# Chapter 5

# Anti-synchronization of inertial neural networks with quaternion-valued and unbounded delays: Non-reduction and non-separation approach

## 5.1 Introduction

This chapter investigates the anti-synchronization problem for QVINNs with unbounded time delays. Using two different types of control strategies (feedback and adaptive controllers) and Lyapunov theory, various conditions are established to guarantee the anti-synchronization of QVINNs. Up to now, there are lots of works on stability, synchronization, etc., for QVINNs [124, 125, 126, 127, 128, 129, 130]. Most of these research works either adopted the separation approach, i.e., the systems under consideration are divided into four inertial RVNNs, or the considered second-order systems were transformed into the first order systems by applying some suitable variable substitution to reach the desired results. The separation eliminates the non-commutativity of quaternion multiplication. However, the separation and order-reduction approaches need many substitution processes, for example, more formulas and scaling, which may increase the dimensions of considered systems and, as a result, may cause theoretical complexity. Additionally, the present chapter has focused on the non-reduction and non-separation approach for QVINNs to preserve originality and improve computation complexity.

Recently, anti-synchronization results have been used in several areas, including information systems and communication processing [131, 132, 133]. When the sum of two signals converges to zero, anti-synchronization is observed. In chaotic systems, it is an observable phenomenon with significant practical implications. Using anti-synchronization to lasers, one may generate not only drop-outs in intensity as with ordinary low-frequency fluctuations but also short pulses of high intensity, which offer new ways of generating pulses of special shapes. The security and secrecy of digital signals can be strengthened by continuously transforming between synchronization and anti-synchronization when using anti-synchronization in communication systems. Most of the results available for QVINNs are based on the variable substitution approach, which reduces the order of the original second-order system into the first-order system. Here, the QVINNs have unbounded time-varying delays since neurons' actions are related to their previous states. Hence, the approach adopted here is more realistic and easy to deal with the QVINNs with unbounded time delays, which no researcher has yet considered. Finally, two numerical examples are provided to demonstrate the efficiency and effectiveness of the proposed approach. In the first example, the present theoretical results have been validated, whereas the second example contains the application of QVNNs related to associative memory to demonstrate their capacity to restore true color image patterns accurately.

## 5.2 Model Description and Preliminaries

Consider a class of QVINNs with unbounded time-varying delays as

$$\frac{d^2q_j(t)}{dt^2} = -\alpha_j\frac{dq_j(t)}{dt} - \beta_j q_j(t) + \sum_{l=1}^{n} a_{jl}F_l(q_l(t)) + \sum_{l=1}^{n} b_{jl}G_l(q_l(t-\tau_l(t))), \quad (5.1)$$

where $j \in I$, $q_j((t))$ represents the state of the $j$th neurons at a time $t$, the second-order derivative $\frac{d^2q}{dt^2}$ is called the inertial term, $\alpha_j, \beta_j \in \mathbf{R}^+$, $a_{jl}, b_{jl} \in \mathbf{H}$ are weight connection matrices, $F_l(q_l(t)), G_l(q_l(t-\tau_l(t))$ are the activation functions of the $l$-th neurons at a time $t$ and $t-\tau_l(t)$, $\tau_l(t) > 0$ is the time delay of $l$-th neuron.
The initial conditions of the proposed model (5.1) are given by $q_j(t_0) = \varsigma_j(t_0)$, $\dot{q}_j(t_0) = \varpi_j(t_0)$, $t_0 \in (-\infty, 0]$, where $\varsigma_j(t_0)$ and $\varpi_j(t_0)$ are continuous $\forall\ j \in I$.
Let the system (5.1) be the drive system, and the corresponding response system is

$$\begin{aligned}\frac{d^2v_j(t)}{dt^2} =& -\alpha_j\frac{dv_j(t)}{dt} - \beta_j v_j(t) + \sum_{l=1}^{n} a_{jl}F_l(v_l(t)) + \sum_{l=1}^{n} b_{jl}G_l(v_l(t-\tau_l(t))) \\ &+ U_j(t),\end{aligned} \quad (5.2)$$

where $u \in I$, $v_j((t)) \in \mathbf{H}$ represents the state of the $u$-th neurons at a time $t$ , $U_j(t)$ is an appropriate control input of $j$-th neuron at a time $t$.The other notations viz.,

$\alpha_j, \beta_j, a_{jl}, b_{jl}, F_l, G_l$ are same as model (5.1).

The initial conditions of model (5.2) are given as $v_j(t_0) = \tilde{\varsigma}_j(t_0), \quad \dot{v}_j(t_0) = \tilde{\varpi}_j(t_0), \quad t_0 \in (-\infty, 0]$, where $j \in I$, $\tilde{\varsigma}_j(t_0)$ and $\tilde{\varpi}_j(t_0)$ are continuous.

**Remark 5.2.1.** *The majority of the available results on synchronization, anti-synchronization, and stability results for QVINNs are based on the separation approach [124, 125, 126, 127, 128, 129, 130]. This will lead to an increase in variables and dimensions as well as an increase in computational complexity. Instead of using this approach, let us perform the task directly, which preserves the originality of the addressed systems and better satisfies the demands of practical applications.*

**Definition 5.2.1.** *Let there exist positive constants $M$ and $\lambda > 0$ such that for any two solutions*
$\varsigma_*(t_0) = (\varsigma_1(t_0), \varsigma_2(t_0), ..., \varsigma_n(t_0), \varpi_1(t_0), \varpi_2(t_0), ..., \varpi_n(t_0))^T$ *and*
$\varpi_*(t_0) = (\tilde{\varsigma}_1(t_0), \tilde{\varsigma}_2(t_0), ..., \tilde{\varsigma}_n(t_0), \tilde{\varpi}_1(t_0), \tilde{\varpi}_2(t_0), ..., \tilde{\varpi}_n(t_0))^T$. *Then the systems (5.1) and (5.2) are said to be globally exponential anti-synchronized if the following condition is satisfied.*

$$\|q(t) + v(t)\| \leq M\|\varsigma_*(t_0) + \varpi_*(t_0)\|e^{-\lambda t}, \qquad \forall t \geq 0.$$

**Lemma 5.2.1.** *[134] For any $x, y \in Q$, the two properties hold*
*(I) $\overline{x+y} = \overline{x} + \overline{y}$; (II) $\overline{xy} = \overline{y}\,\overline{x}$.*

**Lemma 5.2.2.** *[134] If $x, y \in Q$ and $\epsilon \in \mathbf{R}^+$, then*

$yx + \overline{yx} \leq \epsilon \overline{x} x + \frac{1}{\epsilon} y \overline{y}$.

**Lemma 5.2.3.** *[135] If the function $H(t)$ is uniformly continuous, bounded and $\lim_{t\to+\infty} \int_0^t H(t)dt$ exists, then $\lim_{t\to+\infty} H(t) = 0$.*

**Assumption 5.2.1.** *The time-varying delays $\tau_l(t),\ l \in I$ are unbounded, differentiable and satisfying $\dot{\tau}_l(t) \leq \tau < 1$, where $\tau$ is constant.*

**Assumption 5.2.2.** *For any $w \in I, \forall s_1, s_2 \in \mathbf{H}$, the nonlinear activation functions are odd, and there exist non-negative constants $K_l > 0,\ L_l > 0$ such that*

$$
\begin{aligned}
&|f_l(s_1) - f_l(s_2)| \leq K_l|s_1 - s_2|,\\
&|g_l(s_1) - g_l(s_2)| \leq L_l|s_1 - s_2|e^{-\tau_l(t)},
\end{aligned}
$$

*where $\tau_l(t)$ are the time delays.*

**Remark 5.2.2.** *It is important to note that Assumption 5.2.1 incorporates various types of time delay as a particular case, for example, constant discrete-delays [127, 128, 114], proportional delays [136], and discrete-time delays with bounds [124, 125, 126, 127, 128, 129, 130, 137, 138]. Therefore, the proposed model is more general and more flexible. On the other hand, studying exponential synchronization is typically challenging for unbounded discrete time delays. However, a novel theory is presented in this chapter for the activation function $g_l()$, which makes this challenge simple to resolve.*

## 5.3 Main results

This section focuses on the examination of exponential and adaptive antisynchronizations for the NNs (5.1) and (5.2), as opposed to the traditional reduced order and separation technique.

Firstly, the following appropriate linear feedback control system has been developed to address global exponential anti-synchronization as

$$U_j(t) = -\omega_j \varepsilon_j(t) - \gamma_j \dot{\varepsilon}_j(t), \quad j \in I, \tag{5.3}$$

where $\omega_j,\ \gamma_j \in \mathbf{R}^+$ are suitable control gains. Let the errors for anti-synchronization are $\varepsilon_j(t) = v_j(t) + q_j(t), j \in I$.

Then, the error system becomes

$$\begin{aligned}\frac{d^2\varepsilon_j}{dt^2} =& -(\alpha_j+\gamma_j)\frac{d\varepsilon_j}{dt} - (\beta_j+\omega_j)\varepsilon_j(t) + \sum_{l=1}^{n} a_{jl} f_l(\varepsilon_l(t)) \\ &+ \sum_{l=1}^{n} b_{jl} g_l(\varepsilon_l(t-\tau_l(t))), j \in I,\end{aligned} \tag{5.4}$$

where $f_l(\varepsilon_l(t)) = F_l(q_l(t)) + F_l(v_l(t)), \quad g_l(\varepsilon_l(t-\tau_l(t))) = G_l(q_l(t-\tau_l(t)) + G_l(v_l(t-\tau_l(t))$.

This section uses feedback and adaptive controllers to discuss the global antisynchronization of the systems (5.1) and (5.2).

**Theorem 5.3.1.** *Presume that the Assumption 5.2.1-5.2.2 hold, then the systems* (5.1) *and* (5.2) *are anti-synchronized on behalf of feedback controllers* (5.3) *if the following conditions hold.*

$\tilde{C}_j \leq 0, \qquad \tilde{A}_j \leq 0, \qquad \tilde{B}_j^2 \leq \tilde{A}_j \tilde{C}_j,$

*where* $\tilde{A}_j = 2\lambda + 2\lambda\mu_j^2 - 2\sigma_j\mu_j(\beta_j+\omega_j) + \sum_{l=1}^{n}\Big(\epsilon_1\sigma_l^2K_j^2 + \epsilon_2\sigma_l\mu_lK_j^2 + \frac{1}{\epsilon_2}\sigma_j\mu_j a_{jl}\overline{a}_{jl} + \frac{1}{\epsilon_4}\sigma_j\mu_j b_{jl}\overline{b}_{jl} + \frac{(\epsilon_3+\epsilon_4)L_j^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\Big),$

$\tilde{B}_j = 1 + 2\lambda\sigma_j\mu_j - \sigma_j\mu_j(\alpha_j+\gamma_j) + \mu_j^2 - \sigma_j^2(\beta_j+\omega_j),$

$\tilde{C}_j = 2\lambda\sigma_j^2 - 2\sigma_j^2(\alpha_j+\gamma_j) + 2\mu_j\sigma_j + \sum_{l=1}^n \sigma_j^2\big(\frac{1}{\epsilon_1}a_{jl}\overline{a}_{jl} + \frac{1}{\epsilon_3}b_{jl}\overline{b}_{jl}\big)$.

*Proof.* In order to prove this Theorem, let us construct the following Lyapunov function as

$$V(t) = \sum_{j=1}^n \overline{\varepsilon}_j((t))\varepsilon_j((t))e^{2\lambda t} + \sum_{j=1}^n \left(\overline{\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t)}\right)(\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t))\,e^{2\lambda t}$$
$$+ \sum_{j=1}^n\sum_{l=1}^n \frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\int_{t-\tau_l(t)}^t \overline{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda\theta}d\theta, \tag{5.5}$$

where $\sigma_j, \mu_j, \epsilon_3$, and $\epsilon_4$ are positive constants.

Taking derivative along (5.4), we have

$$\frac{dV(t)}{dt} = \sum_{j=1}^n \left(2\lambda\overline{\varepsilon}_j(t)\varepsilon_j(t) + \dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\right)e^{2\lambda t} + \sum_{j=1}^n \Bigg((2\lambda(\sigma_j\dot{\overline{\varepsilon}}_j(t)+$$
$$\mu_j\overline{\varepsilon}_j(t))\,(\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t))\,e^{2\lambda t} + (\sigma_j\ddot{\overline{\varepsilon}}_j(t)+\mu_j\dot{\overline{\varepsilon}}_j(t))(\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t))e^{2\lambda t}\Bigg)$$
$$+ \left(\sigma_j\dot{\overline{\varepsilon}}_j(t)+\mu_j\overline{\varepsilon}_j(t)\right)(\sigma_j\ddot{\varepsilon}_j(t)+\mu_j\dot{\varepsilon}_j(t))\,e^{2\lambda t}$$
$$+ \sum_{j=1}^n\sum_{l=1}^n \frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\Bigg(\overline{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t} - \overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))$$
$$\times (1-\dot{\tau}_l(t))e^{2\lambda(t-\tau_l(t))}\Bigg). \tag{5.6}$$
$$= \sum_{j=1}^n \Bigg[2\lambda\overline{\varepsilon}_j(t)\varepsilon_j(t) + \dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + 2\lambda(\sigma_j\dot{\overline{\varepsilon}}_j(t)+\mu_j\overline{\varepsilon}_j(t))\big(\sigma_j\dot{\varepsilon}_j(t)$$
$$+\mu_j\varepsilon_j(t))\Bigg]e^{2\lambda t} + \sum_{j=1}^n \sigma_j\Bigg\{-(\alpha_j+\gamma_j)\dot{\overline{\varepsilon}}_j(t) - (\beta_j+\omega_j)\overline{\varepsilon}_j(t) + \sum_{l=1}^n \overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}$$

$$+ \sum_{l=1}^n \overline{g}_l(\varepsilon_l(t-\tau_l(t)))\overline{b}_{jl}\Bigg\}\Bigg(\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t)\Bigg)e^{2\lambda t} + \sum_{j=1}^n \mu_j\dot{\varepsilon}_j(t)(\sigma_j\dot{\varepsilon}_j(t)$$
$$+\mu_j\dot{\overline{\varepsilon}}_j(t))\Bigg(\sigma_j\dot{\varepsilon}_j(t)+\mu_j\varepsilon_j(t)\Bigg) + \sum_{j=1}^n(\sigma_j\dot{\overline{\varepsilon}}_j(t)+\mu_j\overline{\varepsilon}_j(t))\sigma_j\Bigg\{-(\alpha_j+\gamma_j)\dot{\varepsilon}_j(t)$$

$$
\begin{aligned}
&- (\beta_j + \omega_j)\varepsilon_j(t) + \sum_{l=1}^{n} a_{jl} f_l((\varepsilon_l(t)) + \sum_{l=1}^{n} b_{jl} g_l((\varepsilon_l(t-\tau_l(t)))\Big\} + \sum_{j=1}^{n}(\sigma_j \bar{\dot{\varepsilon}}_j(t) \\
&+ \mu_j \bar{\varepsilon}_j(t))\mu_j \dot{\varepsilon}_j(t) + \sum_{j=1}^{n}\sum_{l=1}^{n} \frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\Big(\bar{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t} \\
&- \bar{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))(1-\dot{\tau}_l(t))e^{2\lambda(t-\tau_l(t))}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV(t)}{dt} =& e^{2\lambda t}\sum_{j=1}^{n}\Big[2\lambda\bar{\varepsilon}_j(t)\varepsilon_j(t) + \bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t) + \bar{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + 2\lambda\Big(\sigma_j^2\bar{\dot{\varepsilon}}_j\dot{\varepsilon}_j(t) + \sigma_j\mu_j\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t) \\
&+ \sigma_j\mu_j\bar{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + \mu_j^2\bar{\varepsilon}_j(t)\varepsilon_j(t)\Big) - \sigma_j^2(\alpha_j+\gamma_j)\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t) - \sigma_j^2(\beta_j+\omega_j) \\
&\times \bar{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + \sigma_j^2\sum_{l=1}^{n}\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\dot{\varepsilon}_j(t) + \sum_{l=1}^{n}\sigma_j^2\overline{g}_l(\varepsilon_l(t-\tau_l))\overline{b}_{jl}\dot{\varepsilon}_j(t) - \sigma_j\mu_j(\alpha_j \\
&+ \gamma_j)\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t) - \sigma_j(\beta_j+\omega_j)\bar{\varepsilon}_j(t)\varepsilon_j(t) + \sum_{l=1}^{n}\sigma_j\mu_j\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\varepsilon_j(t) \\
&+ \sum_{l=1}^{n}\sigma_j\mu_j\overline{g}_l(\varepsilon_l(t-\tau_l(t))\overline{b}_{jl}\varepsilon_j(t) + \mu_j\sigma_j\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t) + \mu_j^2\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t) - \sigma_j^2(\alpha_j \\
&+ \gamma_j)\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t) - \sigma_j^2(\beta_j+\omega_j)\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t) + \sum_{l=1}^{n}\sigma_j^2\bar{\dot{\varepsilon}}_j(t)a_{jl}f_l(\varepsilon_l(t)) \\
&+ \sum_{l=1}^{n}\sigma_j^2\bar{\dot{\varepsilon}}_j(t)b_{jl}g_l(\varepsilon_l(t-\tau_l(t))) - \mu_j\sigma_j(\alpha_j+\gamma_j)\bar{\varepsilon}_j(t)\dot{\varepsilon}_j(t) \\
&\times \mu_j\sigma_j(\beta_j+\omega_j)\bar{\varepsilon}_j(t)\varepsilon_j(t) + \sum_{l=1}^{n}\mu_j\sigma_j\bar{\varepsilon}_j(t)a_{jl}f_l(\varepsilon_l(t)) + \sum_{l=1}^{n}\mu_j\sigma_j\bar{\varepsilon}_j(t)b_{jl} \\
&\times g_l(\varepsilon_l(t-\tau_l(t))) + \sigma_j\mu_j\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t) + \mu_j^2\bar{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\Big] \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\Big(\bar{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t} - \bar{\varepsilon}_l(t-\tau_l(t)) \\
&\times \varepsilon_l(t-\tau_l(t))(1-\dot{\tau}_l(t))e^{2\lambda(t-\tau_l(t))}\Big). \qquad (5.7)
\end{aligned}
$$

Now, by the Assumption 5.2.1, one can write $-\frac{1-\tau_l(t)}{(1-\tau)} \leq -1$, and as a result equation (5.7) reduces to

$$
\frac{dV}{dt} \leq e^{2\lambda t}\sum_{j=1}^{n}\Big[\left(2\lambda + 2\lambda\mu_j^2 - 2\sigma_j\mu_j(\beta_j+\omega_j)\right)\bar{\varepsilon}_j(t)\varepsilon_j(t) + \Big(1 + 2\lambda\sigma_j\mu_j - \sigma_j\mu_j(\alpha_j
$$

$$
\begin{aligned}
&+\gamma_j+\mu_j^2-\sigma_j^2(\beta_j+\omega_j)\Big)\overline{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\Big(1+2\lambda\sigma_j\mu_j-\sigma_j^2(\beta_j+\omega_j)-\mu_j\sigma_j\\
&\times(\alpha_j+\gamma_j)+\mu_j^2\Big)\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\big(2\lambda\sigma_j^2-2\sigma_j^2(\alpha_j+\gamma_j)+2\mu_j\sigma_j\big)\overline{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)\\
&+\sum_{l=1}^{n}\sigma_j^2\left(\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\dot{\varepsilon}_j(t)+\overline{\dot{\varepsilon}}_j(t)a_{jl}f_l(\varepsilon_l(t))\right)+\sum_{l=1}^{n}\sigma_j^2\Big(\overline{g}_l(\varepsilon_l(t-\tau_l(t)))\overline{b}_{jl}\\
&\times\dot{\varepsilon}_j(t)+\overline{\dot{\varepsilon}}_j(t)b_{jl}g_l(\varepsilon_l(t-\tau_l(t)))\Big)+\sum_{l=1}^{n}\sigma_j\mu_j\Big(\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\varepsilon_j(t)+\overline{\varepsilon}_j(t)a_{jl}\\
&\times f_l(\varepsilon_l(t))\Big)+\sum_{l=1}^{n}\sigma_j\mu_j\Big(\overline{g}_l(\varepsilon_l(t-\tau_l(t)))\overline{b}_{jl}\varepsilon_j(t)+\overline{\varepsilon}_j(t)b_{jl}g_l(\varepsilon_l(t-\tau_l(t)))\Big)\Big]\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\overline{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t}-\sum_{j=1}^{n}\sum_{l=1}^{n}(\epsilon_3+\epsilon_4)L_l^2\\
&\times(\sigma_j^2+\sigma_j\mu_j)\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t)))e^{2\lambda(t-\tau_l(t))}.
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
\frac{dV}{dt}\leq e^{2\lambda t}\sum_{j=1}^{n}\Bigg[&\left(2\lambda+2\lambda\mu_j^2-2\sigma_j\mu_j(\beta_j+\omega_j)\right)\overline{\varepsilon}_j(t)\varepsilon_j(t)+\Big(1+2\lambda\sigma_j\mu_j-\sigma_j\mu_j(\alpha_j\\
&+\gamma_j+\mu_j^2-\sigma_j^2(\beta_j+\omega_j)\Big)\overline{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\Big(1+2\lambda\sigma_j\mu_j-\sigma_j^2(\beta_j+\omega_j)\\
&-\mu_j\sigma_j(\alpha_j+\gamma_j)+\mu_j^2\Big)\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\big(2\lambda\sigma_j^2-2\sigma_j^2(\alpha_j+\gamma_j)+2\mu_j\sigma_j\big)\\
&\times\overline{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\sum_{l=1}^{n}\sigma_j^2\Big(\epsilon_1\overline{f}_l(\varepsilon_l(t))f_l(\varepsilon_l(t))+\frac{1}{\epsilon_1}\overline{\dot{\varepsilon}}_j(t)a_{jl}\overline{a}_{jl}\dot{\varepsilon}_j(t)\Big)\\
&+\sum_{l=1}^{n}\sigma_j\mu_j\Big(\epsilon_2\overline{f}_l(\varepsilon_l(t))f_l(\varepsilon_l(t))+\frac{1}{\epsilon_2}\overline{\varepsilon}_j(t)a_{jl}\overline{a}_{jl}\varepsilon_j(t)\Big)\\
&+\sum_{l=1}^{n}\sigma_j^2\Big(\epsilon_3\overline{g}_l(\varepsilon_l(t-\tau_l(t)))g_l(\varepsilon_l(t-\tau_l(t)))+\frac{1}{\epsilon_3}\overline{\dot{\varepsilon}}_j(t)b_{jl}\overline{b}_{jl}\dot{\varepsilon}_j(t)\Big)
\end{aligned}
$$

$$
+\sum_{l=1}^{n}\sigma_j\mu_j\Big(\epsilon_4\overline{g}_l(\varepsilon_l(t-\tau_l(t)))g_l(\varepsilon_l(t-\tau_l(t)))+\frac{1}{\epsilon_4}\overline{\varepsilon}_j(t)b_{jl}\overline{b}_{jl}\varepsilon_j(t)\Big)\Bigg]
$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\overline{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t}-\sum_{j=1}^{n}\sum_{l=1}^{n}(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2$$
$$+\sigma_j\mu_j)\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_j(t-\tau_l(t)))e^{2\lambda(t-\tau_l(t))},$$

Again by using the Assumption 5.2.2, we have

$$\begin{aligned}\frac{dV}{dt}\leq e^{2\lambda t}\Bigg[&\sum_{j=1}^{n}\left(2\lambda+2\lambda\mu_j^2-2\sigma_j\mu_j(\beta_j+\omega_j)\right)\overline{\varepsilon}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}\Bigg(1+2\lambda\sigma_j\mu_j-\sigma_j\mu_j\\
&\times(\alpha_j+\gamma_j+\mu_j^2-\sigma_j^2(\beta_j+\omega_j)\Bigg)\overline{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}\Bigg(1+2\lambda\sigma_j\mu_j-\sigma_j^2(\beta_j\\
&+\omega_j)-\mu_j\sigma_j(\alpha_j+\gamma_j)+\mu_j^2\Bigg)\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}\left(2\lambda\sigma_j^2-2\sigma_j^2(\alpha_j+\gamma_j)\right.\\
&\left.+2\mu_j\sigma_j\right)\overline{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}\sum_{l=1}^{n}\epsilon_1\sigma_l^2K_j^2\overline{\varepsilon}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{1}{\epsilon_1}\sigma_j^2\overline{\dot{\varepsilon}}_j(t)a_{jl}\overline{a}_{jl}\\
&\times\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}\sum_{l=1}^{n}\epsilon_2\sigma_l\mu_lK_j^2\overline{\varepsilon}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{1}{\epsilon_2}\sigma_j\mu_j\overline{\varepsilon}_j(t)a_{jl}\overline{a}_{jl}\varepsilon_j(t)\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\sigma_j^2\left(\epsilon_3L_l^2\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))e^{-2\tau_l(t)}+\frac{1}{\epsilon_3}\overline{\dot{\varepsilon}}_j(t)b_{jl}\overline{b}_{jl}\dot{\varepsilon}_j(t)\right)\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\sigma_j\mu_j\left(\epsilon_4L_l^2\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))e^{-2\tau_l(t)}+\frac{1}{\epsilon_4}\overline{e}_j(t)b_{jl}\overline{b}_{jl}\varepsilon_j(t)\right)\Bigg]\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\overline{\varepsilon}_l(t)\varepsilon_l(t)e^{2\lambda t}-\sum_{j=1}^{n}\sum_{l=1}^{n}(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2\\
&+\sigma_j\mu_j)\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t)))e^{2\lambda(t-\tau_l(t))},\end{aligned}\tag{5.9}$$

$$\begin{aligned}\leq e^{2\lambda t}\Bigg[&\sum_{j=1}^{n}\Bigg(2\lambda+2\lambda\mu_j^2-2\sigma_j\mu_j(\beta_j+\omega_j)+\sum_{l=1}^{n}\Bigg(\epsilon_1\sigma_l^2K_j^2+\epsilon_2\sigma_l\mu_lK_j^2\\
&+\frac{1}{\epsilon_2}\sigma_j\mu_ja_{jl}\overline{a}_{jl}+\frac{1}{\epsilon_4}\sigma_j\mu_jb_{jl}\overline{b}_{jl}+\frac{2(\epsilon_3+\epsilon_4)L_j^2(\sigma_j^2+2\lambda\sigma_j\mu_j)}{(1-\tau)}\Bigg)\Bigg)\overline{\varepsilon}_j(t)\varepsilon_j(t)\\
&+\sum_{j=1}^{n}\Bigg(1+2\lambda\sigma_j\mu_j-\sigma_j\mu_j(\alpha_j+\gamma_j)+\mu_j^2-\sigma_j^2(\beta_j+\omega_j)\Bigg)\overline{\dot{\varepsilon}}_j(t)\varepsilon_j(t)\end{aligned}$$

$$+\sum_{j=1}^{n}\Big(1+2\lambda\sigma_j\mu_j-\sigma_j^2(\beta_j+\omega_j)\mu_j\sigma_j(\alpha_j+\gamma_j)+\mu_j^2)\Big)\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)$$
$$+\sum_{j=1}^{n}\Big(2\lambda\sigma_j^2-2\sigma_j^2(\alpha_j+\gamma_j)+2\mu_j\sigma_j+\sum_{l=1}^{n}\sigma_j^2(\frac{1}{\epsilon_1}a_{jl}\overline{a}_{jl}+\frac{1}{\epsilon_3}b_{jl}\overline{b}_{jl})\Big)$$
$$\times\,\overline{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)\Big].$$

$$\frac{dV(t)}{dt}\leq e^{2\lambda t}\sum_{j=1}^{n}\Big[\tilde{A}_j\overline{\varepsilon}_j(t)\varepsilon_j(t)+\tilde{B}_j\big(\overline{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\tilde{C}_j\dot{\varepsilon}_j(t)\varepsilon_j(t)\big)\Big] \tag{5.10}$$

$$\frac{dV}{dt}\leq e^{2\lambda t}\sum_{j=1}^{n}\Big[\tilde{C}_j\overline{\Big(\dot{\varepsilon}_j(t)+\frac{\tilde{B}_j}{\tilde{C}_j}\varepsilon_j(t)\Big)}\Big(\dot{\varepsilon}_j(t)+\frac{\tilde{B}_j}{C_j}\varepsilon_j(t)\Big)+\Big(\tilde{A}_j-\frac{\tilde{B}_j^2}{\tilde{C}_j}\Big)$$
$$\times\,\overline{\varepsilon}_j(t)\varepsilon_j(t)\Big]. \tag{5.11}$$

Finally, we get

$$\frac{dV}{dt}\leq-\delta\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t)<0, \tag{5.12}$$

where $\delta=\min_{j\in I}\Big(\tilde{A}_j-\frac{\tilde{B}_j^2}{\tilde{C}_j}\Big).$

$$\text{Thus,}\quad\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t)e^{2\lambda t}\leq V(t)\leq V(0),\ \text{for } t>0, \tag{5.13}$$

where

$$V(0)=\sum_{j=1}^{n}\overline{\varepsilon}_j(0)\varepsilon_j(0)+\sum_{j=1}^{n}\Big(\overline{\sigma_j\dot{\varepsilon}_j(0)+\mu_j\varepsilon_j(0)}\Big)\left(\sigma_j\dot{\varepsilon}_j(0)+\mu_j\varepsilon_j(0)\right)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\int_{0-\tau_l(0)}^{0}\overline{\varepsilon}_l(0)\varepsilon_l(0)e^{2\lambda\theta}d\theta, \tag{5.14}$$

$$\leq\Big[(1+2\overline{\eta})+\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)L_l^2\overline{\eta}(\overline{\eta}+1)}{(1-\tau)}\Big]\|(\tilde{\varsigma}(t_0)+\varsigma(t_0))\|^2+2\overline{\eta}\|(\tilde{\varpi}(t_0)+\varpi(t_0))\|^2 \tag{5.15}$$

with $\|(\tilde{\varsigma}(t_0)+\varsigma(t_0))\|^2 = \sup_{(-\infty,0]}\sum_{j=1}^{n}(\|(\tilde{\varsigma}_j(t_0)+\varsigma_j(t_0))\|^2)$ and $\|(\tilde{\varpi}(t_0)+\varpi(t_0))\|^2 = \sup_{(-\infty,0]}\sum_{j=1}^{n}(\|(\tilde{\varpi}_j(t_0)+\varpi_j(t_0))\|^2)$, are followed from equations (5.13), (5.15).

$$\text{Now } \sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t)e^{2\lambda t} \leq M^*\|(\tilde{\varsigma}(t_0)+\varsigma(t_0))\|^2, \tag{5.16}$$

where $M^* = (1+2\overline{\eta}) + \sum_{l=1}^{n}\left(\frac{(\epsilon_3+\epsilon_4)L_l^2\overline{\eta}(\overline{\eta}+1)}{(1-\tau)}\right) + 2\overline{\eta}\frac{\|(\tilde{\varpi}(t_0)+\varpi(t_0))\|^2}{\|(\tilde{\varsigma}(t_0)+\varsigma(t_0))\|^2} > 0.$
Then from equation (5.16), we get

$$\sum_{j=1}^{n}\|(v_j(t)+q_j(t))\| \leq \|(\tilde{\varsigma}(t_0)+\varsigma(t_0))\|e^{-\lambda t}, \quad t \geq 0.$$

Hence by Definition 5.2.1, the systems (5.1), (5.2) are globally anti-synchronized. □

**Assumption 5.3.1.** *The time-varying delays $\tau_l(t)$, $l \in I$ are bounded, differentiable and satisfying $\dot{\tau}_l(t) \leq \tau < 1$, where $\tau$ is constant.*

**Assumption 5.3.2.** *For any $j \in I, \forall s_1, s_2 \in \mathbf{H}$, the nonlinear activation functions are odd, and there exist constants $K_l > 0$, $L_l > 0$ such that*
$|f_l(s_1) - f_l(s_2)| \leq K_l|s_1 - s_2|$,
$|g_l(s_1) - g_l(s_2)| \leq L_l|s_1 - s_2|$.

**Corollary 5.3.1.** *Presume the Assumption 5.3 and 5.3.2 and $\tilde{C}_j \leq 0$, $\tilde{A}_j \leq 0$, and $\tilde{B}_j^2 \leq \tilde{A}_j\tilde{C}_j$ are satisfied. Then the systems (5.1) and (5.2) are globally exponentially anti-synchronized with bounded time-varying delay under the control scheme (5.3), where $\tilde{A}_j, \tilde{B}_j, \tilde{C}_j$ are already stated in Theorem 5.3.*

**Corollary 5.3.2.** *Presume the Assumptions 5.2.1 and 5.3 hold, then the systems* (5.1) *and* (5.2) *are globally asymptotic anti-synchronized via controllers given in* (5.3)*, under the conditions* $\tilde{C}_j \leq 0, \quad \tilde{A}_j \leq 0, \quad \tilde{B}_j^2 \leq \tilde{A}_j\tilde{C}_j,$

*where* $\tilde{A}_j = -2\sigma_j\mu_j(\beta_j+\omega_j)+\sum_{l=1}^{n}\Bigg(\epsilon_1\sigma_l^2K_j^2+\epsilon_2\sigma_l\mu_lK_j^2+\frac{1}{\epsilon_2}\sigma_j\mu_ja_{jl}\overline{a}_{jl}+\frac{1}{\epsilon_4}\sigma_j\mu_jb_{jl}\overline{b}_{jl}+\frac{(\epsilon_3+\epsilon_4)L_j^2(\sigma_j^2+\sigma_j\mu_j)}{(1-\tau)}\Bigg),$

$\tilde{B}_j = 1-\sigma_j\mu_j(\alpha_j+\gamma_j)+\mu_j^2-\sigma_j^2(\beta_j+\omega_j),$

$\tilde{C}_j = 2\mu_j\sigma_j - 2\sigma_j^2(\alpha_j+\gamma_j)+\sum_{l=1}^{n}\sigma_j^2\big(\frac{1}{\epsilon_1}a_{jl}\overline{a}_{jl}+\frac{1}{\epsilon_3}b_{jl}\overline{b}_{jl}\big).$

**Remark 5.3.1.** *The asymptotic synchronization of real-valued INNs was first discussed using a non-reduced order technique in [116]. This chapter analyzes a class of more general QVINN systems, and a quicker convergent error system called exponential-type anti-synchronization, which is handled using a non-reduced order approach.*

**Remark 5.3.2.** *If the control gains are extremely large in the feedback controller, the control cost may be very high, and therefore, the control cost is expensive. A more affordable controller should be designed from a practical standpoint. Therefore, in the remaining part, the following adaptive controllers are used, which are designed as*

$$U_j(t) = -\alpha_j^*(t)\varepsilon_j(t) - \beta_j^*\dot{\varepsilon}_j(t), \tag{5.17}$$

where $\dot{\alpha}_j^*(t) = p_j\bigg(2\overline{\varepsilon}_j(t)\varepsilon_j(t) + \dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\bigg)$, $\dot{\beta}_j^*(t) = r_j\bigg(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t) + \dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\bigg)$,

$\alpha_j^*(t), \beta_j^*(t) \in \mathbf{R}$, and $p_j, r_j$ are the positive constants.

**Theorem 5.3.2.** *Presume the Assumption5.2.1 and 5.3.2, then the QVINNs (5.4) are asymptotically anti-synchronized on behalf of adaptive controller (5.17).*

*Proof.* Let us construct the Lyapunov functional as

$$V(t) = \sum_{j=1}^{n}\delta_j\overline{\varepsilon}_j(t)\varepsilon_j(t) + \sum_{j=1}^{n}\overline{(\dot{\varepsilon}_j(t)+\varepsilon_j(t))}\,(\dot{\varepsilon}_j(t)+\varepsilon_j(t)) + \sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)nL_l^2}{(1-\tau)} \times \int_{t-\tau_l(t)}^{t}\overline{\varepsilon}_l(t)\varepsilon_l(t)d\theta + \sum_{j=1}^{n}\frac{1}{2p_j}(\tilde{\alpha}_j-\alpha_j^*(t))^2 + \sum_{j=1}^{n}\frac{1}{2r_j}(\tilde{\beta}_j-\beta^*(t))^2, \quad (5.18)$$

where $\delta_j, \tilde{\alpha}_j, \tilde{\beta}_j, \epsilon_3,\ \epsilon_4$ and $n$ are positive constants. Taking derivative along the equation (5.4),

$$\begin{aligned}\frac{dV(t)}{dt} =& \sum_{j=1}^{n}\delta_j\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \sum_{j=1}^{n}\delta_j\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + \sum_{j=1}^{n}(\ddot{\overline{\varepsilon}}_j(t)+\dot{\overline{\varepsilon}}_j(t))(\dot{\varepsilon}_j(t)+\varepsilon_j(t)) \\ &+ \sum_{j=1}^{n}(\dot{\overline{\varepsilon}}_j(t)+\overline{\varepsilon}_j(t))(\ddot{\varepsilon}_j(t)+\dot{\varepsilon}_j(t)) + \sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)nL_l^2}{(1-\tau)}\bigg(\overline{\varepsilon}_l(t)\varepsilon_l(t) \\ &- \overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_j(t-\tau_l(t))(1-\dot{\tau}_l(t))\bigg) - \sum_{j=1}^{n}(\tilde{\alpha}_j-\alpha_j^*(t))\big(2\overline{\varepsilon}_j(t)\varepsilon_j(t)+ \\ &\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big) - \sum_{j=1}^{n}(\tilde{\beta}_j-\beta^*(t))\big(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)\end{aligned}$$

$$\begin{aligned}=& \sum_{j=1}^{n}\delta_j\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t) + \sum_{j=1}^{n}\delta_j\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t) + \sum_{j=1}^{n}\bigg\{-(\alpha_j+\beta_j^*(t))\dot{\overline{\varepsilon}}_j(t) - (\beta_j+\alpha_j^*(t))\overline{\varepsilon}_j(t) \\ &+ \sum_{l=1}^{n}\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl} + \sum_{l=1}^{n}\overline{g}_l(\varepsilon_l(t-\tau_l(t)))\overline{b}_{jl}\bigg\}\bigg(\dot{\varepsilon}_j(t)+\varepsilon_j(t)\bigg) + \sum_{j=1}^{n}\dot{\overline{\varepsilon}}_j(t)(\dot{\varepsilon}_j(t)\end{aligned}$$

$$+\varepsilon_j(t))+\sum_{j=1}^{n}(\dot{\overline{\varepsilon}}_j(t)+\overline{\varepsilon}_j(t))\Big\{-(\alpha_j+\beta_j^*(t))\dot{\varepsilon}_j(t)-(\beta_j+\alpha_j^*(t))\varepsilon_j(t)+\sum_{l=1}^{n}a_{jl}$$
$$\times f_l(\varepsilon_l(t))+\sum_{l=1}^{n}b_{jl}g_l(\varepsilon_l(t-\tau_l(t)))\Big\}+\sum_{j=1}^{n}(\dot{\overline{\varepsilon}}_j(t)+\overline{\varepsilon}_j(t))\dot{\varepsilon}_j(t)+\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)nL_l^2}{(1-\tau)}$$
$$\times\Big(\overline{\varepsilon}_l(t)\varepsilon_l(t)-\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_j(t-\tau_l(t))(1-\dot{\tau}_l(t))\Big)-\sum_{j=1}^{n}\tilde{\alpha}_j\big(2\overline{\varepsilon}_j(t)\varepsilon_j(t)$$
$$+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)+\sum_{j=1}^{n}\alpha_j^*(t)\big(2\overline{\varepsilon}_j(t)\varepsilon_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)$$
$$-\sum_{j=1}^{n}\tilde{\beta}_j\big(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)+\sum_{j=1}^{n}\beta_j^*(t)\big(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)$$
$$+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big),$$

$$\begin{aligned}\frac{dV(t)}{dt}=&\sum_{j=1}^{n}\delta_j\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}\delta_j\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}\Bigg[-\alpha_j\Big(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)\\&+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\Big)-\beta_j\Big(\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+2\overline{\varepsilon}_j(t)\varepsilon_j(t)\Big)\\&+\sum_{l=1}^{n}\Big(\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)a_{jl}f_l(\varepsilon_l(t))\Big)\\&+\sum_{l=1}^{n}\Big(\overline{f}_l(\varepsilon_l(t))\overline{a}_{jl}\varepsilon_j(t)+\overline{\varepsilon}_j(t)a_{jl}f_l(\varepsilon_l(t))\Big)\\&+\sum_{l=1}^{n}(\overline{g}_l(\varepsilon_l(t-\tau_l(t))\overline{b}_{jl}\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)b_{jl}g_l(\varepsilon_l(t-\tau_l(t)))\\&+\sum_{l=1}^{n}\big(\overline{g}_l(\varepsilon_l(t-\tau_l(t))\overline{b}_{jl}\varepsilon_j(t)+\overline{\varepsilon}_j(t)b_{jl}g_l(\varepsilon_l(t-\tau_l(t))\big)\\&+2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\Bigg]\\&+\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)nL_l^2}{(1-\tau)}\Big(\overline{\varepsilon}_l(t)\varepsilon_l(t)-\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_j(t-\tau_l(t))(1-\dot{\tau}_l(t))\Big)\\&-\sum_{j=1}^{n}\tilde{\alpha}_j\big(2\overline{\varepsilon}_j(t)\varepsilon_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)\\&-\sum_{j=1}^{n}\tilde{\beta}_j\big(2\dot{\overline{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\dot{\overline{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big).\end{aligned}\tag{5.19}$$

Now using Assumptions 5.2.1, 5.3.2 and Lemma 5.2.2,

$$\begin{aligned}
\frac{dV(t)}{dt} \leq & \sum_{j=1}^{n}(\delta_j+1)\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\sum_{j=1}^{n}(\delta_j+1)\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}2\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)\\
& -\sum_{j=1}^{n}\alpha_j\Big(2\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\Big)\\
& -\sum_{j=1}^{n}\beta_j\Big(\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)+\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+2\overline{\varepsilon}_j(t)\varepsilon_j(t)\Big)\\
& +\sum_{j=1}^{n}\sum_{l=1}^{n}\Big(\epsilon_1K_l^2\overline{\varepsilon}_l(t)\varepsilon_l(t)+\frac{1}{\epsilon_1}\bar{\dot{\varepsilon}}_j(t)a_{jl}\overline{a}_{jl}\dot{\varepsilon}_j(t)\Big)+\sum_{j=1}^{n}\sum_{l=1}^{n}(\epsilon_2K_l^2\overline{\varepsilon}_l(t)\varepsilon_l(t)\\
& +\frac{1}{\epsilon_2}\overline{\varepsilon}_j(t)a_{jl}\overline{a}_{jl}\varepsilon_j(t))+\sum_{j=1}^{n}\sum_{l=1}^{n}\Big(\epsilon_3L_l^2\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))+\frac{1}{\epsilon_3}\bar{\dot{\varepsilon}}_j(t)b_{jl}\\
& \times\overline{b}_{jl}\dot{\varepsilon}_j(t)\Big)+\sum_{j=1}^{n}\sum_{l=1}^{n}\Big(\epsilon_4L_l^2\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_l(t-\tau_l(t))+\frac{1}{\epsilon_4}\overline{\varepsilon}_j(t)b_{jl}\overline{b}_{jl}\varepsilon_j(t)\Big)\\
& +\sum_{l=1}^{n}\frac{(\epsilon_3+\epsilon_4)nL_l^2}{(1-\tau)}\overline{\varepsilon}_l(t)\varepsilon_l(t)-\sum_{l=1}^{n}(\epsilon_3+\epsilon_4)nL_l^2\overline{\varepsilon}_l(t-\tau_l(t))\varepsilon_j(t-\tau_l(t))\\
& -\sum_{j=1}^{n}\tilde{\alpha}_j\big(2\overline{\varepsilon}_j(t)\varepsilon_j(t)+\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)-\sum_{j=1}^{n}\tilde{\beta}_j\big(2\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)\\
& +\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big),\\
\leq & \sum_{j=1}^{n}\Big((\delta_j+1)-\alpha_j-\beta_j-\tilde{\alpha}_j-\tilde{\beta}_j\Big)\big(\bar{\dot{\varepsilon}}_j(t)\varepsilon_j(t)+\overline{\varepsilon}_j(t)\dot{\varepsilon}_j(t)\big)+\sum_{j=1}^{n}\Big(2-2\alpha_j\\
& -2\tilde{\beta}_j+\sum_{l=1}^{n}\Big(\frac{1}{\epsilon_1}|a_{jl}|^2+\frac{1}{\epsilon_3}|b_{jl}|^2\Big)\Big)\bar{\dot{\varepsilon}}_j(t)\dot{\varepsilon}_j(t)+\sum_{j=1}^{n}\Big(\frac{n(\epsilon_3+\epsilon_4)L_l^2}{(1-\tau)}-2\beta_j\\
& -2\tilde{\alpha}_j+\sum_{l=1}^{n}\Big((\epsilon_1+\epsilon_2)K_j^2+\frac{1}{\epsilon_4}|b_{jl}|^2+\frac{1}{\epsilon_2}|a_{jl}|^2\Big)\Big)\overline{\varepsilon}_j(t)\varepsilon_j(t). \qquad (5.20)
\end{aligned}$$

Now for each $j\in I$, we get

$$2\tilde{\alpha}_j=1+\frac{n(\epsilon_3+\epsilon_4)L_l^2}{(1-\tau)}-2\beta_j+\sum_{l=1}^{n}\Big((\epsilon_1+\epsilon_2)K_j^2+\frac{1}{\epsilon_4}|b_{jl}|^2+\frac{1}{\epsilon_2}|a_{jl}|^2+\frac{2L_j^2}{(1-\tau)}\Big), \qquad (5.21)$$

$$2\tilde{\beta}_j = 2 - 2\alpha_j + \sum_{l=1}^{n}\left(\frac{1}{\epsilon_1}|a_{jl}|^2 + \frac{1}{\epsilon_3}|b_{jl}|^2\right), \tag{5.22}$$

$$\delta_j = \alpha_j + \beta_j + \tilde{\alpha}_j + \tilde{\beta}_j - 1, \text{which reduce the equation (5.20) as} \tag{5.23}$$

$$\frac{dV}{dt} \leq -\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t). \tag{5.24}$$

Therefore,

$$\lim_{t\to\infty}\int_0^{\infty}\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t) \leq V(0) < \infty. \tag{5.25}$$

In addition, from (5.24), $V(t) \leq V(0),\ \forall\ t \in [0,\infty)$. This means that both $\varepsilon_j(t)$ and its derivative are bounded $\forall\ t \geq 0,\ j \in I$, hence $\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t)$ has bounded derivative and $\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\ \varepsilon_j(t)$ uniformly continuous. Using Lemma 5.2.3, we get $\lim_{t\to\infty}\sum_{j=1}^{n}\overline{\varepsilon}_j(t)\varepsilon_j(t) = 0$.

Hence, the proposed systems (5.1) and (5.2) are globally anti-synchronized. □

**Corollary 5.3.3.** *Based on Assumptions 5.1 and 5.3, the QVINN model (5.4) with bounded discrete delays realize asymptotic anti-synchronization under the adaptive controller (5.17).*

**Remark 5.3.3.** *It is the first time a non-reduced-order approach has been investigated for the adaptive anti-synchronization for QVINNs. In particular, if the QVINNs (5.1) and (5.2) are transformed to RVINNs, then equation (5.17) is reduced to the following form*

$$U_j(t) = -\alpha_j^*(t)\varepsilon_j(t) - \beta_j^*\dot{\varepsilon}_j(t), \tag{5.26}$$

*where* $\dot{\alpha}_j^*(t) = p_j\bigg(\varepsilon_j^2(t) + \dot{\varepsilon}_j(t)\varepsilon_j(t)\bigg)$, $\dot{\beta}_j^*(t) = r_j\bigg(\dot{\varepsilon}_j^2(t) + \dot{\varepsilon}_j(t)\varepsilon_j(t)\bigg)$, *which has been already discussed in [116] for asymptotic synchronization of RVNNs. Thus, the adaptive controller designed in equation* (5.18) *belongs to the larger domain and is more general and flexible.*

**Remark 5.3.4.** *Because the direct method was used, it is clear that the proofs for Theorems 5.3* and *5.3.2 are relatively straightforward. Only two inequalities and several common real number operations have been used in this method. Additionally, the direct method avoids splitting the quaternion system into four RVSs, significantly reducing the computing complexity as compared to the separation methodology.*

**Remark 5.3.5.** *In [139], the authors have investigated the adaptive synchronization for QVNNs. The quaternion-valued systems were split into four RVSs, and each of the four sub-systems is constructed with four real-valued adaptive controllers. However, in [127], the synchronization for Cohen-Grossberg QVINNs has been examined via a direct quaternion approach, and the original second-order system is transformed into a first-order system by using the variable substitution approach. Unlike these works, the present study is concerned with a class of second-order QVINNs without utilizing the separation method. A quaternion-valued adaptive controller* (5.15) *is developed to analyze the asymptotic anti-synchronization. Thus, it is claimed that the proposed method is easier to use and more effective.*

**Remark 5.3.6.** *Up to now, various kinds of INNs have been studied related to anti-synchronization results on the real or complex domain, [140, 141, 142]. But all these results are based on the variable substitution approach to reduce the second-order system into the first-order systems. Only a few anti-synchronization results for RVINNs by non-reduction order technique are available [143, 144], but there have been no related results for QVINNs till now. Different from the existing research* *work, a class of QVINNs with unbounded delays is introduced in the present chapter, and the global exponential and asymptotic anti-synchronization are discussed by introducing an innovative Lyapunov functional rather than the variable substitution technique. This demonstrates how well the present suggested results complement and advance the earlier ones.*

## 5.4 Numerical Examples

**Example 5.4.1.** *Consider the two-dimensional QVINNs with unbounded delays as the master system as*

$$\frac{d^2q_j}{dt^2} = -\alpha_j\frac{dq_j}{dt} - \beta_j q_j(t) + \sum_{l=1}^{2} a_{jl} f_l(q_l(t)) + \sum_{l=1}^{2} b_{jl} g_l(q_l(t-\tau_l(t))), \quad j = 1, 2. \tag{5.27}$$

*The corresponding response system is*

$$\frac{d^2v_j}{dt^2} = -\alpha_j\frac{dv_j}{dt} - \beta_j v_j(t) + \sum_{l=1}^{2} a_{jl} f_l(v_l(t)) + \sum_{l=1}^{2} b_{jl} g_l(v_l(t-\tau_l(t))) + U_j(t), \ j = 1, 2, \tag{5.28}$$

*where the parametric values are* $\lambda = 0.1,\ \alpha_1 = 10,\ \alpha_2 = 11,\ \beta_1 = 11,\ \beta_2 = 9,\ \gamma_1 = 8,\ \gamma_2 = 9,\ \omega_1 = 8,\ \omega_2 = 9,\ \epsilon_j = 1$ *for* $j = 1, 2, ..., 5.$

$f_l = \frac{1}{4}\big(\tanh\big(q_l^R\big) + i\tanh\big(q_l^I\big) + j\tanh\big(q_l^J\big) + k\tanh\big(q_l^K\big)\big),$

$g_l = \frac{1}{2}\big(\tanh\big(q_l^R\big) + i\tanh\big(q_l^I\big) + j\tanh\big(q_l^J\big) + k\tanh\big(q_l^K\big)\big)e^{-\tau_l(t)},$

$\tau_l(t) = \ln(2+2t),$ *for* $l = 1, 2.$

$$[a]_{2\times 2} = \begin{pmatrix} -0.25 + 0.10i - 0.5j + 0.15k & 0.15 - 0.2i + 0.6j - 0.10k \\ -0.15 + 0.5i - 0.2j - 0.17k & -0.3 - 0.15i - 0.5j + 0.19k \end{pmatrix},$$

$$[b]_{2\times 2} = \begin{pmatrix} 0.1 - 0.2i - 0.45j + 0.2k & -0.1 - 0.1i + 0.20j - 0.3k \\ -0.2 + 0.2i + 0.25j - 0.15k & -0.25 - 0.2i - 0.25j - 0.15k \end{pmatrix}.$$

*Then, by simple computations, one can get* $K_l = 0.25, L_l = 0.5,$ *where* $w = 1, 2.$ $\sigma_1 = \sigma_2 = 0.5, \mu_1 = \mu_2 = 0.5, \tau = 0.5.$

*The values* $\tilde{A}_1 = -8.660, \tilde{C}_1 = -8.3757, \tilde{B}_1 = -7.95$ *satisfy* $\tilde{B}_1^2 \leq \tilde{A}_1\tilde{C}_1,$ *Also* $\tilde{A}_2 = -8.1846, \tilde{B}_2 = -8.2, \tilde{C}_2 = -9.4178$ *satisfy* $\tilde{B}_2^2 \leq \tilde{A}_2\tilde{C}_2.$

*The dynamical behaviors of master system* (5.27) *with the above-mentioned parameters are depicted through Figures 5.1 and 5.2 with the initial conditions* $\varsigma_1 = 0.1+0.9i-0.1j+0.4k,\ \varpi_1 = 0.3+0.6i+0.2j+0.1k,\ \varsigma_2 = 0.4+0.8i+0.3j+0.4k,\ \varpi_2 = 0.3+0.2i+0.5j+0.1k,\ \ t_0 \in (-\infty, 0].$

*Firstly, the exponential anti-synchronization between the systems* (5.27) *and* (5.28) *via feedback controllers* (5.3) *are depicted through Figures 5.3-5.6.*

*The anti-synchronization errors* $\varepsilon_1(t), \varepsilon_2(t)$ *for the systems* (5.27) *and* (5.28) *under the feedback controllers* (5.3) *are depicted through the Figures 5.7 and 5.8.*

*Below, the anti-synchronization between the systems* (5.27) *and* (5.28) *via adaptive controller* (5.18) *has been discussed. Selecting* $p_1 = 0.6, p_2 = 1\ r_1 = 0.5\ r_2 = 1$ *in the controllers* (5.18)*, and using Theorem 5.3.2, the asymptotic anti-synchronization between the systems* (5.27) *and* (5.28) *are reached, which are depicted through the Figures 5.9-5.12. The anti-synchronization errors* $\varepsilon_1(t),\ \varepsilon_2(t)$ *between the master system* (5.27) *and response system* (5.28) *are depicted through the Figures 5.13 and 5.14, and the control gains* $\alpha_j^*, \beta_j^*$ *for* $j = 1, 2$ *of the adaptive controllers* (5.18) *are depicted through the Figures 5.15.*

**Example 5.4.2.** *In order to show the application of QVNNs, let us consider the traditional NNs rather than INNs for convenience.*

*Consider the size of* $12 \times 12$ *pixels image pattern "S," whose color image pattern is depicted in Figure 5.16. For this, QVNNs in the form of* (5.29) *are designed with* 144 *neurons, which have* 144*-dimensional equilibrium points for storing colored patterns of "S." Let us consider the traditional QVNNs*

$$\frac{dq_j}{dt} = -\beta_j q_j(t) + \sum_{l=1}^{n} a_{jl} f_l(q_l(t)) + \sum_{l=1}^{n} b_{jl} g_l(q_l(t - \tau_l((t)))) + J_j, \quad j \in I, \tag{5.29}$$

*where* $\beta_j > 0,\ a_{jl}, b_{jl}$ *are weight connection matrices and* $J_j$ *is the external input of the system. The parametric values of QVNNs* (5.29) *are considered as*

$$\beta_j = 1, \tag{5.30}$$

$$a_{jl} = \begin{cases} 4.0 + 4.0 \times 10^{-1} i - 3.0 \times 10^{-1} j + 5.0 \times 10^{-1} k, \quad j = l \\ 4.0 \times 10^{-1} - 5.0 \times 10^{-1} i + 5.0 \times 10^{-1} j - 3.0 \times 10^{-1} k, \quad j \neq l, \end{cases} \tag{5.31}$$

$$b_{jl} = \begin{cases} -2.0 \times 10^{-1} + 2.0 \times 10^{-1} i - 5.0 \times 10^{-1} j + 4.0 \times 10^{-1} k, \quad j < l \\ 2 + 3.0 \times 10^{-1} i + 2.0 \times 10^{-1} j - 3.0 \times 10^{-1} k, \quad j = l \\ -1.0 \times 10^{-1} + 2.0 \times 10^{-1} i + 3.0 \times 10^{-1} j - 5.0 \times 10^{-1} k, \quad j > l, \end{cases} \tag{5.32}$$

$$f_l^x(q) = \tanh(q), \tag{5.33}$$

*where* $j, l = 1, 2, ..., 144$ *and* $x = 0, 1, 2, 3$*. The equilibrium point of the proposed QVNNs must be* $q = (q_1, q_2, ..., q_1 44)^T \in \mathbf{H}^{144}$ *to recall the image pattern "S", where* $q_1 = 0 + 15.0 \times 10^{-2} i + 3.0 \times 10^{-1} j + 15.0 \times 10^{-2} k,\ q_2 = 0 + 15.0 \times 10^{-2} i + 3.0 \times 10^{-1} j + 14.0 \times 10^{-2} k, ..., q_1 44 = 0 + 15.0 \times 10^{-2} i + 0j + 0k,$ *which correspond to the color* $(15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 0.150), (15.0 \times 10^{-2}, 3.0 \times 10^{-1}, 14.0 \times 10^{-2}), ..., (15.0 \times$

$10^{-2}, 0, 0)$ *of pixels of pattern "S". Figure 5.17 shows a simulation with random initial values. Now for above* $q$ *the external input* $J$ *is calculated as*

$J = (J_1, J_2, ..., J_{144}) \in \mathbf{H}^{144}$,

*where* $J_1 = -628.0 \times 10^{-1} + 2275.0 \times 10^{-2} i - 905.0 \times 10^{-1} j - 725.0 \times 10^{-2} k$, $J_2 = -63 + 2095.0 \times 10^{-2} i - 905.0 \times 10^{-1} j - 566.0 \times 10^{-2} k, ..., J_{144} = -914.0 \times 10^{-1} - 2346.0 \times 10^{-1} i - 908.0 \times 10^{-1} j + 2214.0 \times 10^{-1} k$. *Due to space limitation, only three members of* $q$ *and* $J$ *have been written. The simulation part with arbitrary initial data is depicted in Figure 5.17. Parameters from equations* (5.30)-(5.33) *show that the considered system* (5.29) *has the ability to retrieve the above pattern "S" reliably.*

**Remark 5.4.1.** *It is observed from the Example 5.2 that the proposed system* (5.29)*, whose parametric values are mentioned by equations* (5.30)-(5.32)*, require* 144.00 *neurons to store* $12 \times 12$ *pixels image pattern. However, this article [145] suggested that the storage capacity of the QVNNs is larger as compared with the CVNNs because storing* $12 \times 12$ *pixel color image requires* 432 *neurons in CVNNs, which is much larger as mentioned in [145]. However, in the present chapter, the QVNNs with* $'\tanh'$ *activation functions are considered, and one can retrieve the image "S" in approximated time* $t = 2.5$.

**Remark 5.4.2.** *The work can be extended in the near future following the research works given in [146, 147] for QVINNs. These works demonstrate that image encryption can be achieved through the synchronization of NNs. It is a technique that utilizes the complex dynamics of NNs to encrypt and decrypt digital images effectively. By utilizing the computational strength and complexity of NNs, this approach ensures robust security for image data. Anti-synchronization, as a result, image encryption based on NNs' synchronization holds significant promise for various practical applications, including secure communication, image authentication, privacy protection,* *cloud security, and many more. These potential applications have inspired me to extend the work to the applications for QVINNs.*

## 5.5 Conclusion

The anti-synchronization problem of QVINNs with unbounded time-varying delays has been solved in the present chapter. Unlike the traditional variable substitution for the reduced-order approach for the second-order neural systems and the separation approach in which QVNNs are separated into four equivalent RVSs, some innovative Lyapunov functionals have been constructed to deal with the global exponential and adaptive anti-synchronization of QVINNs directly. Particularly, in Theorem 5.3, the multi-parameter conditions have been derived by Lyapunov functional and linear quaternion feedback controller to ensure the global exponential anti-synchronization for QVINNs. Additionally, an AC has been developed on the quaternion domain to attain asymptotic anti-synchronization given in Theorem 5.3.2, which is direct and simpler as compared to the four real-valued adaptive controllers designed for separated real-valued sub-systems given in [139]. The results of this chapter are more compact and less calculative due to the non-separation, non-reduction approach as compared to the results given in [124, 125, 126, 127, 128, 129, 130]. The unbounded time-varying delays have never been taken for the QVINNs. However, in the present chapter, unbounded time delays have been considered in this chapter, which makes it more reliable as compared to the existing research [136, 134, 148, 145, 149, 124, 125, 126, 127, 128, 129, 130, 137, 138]. Lastly, two examples demonstrate the accuracy and advantage of the proposed results and the applications of QVNNs that can retrieve true color images.

The proposed approach is totally different from the separation approach as well as the order-reduction method with bounded time delays. Therefore, it is concise and more effective.

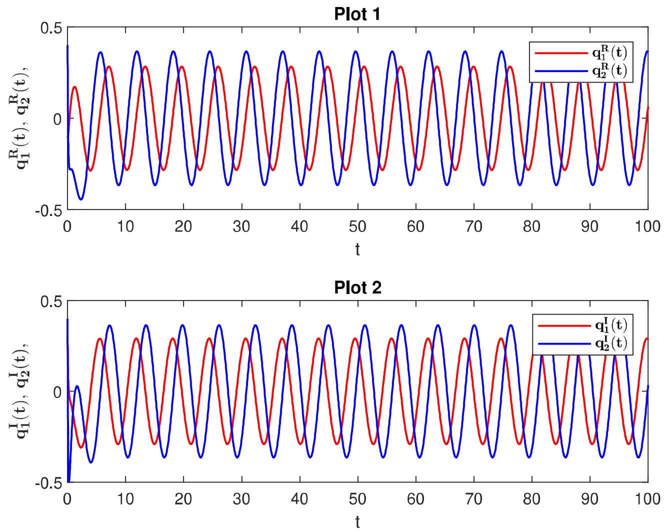


FIGURE 5.1: Time behaviour of state variables $q_1^R, q_2^R$ and $q_1^I, q_2^I$ of the system (5.27)

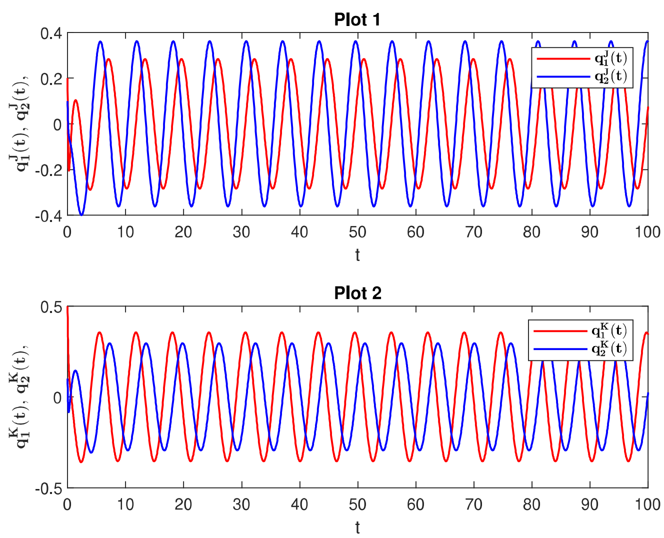


FIGURE 5.2: Time behaviour of state variables $q_1^J, q_2^J$ and $q_1^K, q_2^K$ of the system (5.27)

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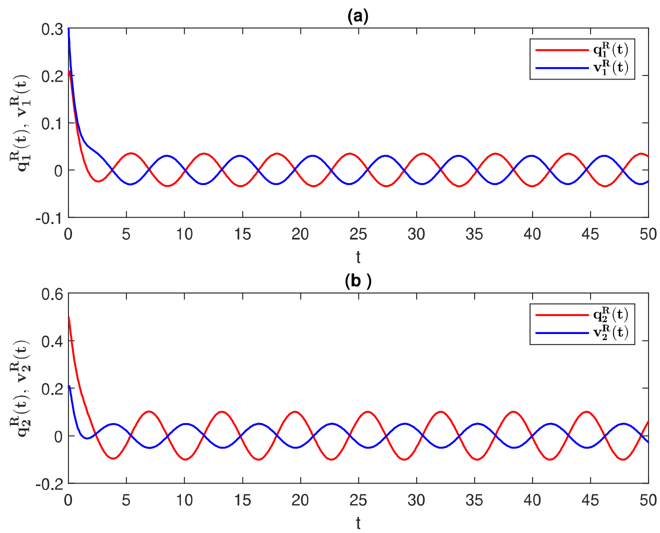


FIGURE 5.3: Anti-synchronization of $q_1^R, v_1^R$ and $q_2^R, v_2^R$ with control(5.3)

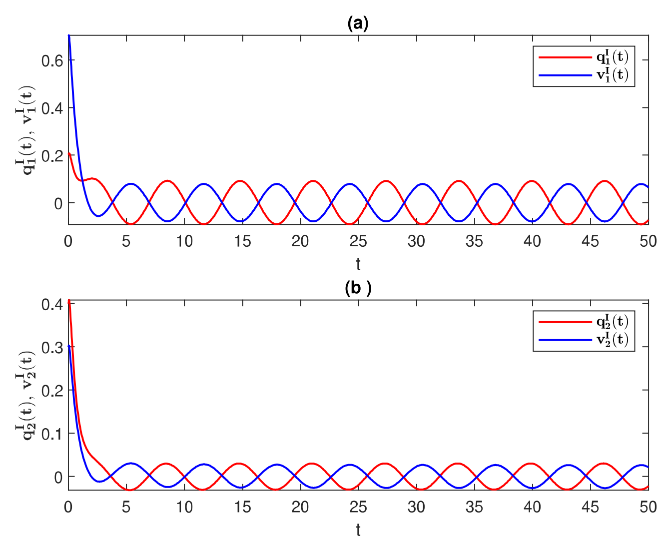


FIGURE 5.4: Anti-synchronization of $q_1^R, v_1^R$ and $q_2^R, v_2^R$ with control (5.3)

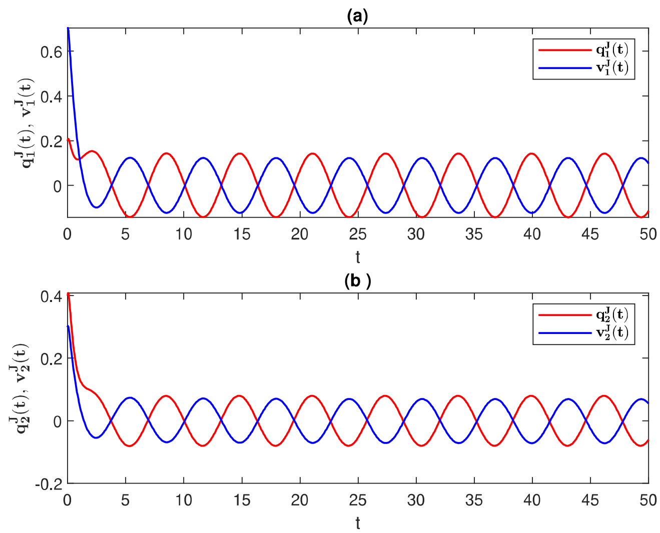


FIGURE 5.5: Anti-synchronization of $q_1^I, v_1^I$ and $q_2^I, v_2^I$ with control (5.3)

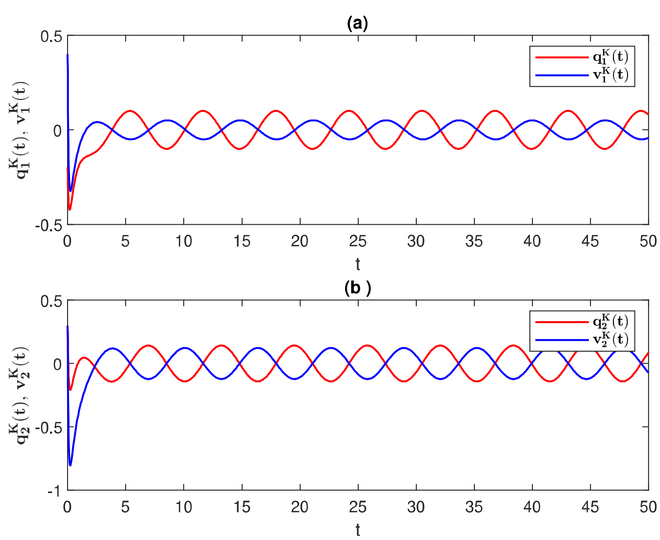


FIGURE 5.6: Anti-synchronization of $q_1^J, v_1^J$ and $q_2^J, v_2^J$ with control (5.3)

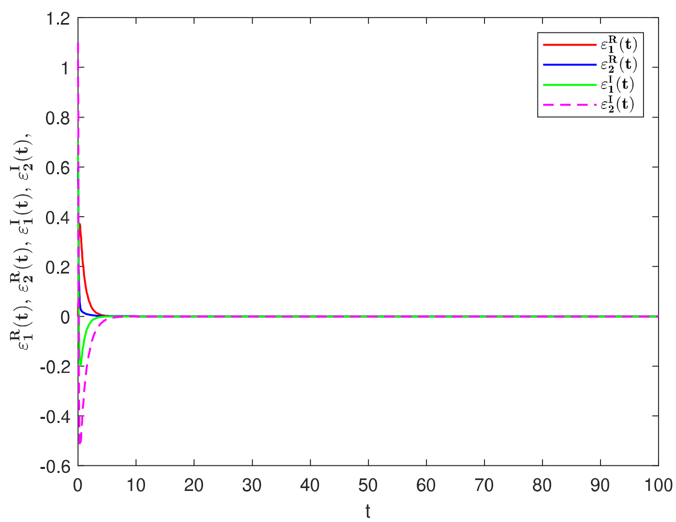


FIGURE 5.7: Error $\varepsilon_1^R, \varepsilon_2^R$ and $\varepsilon_1^I, \varepsilon_2^I$ with the control (5.3)

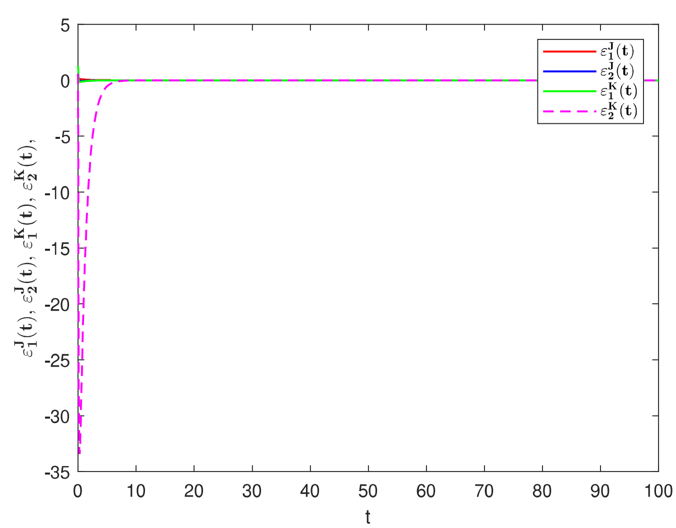


FIGURE 5.8: Error $\varepsilon_1^J, \varepsilon_2^J$ and $\varepsilon_1^K, \varepsilon_2^K$ with the control (5.3)

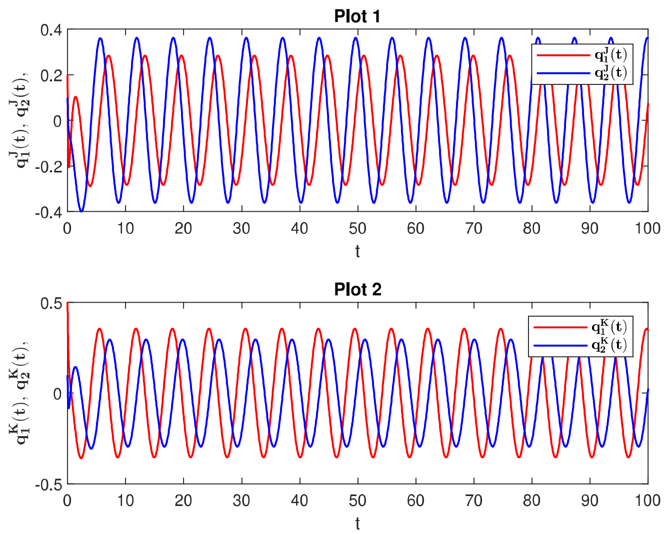


FIGURE 5.9: Anti-synchronization of $q_1^R, v_1^R$ and $q_2^R, v_2^R$ with control (5.17)

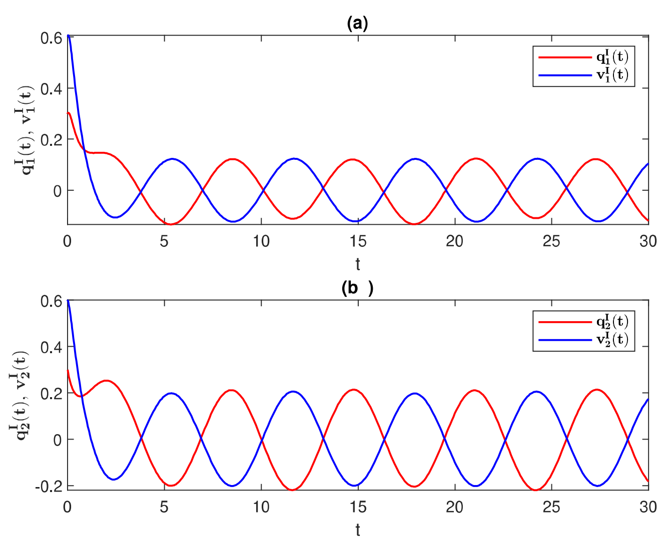


FIGURE 5.10: Anti-synchronization of $q_1^I, v_1^I$ and $q_2^I, v_2^I$ with control (5.17)

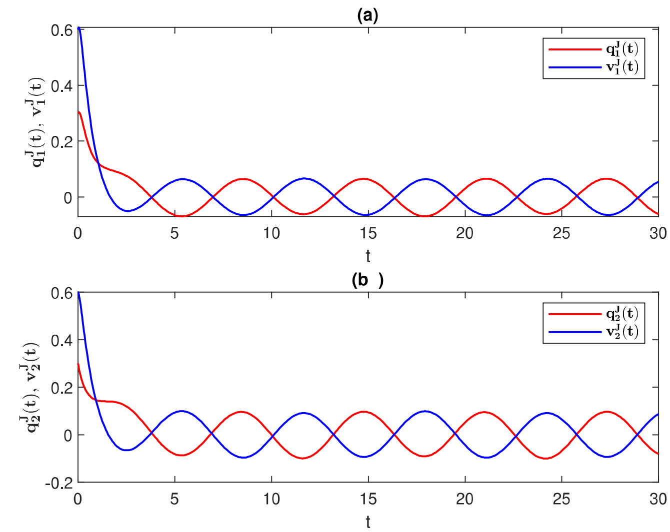


FIGURE 5.11: Anti-synchronization of $q_1^J, v_1^j$ and $q_2^J, v_2^J$ with control (5.17)

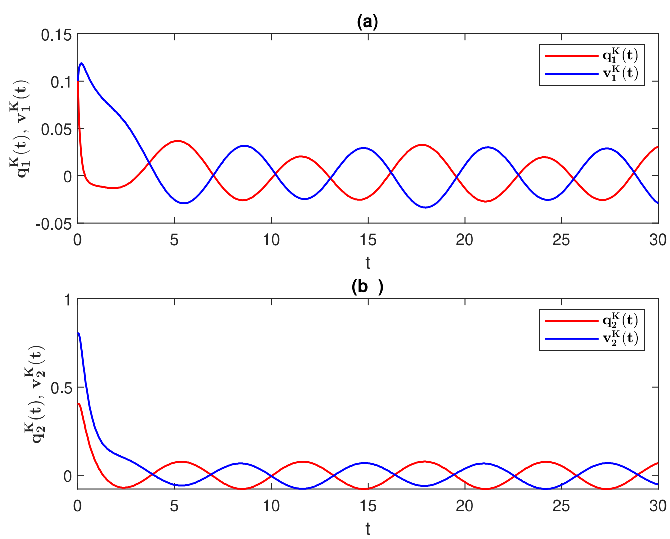


FIGURE 5.12: Anti-synchronization of $q_1^K, v_1^K$ and $q_2^K, v_2^K$ with control (5.17)

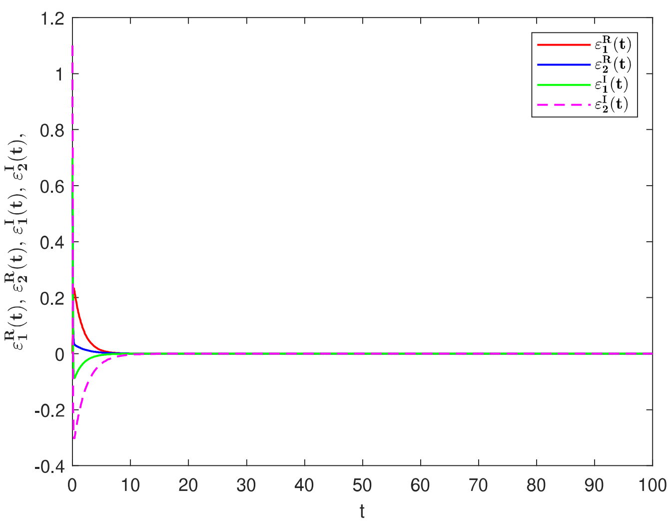

FIGURE 5.13: Error $\varepsilon_1^R, \varepsilon_2^R$ and $\varepsilon_1^I, \varepsilon_2^I$ with the control (5.17)

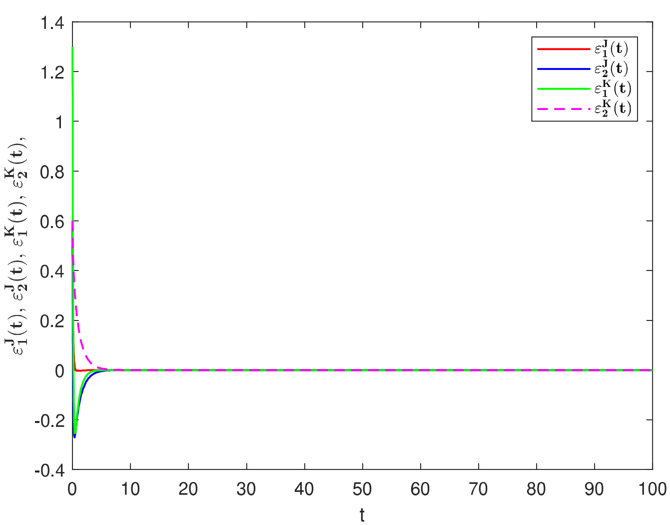

FIGURE 5.14: Error $\varepsilon_1^J, \varepsilon_2^J$ and $\varepsilon_1^K, \varepsilon_2^K$ with the control (5.17)

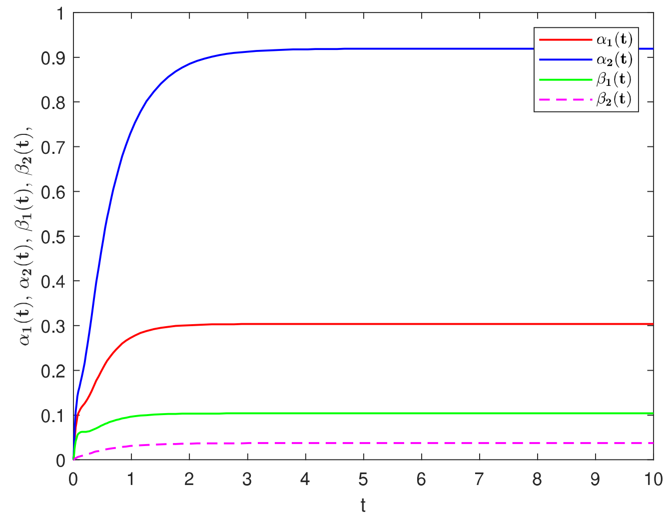


FIGURE 5.15: Time response of control gains $\alpha_1, \alpha_2, \beta_1, \beta_2$ (5.17)

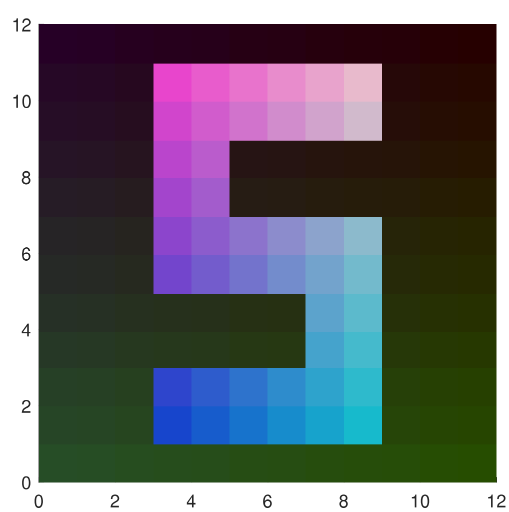

FIGURE 5.16: Original color image of pattern "S".

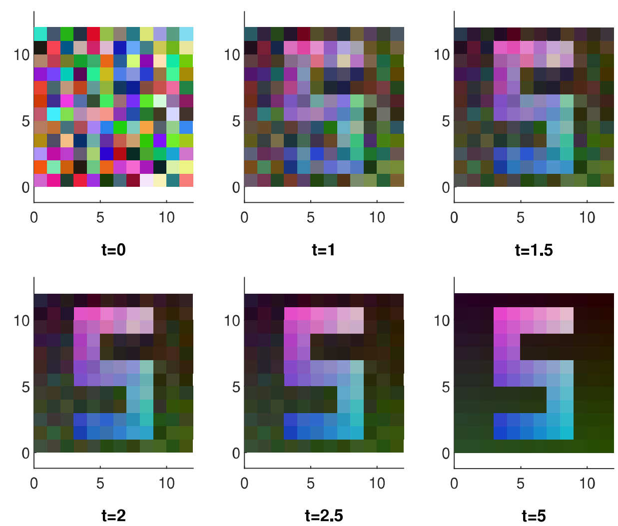


FIGURE 5.17: Simulation of retrieving image "S" with random initial values of time $t$.