrt{\eta\Gamma}yzdW \tag{6b}$$

$$dz = -uxdt + \sqrt{\eta\Gamma}(1 - z^2)dW \tag{6c}$$

This completes the description of the spin-$\frac{1}{2}$ system.

The infinitesimal generator associated with the system (3) is denoted as $\mathcal{L}$. Assume twice continuously differentiable function in ρ and once in t given as $V(\rho, t) : \mathcal{D} \times \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\mathcal{L}V(\rho, t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \rho}g(\rho) + \frac{1}{2}\frac{\partial^2 V}{\partial \rho^2}(h(\rho), h(\rho)) \tag{7}$$

Using Ito's formula [5], we have

$$dV(\rho, t) = \mathcal{L}V(\rho, t)dt + \frac{\partial V(\rho, t)}{\partial \rho}h(\rho)dW$$

3. Main results

The following Lemma offers sufficient conditions for ensuring the practical finite-time stability of stochastic systems. While this system was studied in [34], the authors did not provide specific sufficient conditions.

Lemma 2. *Let $V(q)$ be a positive definite twice continuously differentiable function. Assume there exist constants $c > 0, \delta > 0$, $0 < \alpha < 1$ and K_∞ convex functions K_1 and K_2 such that*

1. $K_1(\|q\|^2) \leq V(q) \leq K_2(\|q\|^2)$
2. $\mathcal{L}V(q) \leq -cV^\alpha + \delta$,

then there exists a real number $\bar{T}$ and a constant $\epsilon > 0$ such that the solution $q(t)$ of system (1) is practical finite-time stable in mean square with $\epsilon = K_1^{-1}\left[\left(\frac{\delta}{(1-\beta)c}\right)^{\frac{1}{\alpha}}\right]$ and $\mathbb{E}[\bar{T}] = \frac{1}{(1-\alpha)\beta c}\left[(\mathbb{E}[V(q(0))])^{1-\alpha} - \left(\frac{\delta}{(1-\beta)c}\right)^{\frac{1-\alpha}{\alpha}}\right]$, where $\beta \in (0, 1)$.

Proof. We define the stopping time

$$\tau_m := \inf\left\{t \geq 0 \,\middle|\, \|q(t)\| \geq m\right\}, \quad \text{for } m \in \mathbb{N}.$$

This ensures the solution remains in a compact set up to time τ_m, which guarantees that all stochastic integrals and expectations are well-defined.

Let $\bar{V}(q) := V(q)^{1-\alpha}$. Then, by Itô's formula, we compute

$$\begin{aligned} \mathcal{L}\bar{V}(q) &= (1-\alpha)V(q)^{-\alpha}\mathcal{L}V(q) \\ &\quad - \frac{1}{2}(1-\alpha)\alpha V(q)^{-\alpha-1}\,\mathrm{tr}\left[\nabla V(q)^T g(q)^T g(q)\nabla V(q)\right] \end{aligned}$$

$$\leq (1-\alpha)V(q)^{-\alpha}\mathcal{L}V(q), \quad \text{(since the trace term is nonnegative)}$$
$$\leq (1-\alpha)V(q)^{-\alpha}\left(-cV(q)^{\alpha}+\delta\right)$$
$$= -c(1-\alpha)+(1-\alpha)\delta V(q)^{-\alpha}.$$

Now, for $q \notin \Omega := \left\{ q \mid V(q) \leq \left(\frac{\delta}{(1-\beta)c}\right)^{1/\alpha} \right\}$, we have

$$V(q)^{-\alpha} \leq \left(\frac{(1-\beta)c}{\delta}\right) \Rightarrow (1-\alpha)\delta V(q)^{-\alpha} \leq c(1-\alpha)(1-\beta).$$

Hence,

$$\mathcal{L}\bar{V}(q) \leq -c(1-\alpha)\beta, \quad \text{for } q \notin \Omega.$$

Let us define the truncated stopping time

$$T_m := \bar{T} \wedge \tau_m,$$

where

$$\bar{T} := \inf \left\{ t \geq 0 \,\middle|\, V(q(t)) \leq \left(\frac{\delta}{(1-\beta)c}\right)^{1/\alpha} \right\}.$$

Applying Itô's formula over $[0, T_m]$

$$\mathbb{E}[\bar{V}(q(T_m))] = \mathbb{E}[\bar{V}(q_0)] + \mathbb{E}\left[\int_0^{T_m} \mathcal{L}\bar{V}(q(s))\,ds\right]$$
$$\leq \mathbb{E}[\bar{V}(q_0)] - c\beta(1-\alpha)\mathbb{E}[T_m].$$

Rearranging the above gives

$$\mathbb{E}[T_m] \leq \frac{1}{\beta c(1-\alpha)}\left(\mathbb{E}[V(q_0)^{1-\alpha}] - \mathbb{E}[V(q(T_m))^{1-\alpha}]\right).$$

Now, at $t = \bar{T}$, we have

$$V(q(\bar{T})) \leq \left(\frac{\delta}{(1-\beta)c}\right)^{1/\alpha} \Rightarrow \bar{V}(q(\bar{T})) = V(q(\bar{T}))^{1-\alpha} \leq \left(\frac{\delta}{(1-\beta)c}\right)^{\frac{1-\alpha}{\alpha}}.$$

So taking $m \to \infty$, i.e., $\tau_m \to \infty$ a.s., and applying monotone convergence theorem

$$\mathbb{E}[\bar{T}] \leq \lim_{m\to\infty} \mathbb{E}[T_m] \leq \frac{1}{\beta c(1-\alpha)}\left(\mathbb{E}[V(q_0)^{1-\alpha}] - \left(\frac{\delta}{(1-\beta)c}\right)^{\frac{1-\alpha}{\alpha}}\right).$$

Finally, since, $K_1(\|q\|^2) \leq V(q)$, we have $\mathbb{E}[K_1(\|q\|^2)] \leq \mathbb{E}[V(q)]$. Using Jensen's inequality (as K_1 is convex), we can write $K_1(\mathbb{E}[\|q\|^2]) \leq \mathbb{E}[K_1(\|q\|^2)]$ and hence,

$$K_1\left(\mathbb{E}[\|q(t)\|^2]\right) \leq \mathbb{E}[V(q(t))] \leq \left(\frac{\delta}{(1-\beta)c}\right)^{1/\alpha},$$

so that

$$\mathbb{E}[\|q(t)\|^2] \leq K_1^{-1}\left(\left(\frac{\delta}{(1-\beta)c}\right)^{1/\alpha}\right) = \epsilon, \quad \forall t \geq \bar{T}.$$

This concludes the proof. □

Based on the previous Lemma, we will derive the control law that strictly follows the Lyapunov based sufficient conditions for practical finite-time convergence.

3.1. State feedback controller design

In this section, we develop a finite-time controller designed to steer a quantum system's state into close proximity to the target eigenstate within a finite time $\bar{T}$. We refer to this as a *strict Lyapunov control* strategy, wherein the control law is designed such that the closed-loop system admits a Lyapunov function $V(\rho)$ satisfying the condition $\mathcal{L}V(\rho) \leq -cV^{\alpha} + \delta$, thereby ensuring practical finite-time convergence under stochastic dynamics. We have summarized the result in the following theorem.

Theorem 2. *Consider a two-level open quantum system defined using SME (3). Define a subset of $\mathcal{D}$*

$$\mathcal{S} := \left\{ \rho \in \mathcal{D} : |\mathrm{tr}(\rho\sigma_x)| \geq \frac{\Gamma_1 V^{\frac{\alpha}{2}}}{\Gamma_2 + V^{\frac{\alpha}{2}}} \right\}$$

We assume $\rho_0 \in \mathcal{S}$. The target state is an eigenstate of the free Hamiltonian defined as ρ_e such that $\rho_e = |1\rangle\langle 1|$ or $|0\rangle\langle 0|$. Let the Lyapunov function as

$$V(\rho) = 4(1-\mathrm{tr}(\rho\rho_e))^2 \tag{8}$$

with the feedback control $u(\rho)$ as $u = u_1 u_2$ such that

$$u_1 = -\mathit{sign}(\mathrm{tr}(i[\sigma_y,\rho]\rho_e))$$
$$u_2 = \frac{2a_3 + a_4 V^{\frac{\alpha}{2}}}{2a_1} \tag{9}$$

then there exists $a_1, a_2, a_3, a_4 > 0$, $0 < \alpha < 1$, $\Gamma_1 = \frac{a_1(a_4-2a_2)}{a_4}$, $\Gamma_2 = \frac{2a_3}{a_4}$ such that if

$$-u_2|\mathrm{tr}(i[\sigma_y,\rho]\rho_e)| \leq -a_1 u_2 + a_2 V^{\frac{\alpha}{2}} + a_3 \tag{10}$$

where $a_4 > 2a_2$, then u stabilizes system (3) practically in finite time to the target state ρ_e.

Proof. Before establishing practical finite-time convergence, we first ensure that the closed-loop stochastic master equation (SME) admits a unique strong solution for all $t \geq 0$. The closed-loop SME is globally Lipschitz continuous except on the set where $\mathrm{tr}(\rho\sigma_x) = 0$, due to the structure of the feedback control. However, the design of the control law ensures that the initial state $\rho_0 \in \mathcal{S}$ satisfies $|\,\mathrm{tr}(\rho\sigma_x)| \geq \frac{\Gamma_1 V^{\alpha/2}}{\Gamma_2 + V^{\alpha/2}} > 0$, and this inequality prevents the trajectory from remaining in a region arbitrarily close to this non-Lipschitz set. In particular, the lower bound implies that the dynamics avoid the set $\{\rho : \mathrm{tr}(\rho\sigma_x) = 0\}$ except possibly at isolated time instants. Even if at some isolated instant t_1, the condition $\mathrm{tr}(\rho(t_1)\sigma_x) = 0$ is met, it does not hinder the forward propagation of the system because the drift term corresponding to the free Hamiltonian (which is unaffected by the control) ensures nontrivial evolution of the state. Consequently, the trajectory leaves the non-Lipschitz set due to the nonzero influence of the free dynamics. Therefore, the overall system remains well-posed, and the SME admits a unique strong solution for all $t \geq 0$.

Now, to show practical finite-time convergence, we can write the infinitesimal generator of (8) as

$$\mathcal{L}V(\rho) = 2u\,\mathrm{tr}(i[\sigma_y,\rho]\rho_e)\sqrt{V} + 4\eta\Gamma\,\mathrm{tr}^2(\rho\rho_e)V \tag{11}$$

For the initial state $\rho_0 \in \mathcal{S}$, and u_1 following (9) we have

$$\mathcal{L}V = -2u_2|\mathrm{tr}(i[\sigma_y,\rho]\rho_e)|\sqrt{V} + 4\eta\Gamma\,\mathrm{tr}^2(\rho\rho_e)V$$

We can write condition (10) by substituting u_2 on the right-hand side as

$$|u_2\,\mathrm{tr}(\rho\sigma_x)| \geq \left(\frac{a_4}{2} - a_2\right)V^{\frac{\alpha}{2}} \tag{12}$$

This translates to the set where $\rho \in \mathcal{D}$ such that

$$|\mathrm{tr}(\rho\sigma_x)| \geq \frac{\Gamma_1 V^{\frac{\alpha}{2}}}{\Gamma_2 + V^{\frac{\alpha}{2}}} \tag{13}$$

where $\Gamma_1 = \frac{a_1(a_4-2a_2)}{a_4}$ and $\Gamma_2 = \frac{2a_3}{a_4}$. Thus, for $\rho \in \mathcal{S}$ where $\mathcal{S} := \left\{ \rho \in \mathcal{D} : |\mathrm{tr}(\rho\sigma_x)| \geq \frac{\Gamma_1 V^{\frac{\alpha}{2}}}{\Gamma_2 + V^{\frac{\alpha}{2}}} \right\}$, we can write (11) using (12) as

$$\mathcal{L}V \leq -2a_1\sqrt{V}u_2 + 2a_2 V^{\frac{1+\alpha}{2}} + 2a_3\sqrt{V} + 4\eta\Gamma\,\mathrm{tr}^2(\rho\rho_e)V$$
$$= (2a_2 - a_4)V^{\frac{1+\alpha}{2}} + 4\eta\Gamma\,\mathrm{tr}^2(\rho\rho_e)V \tag{14}$$

In the Bloch coordinates, since $x^2 + y^2 + z^2 \leq 1$, we have $4\,\mathrm{tr}^2(\rho\rho_e)V = (1-z^2)^2 \leq 1$, therefore

$$\mathcal{L}V \leq -cV^{\frac{1+\alpha}{2}} + \delta \tag{15}$$

where $c = a_4 - 2a_2$ and $\delta = \eta\Gamma$. Let $\rho_e = |1\rangle\langle 1|$, then $V(z) = (1+z)^2$. Further, we assign $(1+z) = \zeta$. Using Ito's formula, we can write

$$\mathbb{E}[V(\rho)] = \mathbb{E}[V(\rho_0)] + \int_0^t \mathbb{E}[\mathcal{L}V(\rho)]ds$$
$$\mathbb{E}[\zeta^2] \le \mathbb{E}[\zeta_0^2] + \int_0^t \{-c(\mathbb{E}[\zeta^{1+\alpha}])\}ds + \delta t$$

Thus, conditions 1 and 2 of Lemma 2 are satisfied, and hence we can write $\mathbb{E}[V] \le \epsilon \ \forall\ t \ge \bar{T}$, where $\bar{T} = \frac{2}{(1-\alpha)\beta c}\left[(\mathbb{E}[V(\rho_0)])^{\frac{1-\alpha}{2}} - \left(\frac{\delta}{(1-\beta)c}\right)^{\frac{1-\alpha}{1+\alpha}}\right]$, $\epsilon = \left(\frac{\delta}{(1-\beta)c}\right)^{\frac{2}{1+\alpha}}$ and $\beta \in (0,1)$. To show practical finite-time stability, we transform the states $(x,y,z)^T$ to $X = (x,y,\sqrt{1+z})^T$. We again set $\tilde{\zeta} = \sqrt{1+z}$. Then, using the Jensen's inequality, we have,

$$(\mathbb{E}[\tilde{\zeta}^2])^2 \le \mathbb{E}[\tilde{\zeta}^4]. \tag{16}$$

$$\begin{aligned} x^2 + y^2 + \tilde{\zeta}^2 &\le 2\tilde{\zeta}^2 + \tilde{\zeta}^2 \\ &= 3\tilde{\zeta}^2. \end{aligned} \tag{17}$$

Using (16), we can write, $\mathbb{E}[\tilde{V}] \le \sqrt{\epsilon}$, where $\tilde{V} = \tilde{\zeta}^2$. Further, we can write Finally, to see the boundedness of $\tilde{V}$, we can write $\tilde{V} \le (x^2 + y^2 + \tilde{\zeta}^2)$ and from (17), we have $\tilde{V} \ge \frac{(x^2+y^2+\tilde{\zeta}^2)}{3}$. Therefore, all the conditions of Lemma 2 are satisfied; we conclude $\mathbb{E}[\|X\|^2] < \epsilon' \ \forall\ t \ge \bar{T}$, where $\|X\| = \sqrt{x^2+y^2+\tilde{\zeta}^2}$.

This completes the proof. □

Remark 1. When $\rho_0 \notin \mathcal{S}$, the system is first perturbed using a bounded control. For two-level system, $u = \sin(w_{12}t)$ is applied for some time $t = [0, t')$ such that $\rho \in \mathcal{S}$. Then, for $t \ge t'$, the control is switched to (9) for reaching the target eigenstate.

Remark 2. Since the proposed control exhibits infinite switching near the target state, it is recommended to switch the control to $u = -\operatorname{tr}(i[\sigma_y, \rho]\rho_e)$ for $t > \bar{T}$.

3.2. Exponential stability

In this section, we will show that when condition (10) is not satisfied, then the quantum system (3) reaches the target state asymptotically. Consider the Lyapunov candidate as $V = \sqrt{1 - \operatorname{tr}(\rho\rho_e)}$, and its infinitesimal generator can be written as

$$\mathcal{L}V(\rho) = \frac{u\operatorname{tr}(i[\sigma_y,\rho]\rho_e)}{4V} - \frac{\eta\Gamma}{2}\operatorname{tr}^2(\rho\rho_e)V \tag{18}$$

Inserting control (9) with $\alpha \in \left[\frac{1}{2}, 1\right)$ we get

$$\begin{aligned} \mathcal{L}V &\le -\frac{|\operatorname{tr}(i[\sigma_y,\rho]\rho_e)|}{4V}(k_1 + k_2V^{2\alpha}) - \frac{\eta\Gamma}{2}\operatorname{tr}^2(\rho\rho_e)V \\ &\le \beta V^{2\alpha} - \frac{\eta\Gamma}{2}\operatorname{tr}^2(\rho\rho_e)V \end{aligned}$$

where $k_1 = \frac{a_3}{a_1}$, $k_2 = \frac{a_4}{4a_1}$ and $\beta = k_2$. Consider the set $\mathcal{D}_\lambda$ such that $\mathcal{D}_\lambda := \{\rho \in \mathcal{D} : 0 < \lambda < \operatorname{tr}(\rho\rho_e) \le 1\}$. Then for all $\rho \in \mathcal{D}_\lambda$, we can write

$$\begin{aligned} \mathcal{L}V &\le -\left(\frac{\eta\Gamma}{2}\lambda^2 - \beta V^{2\alpha-1}\right)V \\ &\le -\left(\frac{\eta\Gamma}{2}\lambda^2 - \beta(1-\lambda)^{\frac{2\alpha-1}{2}}\right)V \end{aligned}$$

If we select $\frac{\eta\Gamma\lambda^2}{2(1-\lambda)^{\frac{2\alpha-1}{2}}} > \beta > 0$, then $\mathcal{L}V$ is nonpositive. Here also, we use the transformed system coordinates as $X = (x, y, \sqrt{1+z})$. Then, we can see that V is lower bounded by $\frac{\|X\|}{\sqrt{3}}$ and upper bounded by $\|X\|$. Further, we can write $\mathcal{L}V \le -\kappa V \le -\frac{\kappa}{\sqrt{3}}\|X\|$, where $\kappa = \frac{\eta\Gamma}{2}\lambda^2 - \beta(1-\lambda)^{\frac{2\alpha-1}{2}}$. Therefore, using Theorem 1, exponential convergence to the target state ρ_e is directly obtained in the sense of the definition of exponential stability.

4. Simulation result and discussion

Consider a two-level system as defined in (3). We simulate the system in the Kraus form so that ρ remains in $\mathcal{D}$ given as

$$\begin{aligned} \rho + d\rho &= \frac{\mathbb{K}\rho\mathbb{K} + \frac{1-\eta\Gamma}{4}\sigma_z\rho\sigma_z dt}{\operatorname{tr}(\mathbb{K}\rho\mathbb{K} + \frac{1-\eta\Gamma}{4}\sigma_z\rho\sigma_z dt)} \\ \mathbb{K} &= \mathbf{I} - \left(\frac{i}{2}(w_{12}\sigma_z + u\sigma_y) + \frac{\Gamma}{8}\mathbf{I}\right)dt + \frac{\sqrt{\eta\Gamma}}{2}\sigma_z dY \\ dY &= dW + \sqrt{\eta\Gamma}\operatorname{tr}(\sigma_z\rho)dt \end{aligned} \tag{19}$$

This Kraus-type evolution framework arises from a discrete-time approximation of the stochastic master equation (SME) using quantum operations formalism. It provides a numerically stable method to simulate quantum trajectories by preserving the essential properties of the quantum state: complete positivity, Hermiticity, and unit trace.

The Kraus operator formalism is rooted in the theory of completely positive trace-preserving (CPTP) maps [38]. These maps describe the most general evolution of open quantum systems undergoing non-unitary dynamics, especially under measurement and decoherence processes. In our formulation, $\mathbb{K}$ plays the role of a measurement-dependent stochastic Kraus operator, capturing both the deterministic Hamiltonian evolution and the stochastic back-action due to continuous measurement.

By evolving the density matrix using this form, we ensure that $\rho(t) \in \mathcal{D}$ for all time t, which is critical for the physical realizability and correctness of the simulation. This simulation strategy has been widely adopted in quantum feedback control literature to approximate the dynamics of systems under continuous monitoring [5,7,13]. Therefore, the Kraus-based numerical evolution not only aligns with theoretical quantum measurement postulates but also provides a platform to validate feedback control strategies under realistic stochastic dynamics.

Let $|1\rangle\langle 1|$ be the target state. The Lyapunov function is taken as (8) that is, $V = (1+z)^2$. In the Bloch sphere, the target state is $(0,0,-1)$. We implement the control (9) with system parameters $w_{12} = 0, \eta = 0.2, \Gamma = 0.2$ and control parameters $\alpha = 0.8, a_1 = 0.3, a_2 = 1, a_3 = 3, a_4 = 3$. The settling time period using Lemma 2 is calculated as $T \approx 90$ a.u., and ϵ can be derived as 0.029. The simulation results show that the vicinity of the target state is reached within 20 a.u, which is consistent with the theoretical finding. Figs. 2(a) and 2(b) show the evolution of z and x axis trajectories of the Bloch sphere. Fig. 2(c) shows the evolution of the Lyapunov function V with time. Fig. 2(e) depicts the quantum state evolving in the Bloch sphere.

Another result is shown in Fig. 3 with initial condition $(0,0,1)$. Since this initial state does not lie in the set $\mathcal{S}$, we have first perturbed the state and then implemented feedback control (9) to steer it to the target state $(0,0,-1)$, with the system parameters $w_{12} = 0, \eta = 0.2, \Gamma = 0.8$ and control parameters $\alpha = 0.6, a_1 = 0.1, a_2 = 1, a_3 = 3, a_4 = 3$. It has been shown that the close vicinity of the target state has been reached in a finite time, and the time of convergence can be calculated using Lemma 2.

Additionally, to validate the analysis of Section 3.2, we have presented simulation results in Fig. 4 with control parameters $a_1 = 5, a_3 = 1, a_4 = 1, \alpha = 0.8, \lambda = 0.5$ and system parameters as $\eta = 0.8, \Gamma = 0.8$. The results verify that the system converges to the target state exponentially. The upper bound of the Lyapunov function in Fig. 4(b) satisfies the derived results. Since the strict Lyapunov condition of Lemma 2 is not satisfied, we cannot explicitly derive the upper bound of convergence time.

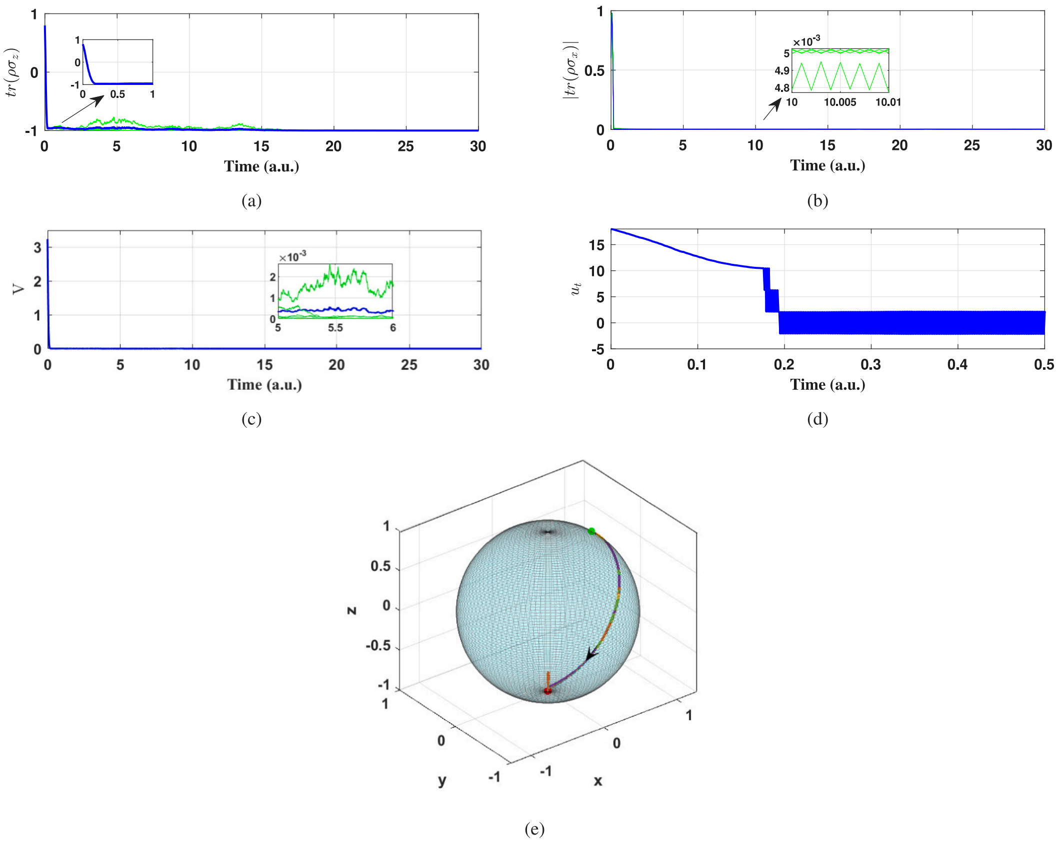

Fig. 2. Finite time stabilization of quantum system with initial state $(0.6, 0, 0.8)$ and target state $(0, 0, -1)$. The blue curve represents the mean value of all the green samples. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

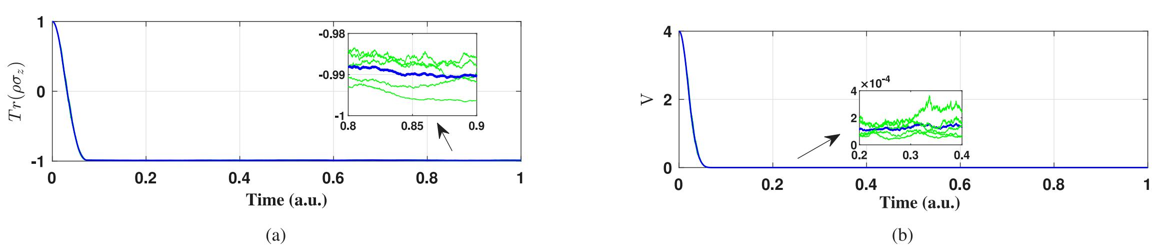

Fig. 3. Finite time stabilization of quantum system with initial state $(0, 0, 1)$ and target state $(0, 0, -1)$.

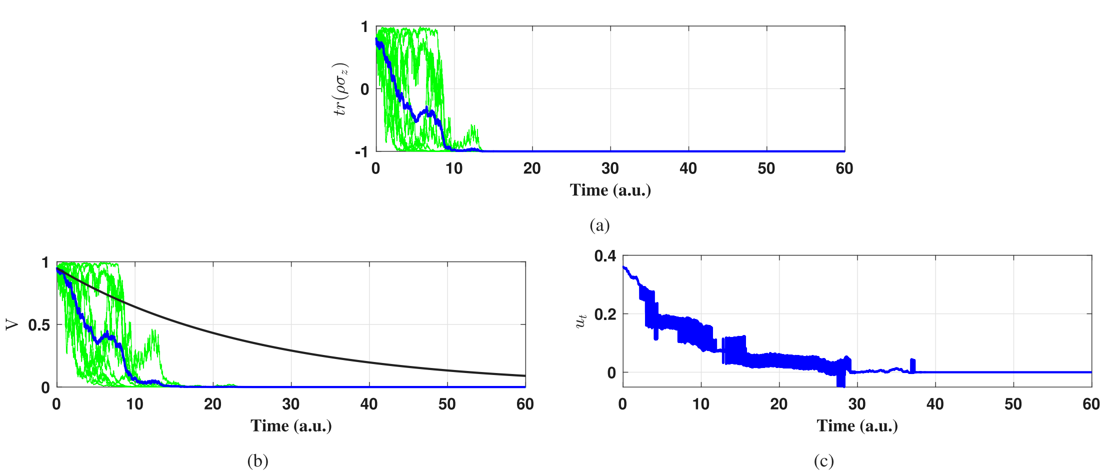

Fig. 4. Exponential stabilization of quantum system with initial state $(0.6, 0, 0.8)$ and target state $(0, 0, -1)$. The black curve represents the bound of the ensemble average of V.

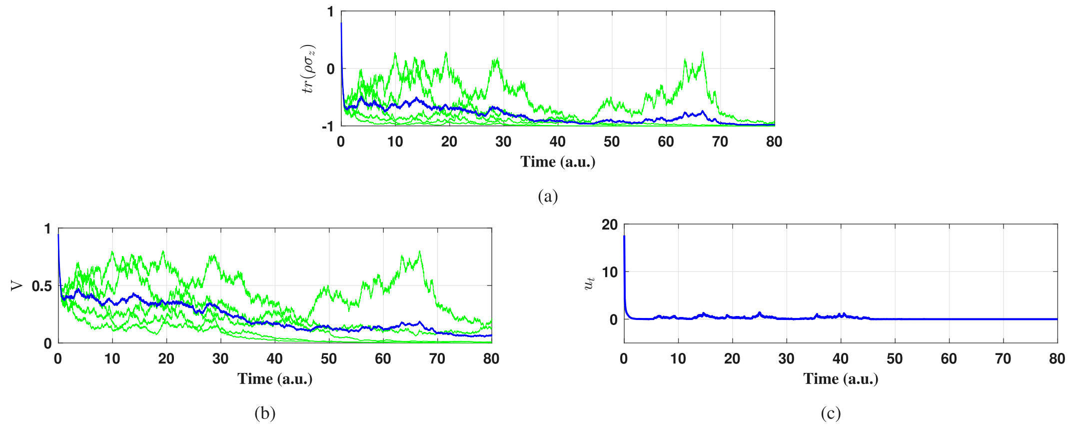

Fig. 5. Exponential stabilization [30] of quantum system with initial state $(0.6, 0, 0.8)$ and target state $(0, 0, -1)$.

Remark 3. The controller parameters can be appropriately tuned to ensure that condition (10) is satisfied, guaranteeing finite-time convergence to the target state. Nevertheless, even if the condition is not satisfied, the controller can still steer the state to the target state, but the finite time cannot be claimed.

Remark 4. Control parameters can be tuned as required. Parameter a_3 directly influences the control amplitude. A smaller value of α leads to high convergence speed. Similarly, the higher the value of $a_4 - 2a_2$, the faster the convergence. a_1 should be chosen small to satisfy condition (10).

For comparative evaluation, we implement the exponential stabilization controller proposed in [30], with its parameters carefully tuned to ensure that the control effort matches that of our proposed method. The corresponding results are shown in Fig. 5. As evident from the plots, the proposed control strategy achieves significantly faster and more accurate convergence to the target state compared to the method in [30]. Moreover, unlike [30], our approach provides an explicit and theoretically justified bound on the convergence time, highlighting its practical advantage and enhanced performance in time-critical applications.

5. Conclusion

In this paper, we have developed a framework for the finite-time stabilization of open quantum systems using continuous-time measurement-based control. Utilizing the quantum stochastic master equation, we have formulated control strategies that ensure convergence to the vicinity of the desired quantum state within a finite time. The results demonstrate that our approach can significantly enhance the efficiency of quantum state preparation. Future research will extend these methods to more complex quantum systems, such as a high-level quantum system, and pursue experimental validation.

CRediT authorship contribution statement

Eram Taslima: Writing – review & editing, Writing – original draft, Validation, Methodology, Conceptualization. **Yanan Liu:** Writing – review & editing, Methodology. **Shyam Kamal:** Supervision, Methodology. **R.K. Saket:** Visualization, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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