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# Abbreviations

<b>FC</b>	<b>Fractional Calculus</b>
<b>1D</b>	<b>One-Dimensional</b>
<b>2D</b>	<b>Two-Dimensional</b>
<b>PDE(s)</b>	<b>Partial Differential Equation(s)</b>
<b>TFCD</b>	<b>Time Fractional Caputo Derivative</b>
<b>TFCDs</b>	<b>Time Fractional Caputo Derivatives</b>
<b>R-L</b>	<b>Riemann-Liouville</b>
<b>GCFD</b>	<b>Generalized Caputo Fractional Derivative</b>
<b>TFDW(s)</b>	<b>Time Fractional Diffusion-Wave(s)</b>
<b>TFMDW(s)</b>	<b>Time Fractional Mixed Diffusion and Diffusion-Wave(s)</b>
<b>FDM(s)</b>	<b>Finite Differenc Method(s)</b>
<b>FD(s)</b>	<b>Fractional Derivative(s)</b>
<b>FDE(s)</b>	<b>Fractional Differential Equation(s)</b>
<b>ADI</b>	<b>Alternating Direction Implicit</b>
<b>OC</b>	<b>Order of Convergence</b>
<b>EOC</b>	<b>Expected Order of Convergence</b>
<b>LTE</b>	<b>Local Truncation Error</b>



# Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}^+$	Set of positive integers
$\gamma \geq 1$	Mesh grading parameter
$\Gamma$	Euler gamma function
$Re(\alpha)$	Real-part of $\alpha$
$AC^n[a, b]$	$n$ th-time absolutely continuous on $[a, b]$
$C[a, b]$	Continuous on $[a, b]$
$C^n[a, b]$	$n$ th derivative continuous on $[a, b]$
$L^p(p \in [1, \infty])$	$L^p$ space
$[x]$	greatest integer function
$\lceil x \rceil$	ceiling function



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