Mathematical Study of Crack Problems Arising in Orthotropic Composites

The thesis submitted in partial fulfilment

for the Award of Degree

DOCTOR OF PHILOSOPHY

by

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I, **Anshika Tanwar**, affirm that this thesis represents my original and genuine work, conducted under the guidance of **Prof. Subir Das** from **January 2020** to **December 2024** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The content of this thesis has not been presented for any other degree or diploma. I confirm that due credit has been given to the respective researchers by citing their work wherever referenced in my thesis. I explicitly state that I have not intentionally replicated or incorporated the work, passages, text, data, results, etc., from any other source, such as journals, books, magazines, reports, dissertations, theses, websites, and have not portrayed them as my own in this thesis.

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Vidya dadati vinyam
(knowledge imparts humility)

DEDICATED

To

MY GRANDMOTHER

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Anshika Tanwar

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Symbols and Abbreviations

a	half-crack length
FM	Fracture Mechanics
SIF	Stress intensity factor
NSIF	Normalised SIF
K_a	stress intensity factor at tip $x = a$.
$K_{I/II/III}$	stress intensity factors of mode-I/II/III, respectively
LEFM	Linear elastic fracture mechanics
σ	stress tensor
σ_{ii}	normal stress component in $i = x,\ y,\ z$ plane
τ_{ij}	shear stress component in i plane and j
N	S.I. unit of force, Newton
M	notation for S.I. unit meter
ϵ	strain tensor with components in the form ϵ_{ij}
(r, θ)	polar coordinate system denoting distance from the crack tip and angle from crack surface, respectively
π	mathematical constant symbol Pi
q_i	heat flux in direction i
k_i	thermal conductivity coefficients in direction i
q_o	constant heat flux flowing through the plane
$(u, v) = (u(x, y), v(x, y))$	displacements in $x,\ y$ directions, respectively
$_{,i}$	subscript denoting partial derivative with respect to variable i
$_{,ij}$	subscript denoting second order partial derivative with respect to variables $i,\ j$
ρ	material density
ι	imaginary unit number, $\iota = \sqrt{-1}$

$T_n(x)$	Chebyshev polynomial of the first kind of order n
$U_n(x)$	Chebyshev polynomial of the second kind of order n
P_n	Jacobi polynomial of the first kind of order n
$\binom{n}{m}$	binomial coefficient with formula $\frac{n!}{m!(n-m)!}$
$J_n(x)$	Bessel function
$K(x,y)$	kernel function
δ_{nm}	Kronecker delta function
$w(x)$	weight function
∇	gradient vector
Δ	delta
$\cdot$	dot product
$T(x,y)$	temperature field
t	time
$\mathbb{R}$	set of real numbers

Note: All the mathematical symbols have usual meanings unless stated otherwise

PREFACE

The thesis addresses multiple problems of fracture mechanics based on the singular integral equation theory. Different chapters have addressed problems with insulated and conducting finite crack models, moving semi-infinite cracks under thermal and mechanical loadings. The thesis delves into the basics of fracture mechanics, presenting a detailed literature survey and fundamentals of elasticity theory through Chapter 1. The chapter has been divided carefully into various sections and subsections, providing the details of preliminary mathematical concepts, definitions, and theorems to the thesis readers. It provides the broad objectives of the thesis and the procedures followed. A brief historical background and overview of the subject, Fracture Mechanics, have also been discussed, mentioning the classical works done in the area. Different methodologies applied in the chapters and their analytical and numerical aspects have been discussed in detail. Starting from insulated cracks present at a fixed distance from the origin and at a fixed height in an elastic body under thermo-mechanical loadings, the thesis delves into problem formulation in anti-plane with finite length cracks and the problem with a moving semi-infinite crack. Chapter 2 investigates the problem of insulated cracks in an orthotropic plane under thermo-mechanical loading, focusing on the interaction between two offset cracks and a central crack. The chapter outlines the problem's geometry, mathematical formulation, and governing equations for the heat conduction as well as the displacement field equations, leading to dual Cauchy-type singular integral equations. The numerical solutions are derived using the expansion collocation method, and the calculated Stress Intensity Factors (SIFs) at the crack tips are discussed using FRC material for numerical examples. The results illustrate how varying loadings and crack separations affect SIFs, suggesting that greater distances between cracks reduce their mutual influence. The anti-plane shear crack problem provides a simplified yet effective approach to examining solutions to solid mechanics problems. Next, in the context of anti-plane deformations, the displacement remains independent of the axial coordinates and is oriented perpendicular to the plane's direction. This problem, specifically for dissimilar orthotropic medium, has been examined in detail in Chapter 3. A numerical solution is derived using the orthogonality of the Jacobi polynomial, circumventing the need to engage with complex dual integrals.

The graphical depictions of normalised SIFs of mode-III at the crack tips, plotted against the ratio of crack lengths, are presented for varying strip widths. Numerical computations have been executed to elucidate the interactions among the cracks for diverse material combinations. This comprehensive analysis provides a deeper understanding of the behaviour of cracks in dissimilar orthotropic mediums. The findings through numerical computations reveal that the SIFs behave differently at all tips for the particular cases considered. Chapter 4 addresses the problem of a semi-infinite crack moving with a fixed velocity at time t. The classical Wiener-Hopf method has been used in the chapter, along with the Fourier integral transform and the Galilean transform for the time domain. The decomposition of the kernel term into analytic functions in half-planes to analytically compute the expressions for stress intensity factor, stress magnification factor and crack energy has been discussed in the chapter. The numerical results supported by the analytical findings reveal that the SH-wave velocity is the limiting value of crack velocity. Section Numerical results and chapter discussion provide other findings and detailed graphical representations. The study in Chapter 5 has been conducted to examine the thermally conducting cracks in dissimilar orthotropic composite mediums. This study investigates the relationship between the thermal conductivity coefficient and various parameters of heat flux, stresses and temperatures in detail. This involves using orthogonal polynomials to expand the temperature and displacement jumps at the interface. This approach is novel for such types of problems and is employed to solve the integral equations and derive the approximate solutions for the considered problem. The results reveal that the temperature difference tends to zero as the thermal conductivity coefficient increases, and an increase in thermal conductivity decreases the SIFs. Overall, this study paves the way for a deeper understanding of the complexities inherent in composite mediums and offers valuable contributions to this intriguing field of research. In the end, Chapter 6 summarises the contents of the thesis and provides a detailed conclusion section. The chapter also provides insight into the future works that can be carried out by pointing at the limitations of the thesis.