Chapter 2

Fixed time synchronization of quaternion-valued neural networks with mixed time-varying delay

2.1 Introduction

This chapter explores fixed-time synchronization among identical drive-response systems within QVNNs featuring mixed time-varying delays. The achievement of this synchronization is facilitated through the application of the fixed-time stability criterion to the error system. Emphasizing the fixed-time stability aspect, the chapter establishes synchronization within a fixed time frame for drive-response systems. Specifically, the focus lies on fixed-time synchronization for a category of QVNNs endowed with mixed time-varying delays. The approach begins by decomposing QVNNs into four equivalent RVNNs. Subsequently, a novel, well-suited controller is crafted to realize fixed-time synchronization in QVNNs, leveraging the Lyapunov function. Two distinct expressions for settling time emerge through applying two different lemmas. Ultimately, the theoretical findings are validated through numerical simulations employing a specific example.

Some significant contributions of this scientific contribution are listed as follows.

1. It is the first time the fixed time synchronization of QVNNs is discussed with mixed time-varying delays.
2. The controllers are designed so that the settling time is independent of the delay terms.
3. Converting the QVNN into four real parts and by using some suitable controllers along with the Lyapunov function, the synchronization within a fixed time is achieved.

2.2 Model Description and Preliminaries

Consider the model of QVNNs with mixed time-varying delays, taking into account the drive system as

$$\begin{aligned}\dot{z}_p(t) =& -c_p z_p(t) + \sum_{q=1}^{n} a_{pq} f_q(z_q(t)) + \sum_{q=1}^{n} b_{pq} g_q(z_q(t-\tau_1(t))) \\ &+ \sum_{q=1}^{n} d_{pq} \int_{t-\tau_2(t)}^{t} h_q(z_q(s))ds + I_p(t), \end{aligned} \tag{2.1}$$

where $p = 1,\ 2,\ ...,\ n$; $z(t) = [z_1(t),\ z_2(t),\ ...,\ z_n(t)]^T \in \mathbb{H}^n$ is the state vector of the neural networks with n neurons at time $t; C = \text{diag}\{c_1,\ c_2,\ ...,\ c_n\} \in \mathbb{R}^{n\times n}$ is the self-feedback connection weights matrix with $c_p > 0$ for $p = 1,\ 2,\ ...,\ n$; $A =$

$(a_{pq})_{n\times n},\ B = (b_{pq})_{n\times n}$ and $D = (d_{pq})_{n\times n} \in \mathbb{H}^{n\times n}$ for $p = 1,\ 2,\ ...,\ n, q = 1,\ 2,\ ...,\ n,$ are the connection weight matrices; $f_q(.),\ g_q(.)$ and $h_q(.)$ are the activation functions for $q = 1,\ 2,\ ...,\ n$ of suitable dimensions; $I_p(t)$ for $p = 1,\ 2,\ ...,\ n,$ denote the external inputs. $\tau_1(t)$ and $\tau_2(t)$ are the time varying delays.

The following assumptions are considered to establish the main results of this chapter.

Assumption 2.2.1. *Each of the activation functions can be written as*

$$F_p(z_p(t)) = F_p^R(z_p^R(t)) + iF_p^I(z_p^I(t)) + jF_p^J(z_p^J(t)) + kF_p^K(z_p^K(t)), \tag{2.2}$$

where $F_p = f_p,\ g_p,\ h_p$ *for* $p = 1,\ 2,\ ...,\ n;$ $z_p^\gamma \in \mathbb{R}$ *for* $\gamma = R,\ I,\ J,\ K;$ $p = 1,\ 2,\ ...,\ n.$

Assumption 2.2.2. *Each of the four components of every activation function satisfies the Lipschitz condition, i.e., for* $x_1,\ x_2 \in \mathbb{R}$*,* $\exists$ *constants* $l_p^1,\ l_p^2,\ l_p^3 \in \mathbb{R}$*, such that*

$$\begin{aligned}
|f_p^\gamma(x_1) - f_p^\gamma(x_2)| &\le l_p^1|x_1 - x_2|,\\
|g_p^\gamma(x_1) - g_p^\gamma(x_2)| &\le l_p^2|x_1 - x_2|,\\
|h_p^\gamma(x_1) - h_p^\gamma(x_2)| &\le l_p^3|x_1 - x_2|.
\end{aligned}$$

Remark 2.2.1. *The activation functions are absolutely inherent components of the neural networks that influence the dynamical behavior of the designed neural networks. From both the assumptions, the existence and uniqueness of the solution of model (2.1) can be guaranteed due to continuity and Lipschitz condition of the activation functions [89].*

The system (2.1) with Assumption 2.2 can be written in four real-valued systems as

$$
\begin{aligned}
\dot{z}_p^R(t) =& - c_p z_p^R(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^R f_q^R(z_q^R(t)) - a_{pq}^I f_q^I(z_q^I(t)) - a_{pq}^J f_q^J(z_q^J(t)) \\
& - a_{pq}^K f_q^K(z_q^K(t)) + b_{pq}^R g_q^R(z_q^R(t-\tau_1(t))) - b_{pq}^I g_q^I(z_q^I(t-\tau_1(t))) \\
& - b_{pq}^J g_q^J(z_q^J(t-\tau_1(t))) - b_{pq}^K g_q^K(z_q^K(t-\tau_1(t))) \\
& + d_{pq}^R \int_{t-\tau_2(t)}^{t} h_q^R(z_q^R(s))ds - d_{pq}^I \int_{t-\tau_2(t)}^{t} h_q^I(z_q^I(s))ds \\
& - d_{pq}^J \int_{t-\tau_2(t)}^{t} h_q^J(z_q^J(s))ds - d_{pq}^K \int_{t-\tau_2(t)}^{t} h_q^K(z_q^K(s))ds + I_p^R(t) \Bigg), \\
\dot{z}_p^I(t) =& - c_p z_p^I(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^I f_q^R(z_q^R(t)) + a_{pq}^R f_q^I(z_q^I(t)) - a_{pq}^K f_q^J(z_q^J(t)) \\
& + a_{pq}^J f_q^K(z_q^K(t)) + b_{pq}^I g_q^R(z_q^R(t-\tau_1(t))) + b_{pq}^R g_q^I(z_q^I(t-\tau_1(t))) \\
& - b_{pq}^K g_q^J(z_q^J(t-\tau_1(t))) + b_{pq}^J g_q^K(z_q^K(t-\tau_1(t))) \\
& + d_{pq}^I \int_{t-\tau_2(t)}^{t} h_q^R(z_q^R(s))ds + d_{pq}^R \int_{t-\tau_2(t)}^{t} h_q^I(z_q^I(s))ds \\
& - d_{pq}^K \int_{t-\tau_2(t)}^{t} h_q^J(z_q^J(s))ds + d_{pq}^J \int_{t-\tau_2(t)}^{t} h_q^K(z_q^K(s))ds + I_p^I(t) \Bigg), \\
\dot{z}_p^J(t) =& - c_p z_p^J(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^J f_q^R(z_q^R(t)) + a_{pq}^K f_q^I(z_q^I(t)) + a_{pq}^R f_q^J(z_q^J(t)) \\
& - a_{pq}^I f_q^K(z_q^K(t)) + b_{pq}^J g_q^R(z_q^R(t-\tau_1(t))) + b_{pq}^K g_q^I(z_q^I(t-\tau_1(t))) \\
& + b_{pq}^R g_q^J(z_q^J(t-\tau_1(t))) - b_{pq}^I g_q^K(z_q^K(t-\tau_1(t))) \\
& + d_{pq}^J \int_{t-\tau_2(t)}^{t} h_q^R(z_q^R(s))ds + d_{pq}^K \int_{t-\tau_2(t)}^{t} h_q^I(z_q^I(s))ds \\
& + d_{pq}^R \int_{t-\tau_2(t)}^{t} h_q^J(z_q^J(s))ds - d_{pq}^I \int_{t-\tau_2(t)}^{t} h_q^K(z_q^K(s))ds + I_p^J(t) \Bigg), \\
\dot{z}_p^K(t) =& - c_p z_p^K(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^K f_q^R(z_q^R(t)) - a_{pq}^J f_q^I(z_q^I(t)) + a_{pq}^I f_q^J(z_q^J(t)) \\
& + a_{pq}^R f_q^K(z_q^K(t)) + b_{pq}^K g_q^R(z_q^R(t-\tau_1(t))) - b_{pq}^J g_q^I(z_q^I(t-\tau_1(t))) \\
& + b_{pq}^I g_q^J(z_q^J(t-\tau_1(t))) + b_{pq}^R g_q^K(z_q^K(t-\tau_1(t)))
\end{aligned}
$$

$$
\begin{aligned}
&+ d_{pq}^{K} \int_{t-\tau_2(t)}^{t} h_q^{R}(z_q^{R}(s))ds - d_{pq}^{J} \int_{t-\tau_2(t)}^{t} h_q^{I}(z_q^{I}(s))ds \\
&+ d_{pq}^{I} \int_{t-\tau_2(t)}^{t} h_q^{J}(z_q^{J}(s))ds + d_{pq}^{R} \int_{t-\tau_2(t)}^{t} h_q^{K}(z_q^{K}(s))ds + I_p^{K}(t)\Big). \qquad (2.3)
\end{aligned}
$$

The corresponding response system with identical parameters as (2.3) is presented by

$$
\begin{aligned}
\dot{s}_p^{R}(t) =& - c_p s_p^{R}(t) + \sum_{q=1}^{n} \Big(a_{pq}^{R} f_q^{R}(s_q^{R}(t)) - a_{pq}^{I} f_q^{I}(s_q^{I}(t)) - a_{pq}^{J} f_q^{J}(s_q^{J}(t)) \\
&- a_{pq}^{K} f_q^{K}(s_q^{K}(t)) + b_{pq}^{R} g_q^{R}(s_q^{R}(t-\tau_1(t))) - b_{pq}^{I} g_q^{I}(s_q^{I}(t-\tau_1(t))) \\
&- b_{pq}^{J} g_q^{J}(s_q^{J}(t-\tau_1(t))) - b_{pq}^{K} g_q^{K}(s_q^{K}(t-\tau_1(t))) \\
&+ d_{pq}^{R} \int_{t-\tau_2(t)}^{t} h_q^{R}(s_q^{R}(s))ds - d_{pq}^{I} \int_{t-\tau_2(t)}^{t} h_q^{I}(s_q^{I}(s))ds \\
&- d_{pq}^{J} \int_{t-\tau_2(t)}^{t} h_q^{J}(s_q^{J}(s))ds - d_{pq}^{K} \int_{t-\tau_2(t)}^{t} h_q^{K}(s_q^{K}(s))ds + I_p^{R}(t)\Big) + U_p^{R}(t), \\
\dot{s}_p^{I}(t) =& - c_p s_p^{I}(t) + \sum_{q=1}^{n} \Big(a_{pq}^{I} f_q^{R}(s_q^{R}(t)) + a_{pq}^{R} f_q^{I}(s_q^{I}(t)) - a_{pq}^{K} f_q^{J}(s_q^{J}(t)) \\
&+ a_{pq}^{J} f_q^{K}(s_q^{K}(t)) + b_{pq}^{I} g_q^{R}(s_q^{R}(t-\tau_1(t))) + b_{pq}^{R} g_q^{I}(s_q^{I}(t-\tau_1(t))) \\
&- b_{pq}^{K} g_q^{J}(s_q^{J}(t-\tau_1(t))) + b_{pq}^{J} g_q^{K}(s_q^{K}(t-\tau_1(t))) \\
&+ d_{pq}^{I} \int_{t-\tau_2(t)}^{t} h_q^{R}(s_q^{R}(s))ds + d_{pq}^{R} \int_{t-\tau_2(t)}^{t} h_q^{I}(s_q^{I}(s))ds \\
&- d_{pq}^{K} \int_{t-\tau_2(t)}^{t} h_q^{J}(s_q^{J}(s))ds + d_{pq}^{J} \int_{t-\tau_2(t)}^{t} h_q^{K}(s_q^{K}(s))ds + I_p^{I}(t)\Big) + U_p^{I}(t), \\
\dot{s}_p^{J}(t) =& - c_p s_p^{J}(t) + \sum_{q=1}^{n} \Big(a_{pq}^{J} f_q^{R}(s_q^{R}(t)) + a_{pq}^{K} f_q^{I}(s_q^{I}(t)) + a_{pq}^{R} f_q^{J}(s_q^{J}(t)) \\
&- a_{pq}^{I} f_q^{K}(s_q^{K}(t)) + b_{pq}^{J} g_q^{R}(s_q^{R}(t-\tau_1(t))) + b_{pq}^{K} g_q^{I}(s_q^{I}(t-\tau_1(t))) \\
&+ b_{pq}^{R} g_q^{J}(s_q^{J}(t-\tau_1(t))) - b_{pq}^{I} g_q^{K}(s_q^{K}(t-\tau_1(t))) \\
&+ d_{pq}^{J} \int_{t-\tau_2(t)}^{t} h_q^{R}(s_q^{R}(s))ds + d_{pq}^{K} \int_{t-\tau_2(t)}^{t} h_q^{I}(s_q^{I}(s))ds \\
&+ d_{pq}^{R} \int_{t-\tau_2(t)}^{t} h_q^{J}(s_q^{J}(s))ds - d_{pq}^{I} \int_{t-\tau_2(t)}^{t} h_q^{K}(s_q^{K}(s))ds + I_p^{J}(t)\Big) + U_p^{J}(t),
\end{aligned}
$$

$$\begin{aligned}\dot{s}_p^K(t) =& - c_p s_p^K(t) + \sum_{q=1}^{n}\bigg(a_{pq}^K f_q^R(s_q^R(t)) - a_{pq}^J f_q^I(s_q^I(t)) + a_{pq}^I f_q^J(s_q^J(t))\\ &+ a_{pq}^R f_q^K(s_q^K(t)) + b_{pq}^K g_q^R(s_q^R(t-\tau_1(t))) - b_{pq}^J g_q^I(s_q^I(t-\tau_1(t)))\\ &+ b_{pq}^I g_q^J(s_q^J(t-\tau_1(t))) + b_{pq}^R g_q^K(s_q^K(t-\tau_1(t)))\\ &+ d_{pq}^K\int_{t-\tau_2(t)}^{t} h_q^R(s_q^R(s))ds - d_{pq}^J\int_{t-\tau_2(t)}^{t} h_q^I(s_q^I(s))ds\\ &+ d_{pq}^I\int_{t-\tau_2(t)}^{t} h_q^J(s_q^J(s))ds + d_{pq}^R\int_{t-\tau_2(t)}^{t} h_q^K(s_q^K(s))ds + I_p^K(t)\bigg) + U_p^K(t).\end{aligned} \tag{2.4}$$

Let us define the error term as $e_p(t) = s_p(t) - z_p(t)$. Then from equations (2.3) and (2.4), we get

$$\begin{aligned}\dot{e}_p^R(t) =& - c_p e_p^R(t) + \sum_{q=1}^{n}\bigg(a_{pq}^R(f_q^R(s_q^R(t)) - f_q^R(z_q^R(t))) - a_{pq}^I(f_q^I(s_q^I(t))\\ &- f_q^I(z_q^I(t))) - a_{pq}^J(f_q^J(s_q^J(t)) - f_q^J(z_q^J(t))) - a_{pq}^K(f_q^K(s_q^K(t))\\ &- f_q^K(z_q^K(t))) + b_{pq}^R(g_q^R(s_q^R(t-\tau_1(t))) - g_q^R(z_q^R(t-\tau_1(t))))\\ &- b_{pq}^I(g_q^I(s_q^I(t-\tau_1(t))) - g_q^I(z_q^I(t-\tau_1(t)))) - b_{pq}^J(g_q^J(s_q^J(t-\tau_1(t)))\\ &- g_q^J(z_q^J(t-\tau_1(t)))) - b_{pq}^K(g_q^K(s_q^K(t-\tau_1(t))) - g_q^K(z_q^K(t-\tau_1(t))))\\ &+ d_{pq}^R\int_{t-\tau_2(t)}^{t}(h_q^R(s_q^R(s)) - h_q^R(z_q^R(s)))ds - d_{pq}^I\int_{t-\tau_2(t)}^{t}(h_q^I(s_q^I(s)) - h_q^I(z_q^I(s)))ds\\ &- d_{pq}^J\int_{t-\tau_2(t)}^{t}(h_q^J(s_q^J(s)) - h_q^J(z_q^J(s)))ds\\ &- d_{pq}^K\int_{t-\tau_2(t)}^{t}(h_q^K(s_q^K(s)) - h_q^K(z_q^K(s)))ds\bigg) + U_p^R(t),\\ \dot{e}_p^I(t) =& - c_p e_p^I(t) + \sum_{q=1}^{n}\bigg(a_{pq}^R(f_q^I(s_q^I(t)) - f_q^I(z_q^I(t))) + a_{pq}^I(f_q^R(s_q^R(t))\\ &- f_q^R(z_q^R(t))) + a_{pq}^J(f_q^K(s_q^K(t)) - f_q^K(z_q^K(t))) - a_{pq}^K(f_q^J(s_q^J(t))\\ &- f_q^J(z_q^J(t))) + b_{pq}^R(g_q^I(s_q^I(t-\tau_1(t))) - g_q^I(z_q^I(t-\tau_1(t))))\\ &+ b_{pq}^I(g_q^R(s_q^R(t-\tau_1(t))) - g_q^R(z_q^R(t-\tau_1(t)))) + b_{pq}^J(g_q^K(s_q^K(t-\tau_1(t)))\end{aligned}$$

$$
\begin{aligned}
&- g_q^K(z_q^K(t-\tau_1(t)))) - b_{pq}^K(g_q^J(s_q^J(t-\tau_1(t))) - g_q^J(z_q^J(t-\tau_1(t)))) \\
&+ d_{pq}^R \int_{t-\tau_2(t)}^{t} (h_q^I(s_q^I(s)) - h_q^I(z_q^I(s)))ds \\
&+ d_{pq}^I \int_{t-\tau_2(t)}^{t} (h_q^R(s_q^R(s)) - h_q^R(z_q^R(s)))ds \\
&+ d_{pq}^J \int_{t-\tau_2(t)}^{t} (h_q^K(s_q^K(s)) - h_q^K(z_q^K(s)))ds \\
&- d_{pq}^K \int_{t-\tau_2(t)}^{t} (h_q^J(s_q^J(s)) - h_q^J(z_q^J(s)))ds\Bigg) + U_p^I(t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_p^J(t) =& - c_p e_p^J(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^R(f_q^J(s_q^J(t)) - f_q^J(z_q^J(t))) - a_{pq}^I(f_q^K(s_q^K(t)) \\
&- f_q^K(z_q^K(t))) + a_{pq}^J(f_q^R(s_q^R(t)) - f_q^R(z_q^R(t))) + a_{pq}^K(f_q^I(s_q^I(t)) \\
&- f_q^I(z_q^I(t))) + b_{pq}^R(g_q^J(s_q^J(t-\tau_1(t))) - g_q^J(z_q^J(t-\tau_1(t)))) \\
&- b_{pq}^I(g_q^K(s_q^K(t-\tau_1(t))) - g_q^K(z_q^K(t-\tau_1(t)))) + b_{pq}^J(g_q^R(s_q^R(t-\tau_1(t))) \\
&- g_q^R(z_q^R(t-\tau_1(t)))) + b_{pq}^K(g_q^I(s_q^I(t-\tau_1(t))) - g_q^I(z_q^I(t-\tau_1(t)))) \\
&+ d_{pq}^R \int_{t-\tau_2(t)}^{t} (h_q^J(s_q^J(s)) - h_q^J(z_q^J(s)))ds \\
&- d_{pq}^I \int_{t-\tau_2(t)}^{t} (h_q^K(s_q^K(s)) - h_q^K(z_q^K(s)))ds \\
&+ d_{pq}^J \int_{t-\tau_2(t)}^{t} (h_q^R(s_q^R(s)) - h_q^R(z_q^R(s)))ds \\
&+ d_{pq}^K \int_{t-\tau_2(t)}^{t} (h_q^I(s_q^I(s)) - h_q^I(z_q^I(s)))ds\Bigg) + U_p^J(t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_p^K(t) =& - c_p e_p^K(t) + \sum_{q=1}^{n} \Bigg(a_{pq}^R(f_q^K(s_q^K(t)) - f_q^K(z_q^K(t))) + a_{pq}^I(f_q^J(s_q^J(t)) \\
&- f_q^J(z_q^J(t))) - a_{pq}^J(f_q^I(s_q^I(t)) - f_q^I(z_q^I(t))) + a_{pq}^K(f_q^R(s_q^R(t)) \\
&- f_q^R(z_q^R(t))) + b_{pq}^R(g_q^K(s_q^K(t-\tau_1(t))) - g_q^K(z_q^K(t-\tau_1(t)))) \\
&+ b_{pq}^I(g_q^J(s_q^J(t-\tau_1(t))) - g_q^J(z_q^J(t-\tau_1(t)))) - b_{pq}^J(g_q^I(s_q^I(t-\tau_1(t))) \\
&- g_q^I(z_q^I(t-\tau_1(t)))) + b_{pq}^K(g_q^R(s_q^R(t-\tau_1(t))) - g_q^R(z_q^R(t-\tau_1(t))))
\end{aligned}
$$

$$+ d^R_{pq} \int_{t-\tau_2(t)}^{t} (h^K_q(s^K_q(s)) - h^K_q(z^K_q(s)))ds$$
$$+ d^I_{pq} \int_{t-\tau_2(t)}^{t} (h^J_q(s^J_q(s)) - h^J_q(z^J_q(s)))ds$$
$$- d^J_{pq} \int_{t-\tau_2(t)}^{t} (h^I_q(s^I_q(s)) - h^I_q(z^I_q(s)))ds$$
$$+ d^K_{pq} \int_{t-\tau_2(t)}^{t} (h^R_q(s^R_q(s)) - h^R_q(z^R_q(s)))ds\Big) + U^K_p(t). \tag{2.5}$$

Define the controllers as

$$U^\gamma_p(t) = -\xi_{1p}e^\gamma_p(t) - sign(e^\gamma_p(t))\Big(\xi^\gamma_{2p}|e^\gamma_p(t-\tau_1(t))| + \xi^\gamma_{3p}\int_{t-\tau_2(t)}^{t}|e^\gamma_p(s)|ds$$
$$+ \xi_{4p}|e^\gamma_p(t)|^\alpha + \xi_{5p}|e^\gamma_p(t)|^\beta\Big), \tag{2.6}$$

where $\xi_{4p} > 0$, $\xi_{5p} > 0$, $0 < \alpha < 1$, $\beta > 1$. ξ_{1p}, ξ^γ_{2p} and ξ^γ_{3p} are the parameters those are to be defined later.

Remark 2.2.2. *The controller in the present chapter is designed in such a way that it contains linear and nonlinear terms. Nonlinear terms play a significant role in the rate of synchronization. Also, the settling time obtained is independent of the delays.*

Some definitions and lemmas used in this chapter are described below.

Definition 2.2.1. *([31]) The origin of the system (2.5) is said to achieve finite-time stability if there exists a function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ called settling time function such that $\lim_{t\to T(e_0)} \|e(t)\| = 0$ and $e(t) = 0$ for all $t \geq T(e_0)$. Here, e_0 is the initial value of the system (2.5).*

Definition 2.2.2. *([31]) The origin of the system (2.5) is called fixed time stable if it is finite time stable and the settling time function is bounded, i.e., there exists a positive constant $T_{\max}$ such that $T(e_0) < T_{\max}$ for all $e_0 \in \mathbb{H}^n$.*

Remark 2.2.3. *From the definitions of finite-time and fixed-time stabilities, it can be seen that in finite-time stability, the settling time depends on initial conditions, i.e., for every change in initial conditions, we will have different settling time expressions. In fixed-time stability, the settling time is invariant of initial conditions, i.e., the settling time expression is independent of initial conditions.*

Lemma 2.2.1. *[90] Let* $V(.) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ *is a continuous and radially unbounded function. If* $e(t)$ *is any solution of expression* (2.5)*, then the origin of the system* (2.5) *is fixed time stable provided*

(i) $V(e(t)) = 0$ *iff* $e(t) = 0$*;*

(ii) for some $k_1,\ k_2,\ k_3 > 0$,

$$\dot{V}(t) \leq -k_1 V^{\alpha}(e(t)) - k_2 V^{\beta}(e(t)) - k_3 V(e(t)),\ 0 < \alpha < 1 \text{ and } \beta > 1.$$

The settling time expression is given by $T^1_{set} = \frac{1}{k_3(1-\alpha)} ln(1 + \frac{k_3}{k_1}) + \frac{1}{k_3(\beta-1)} ln(1 + \frac{k_3}{k_2})$.

Lemma 2.2.2. *[31] If* $V(.) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ *is a continuous and radially unbounded function and* $e(t)$ *is any solution of* (2.5)*, then the origin of the system* (2.5) *is fixed time stable provided*

(i) $V(e(t)) = 0$ *iff* $e(t) = 0$*;*

(ii) for some $k_1,\ k_2 > 0$,

$$\dot{V}(t) \leq -k_1 V^{\alpha}(e(t)) - k_2 V^{\beta}(e(t)),\ 0 < \alpha < 1 \text{ and } \beta > 1.$$

The settling time expression is given by $T^2_{set} = \frac{1}{k_1(1-\alpha)} + \frac{1}{k_2(\beta-1)}$.

Lemma 2.2.3. *[91] Let $z_p \geq 0$ for $p = 1,\ 2,\ \ldots,\ n$; $0 < r \leq 1$ and $s > 1$. Then, the following inequality holds.*

$$\sum_{p=1}^{n} z_p^r \geq (\sum_{p=1}^{n} z_p)^r,\ \sum_{p=1}^{n} z_p^s \geq n^{1-s}(\sum_{p=1}^{n} z_p)^s.$$

2.3 Main results

Theorem 2.3.1. *The system (2.5) with the Assumption 2.2.2 and the controllers (2.6) gains fixed time stability if it satisfies the following conditions:*

$$c_p + \xi_{1p} - \sum_{q=1}^{n} \big(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\big) l_p^1 > 0,$$
$$\xi_{2p}^{\gamma} - \sum_{q=1}^{n} \big(|b_{qp}^R| + |b_{qp}^I| + |b_{qp}^J| + |b_{qp}^K|\big) l_p^2 > 0,$$
$$\xi_{3p}^{\gamma} - \sum_{q=1}^{n} \big(|d_{qp}^R| + |d_{qp}^I| + |d_{qp}^J| + |d_{qp}^K|\big) l_p^3 > 0.$$

Here, the expression of settling time is
$T_{set}^1 = \frac{1}{c(1-\alpha)} \ln\Big(1 + \frac{c}{\min_p(\xi_{4p})}\Big) + \frac{1}{c(\beta-1)} \ln\Big(1 + \frac{c}{\min_p(\xi_{5p})(4n)^{1-\beta}}\Big),$
where $c = \min_p\Bigg(c_p + \xi_{1p} - \sum_{q=1}^{n}\big(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|\big) l_p^1\Bigg).$

Proof. Consider the Lyapunov functional as

$$V(t) = \sum_{i=1}^{4} V_i(t), \tag{2.7}$$

where $V_1(t) = \sum_{i=1}^{n} |e_p^R(t)|,\ V_2(t) = \sum_{i=1}^{n} |e_p^I(t)|,\ V_3(t) = \sum_{i=1}^{n} |e_p^J(t)|,\ V_4(t) =$

$\sum_{i=1}^{n}|e_p^K(t)|$. Taking the Dini derivative of $V_1(t)$ along the trajectories of the considered system, we get

$$\begin{aligned}
\dot{V}_1(t) =& \sum_{p=1}^{n} sign(e_p^R(t))\dot{e}_p^R(t) \\
=& \sum_{p=1}^{n} sign(e_p^R(t))\Big(- c_p e_p^R(t) + \sum_{q=1}^{n}(a_{pq}^R(f_q^R(s_q^R(t)) - f_q^R(z_q^R(t))) \\
& - a_{pq}^I(f_q^I(s_q^I(t)) - f_q^I(z_q^I(t))) - a_{pq}^J(f_q^J(s_q^J(t)) - f_q^J(z_q^J(t))) \\
& - a_{pq}^K(f_q^K(s_q^K(t)) - f_q^K(z_q^K(t))) + b_{pq}^R(g_q^R(s_q^R(t-\tau_1(t))) \\
& - g_q^R(z_q^R(t-\tau_1(t)))) - b_{pq}^I(g_q^I(s_q^I(t-\tau_1(t))) - g_q^I(z_q^I(t-\tau_1(t)))) \\
& - b_{pq}^J(g_q^J(s_q^J(t-\tau_1(t))) - g_q^J(z_q^J(t-\tau_1(t)))) - b_{pq}^K(g_q^K(s_q^K(t-\tau_1(t))) \\
& - g_q^K(z_q^K(t-\tau_1(t)))) + d_{pq}^R\int_{t-\tau_2(t)}^{t}(h_q^R(s_q^R(s)) - h_q^R(z_q^R(s)))ds \\
& - d_{pq}^I\int_{t-\tau_2(t)}^{t}(h_q^I(s_q^I(s)) - h_q^I(z_q^I(s)))ds - d_{pq}^J\int_{t-\tau_2(t)}^{t}(h_q^J(s_q^J(s)) - h_q^J(z_q^J(s)))ds \\
& - d_{pq}^K\int_{t-\tau_2(t)}^{t}(h_q^K(s_q^K(s)) - h_q^K(z_q^K(s)))ds) + U_p^R(t)\Big) \\
\leq& - \sum_{p=1}^{n} c_p|e_p^R(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Big(|a_{pq}^R|l_q^1|e_q^R(t)| + |a_{pq}^I|l_q^1|e_q^I(t)| + |a_{pq}^J|l_q^1|e_q^J(t)| \\
& + |a_{pq}^K|l_q^1|e_q^K(t)| + |b_{pq}^R|l_q^2|e_q^R(t-\tau_1(t))| + |b_{pq}^I|l_q^2|e_q^I(t-\tau_1(t))| \\
& + |b_{pq}^J|l_q^2|e_q^J(t-\tau_1(t))| + |b_{pq}^K|l_q^2|e_q^K(t-\tau_1(t))| \\
& + |d_{pq}^R|l_q^3\int_{t-\tau_2(t)}^{t}|e_q^R(s)|ds + |d_{pq}^I|l_q^3\int_{t-\tau_2(t)}^{t}|e_q^I(s)|ds \\
& + |d_{pq}^J|l_q^3\int_{t-\tau_2(t)}^{t}|e_q^J(s)|ds + |d_{pq}^K|l_q^3\int_{t-\tau_2(t)}^{t}|e_q^K(s)|ds\Big) \\
& + \sum_{p=1}^{n} sign(e_p^R(t))\Big(- \xi_{1p}e_p^R(t) - sign(e_p^R(t))\Big(\xi_{2p}^R|e_p^R(t-\tau_1(t))| \\
& + \xi_{3p}^R\int_{t-\tau_2(t)}^{t}|e_p^R(s)|ds + \xi_{4p}|e_p^R(t)|^{\alpha} + \xi_{5p}|e_p^R(t)|^{\beta}\Big)\Big) \\
\leq& - \sum_{p=1}^{n}\left(c_p - \sum_{q=1}^{n}|a_{qp}^R|l_p^1 + \xi_{1p}\right)|e_p^R(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\Big(|a_{qp}^I|l_p^1|e_p^I(t)|
\end{aligned}$$

$$
\begin{aligned}
&+ |a_{qp}^J| l_p^1 |e_p^J(t)| + |a_{qp}^K| l_p^1 |e_p^K(t)| + |b_{qp}^R| l_p^2 |e_p^R(t-\tau_1(t))| \\
&+ |b_{qp}^I| l_p^2 |e_p^I(t-\tau_1(t))| + |b_{pq}^J| l_p^2 |e_p^J(t-\tau_1(t))| + |b_{qp}^K| l_p^2 |e_p^K(t-\tau_1(t))| \\
&+ |d_{qp}^R| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^R(s)| ds + |d_{qp}^I| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^I(s)| ds \\
&+ |d_{qp}^J| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^J(s)| ds + |d_{qp}^K| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^K(s)| ds \Bigg) \\
&- \sum_{p=1}^{n} \xi_{2p}^R |e_p^R(t-\tau_1(t))| - \sum_{p=1}^{n} \xi_{3p}^R \int_{t-\tau_2(t)}^{t} |e_p^R(s)| ds \\
&- \sum_{p=1}^{n} \big(\xi_{4p} |e_p^R(t)|^{\alpha} + \xi_{5p} |e_p^R(t)|^{\beta} \big). \qquad (2.8)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\dot{V}_2(t) \leq &- \sum_{p=1}^{n} \Bigg(c_p - \sum_{q=1}^{n} |a_{qp}^R| l_p^1 + \xi_{1p} \Bigg) |e_p^I(t)| \\
&+ \sum_{p=1}^{n} \sum_{q=1}^{n} \Bigg(|a_{qp}^I| l_p^1 |e_p^R(t)| + |a_{qp}^J| l_p^1 |e_p^K(t)| + |a_{qp}^K| l_p^1 |e_p^J(t)| \\
&+ |b_{qp}^R| l_p^2 |e_p^I(t-\tau_1(t))| + |b_{qp}^I| l_p^2 |e_p^R(t-\tau_1(t))| + |b_{pq}^J| l_p^2 |e_p^K(t-\tau_1(t))| \\
&+ |b_{qp}^K| l_p^2 |e_p^J(t-\tau_1(t))| + |d_{qp}^R| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^I(s)| ds \\
&+ |d_{qp}^I| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^R(s)| ds + |d_{pq}^J| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^K(s)| ds \\
&+ |d_{qp}^K| l_p^3 \int_{t-\tau_2(t)}^{t} |e_p^J(s)| ds \Bigg) - \sum_{p=1}^{n} \xi_{2p}^I |e_p^I(t-\tau_1(t))| \\
&- \sum_{p=1}^{n} \xi_{3p}^I \int_{t-\tau_2(t)}^{t} |e_p^I(s)| ds - \sum_{p=1}^{n} \big(\xi_{4p} |e_p^I(t)|^{\alpha} + \xi_{5p} |e_p^I(t)|^{\beta} \big). \\
\dot{V}_3(t) \leq &- \sum_{p=1}^{n} \Bigg(c_p - \sum_{q=1}^{n} |a_{qp}^R| l_p^1 + \xi_{1p} \Bigg) |e_p^J(t)| \\
&+ \sum_{p=1}^{n} \sum_{q=1}^{n} \Bigg(|a_{qp}^J| l_p^1 |e_p^R(t)| + |a_{qp}^K| l_p^1 |e_p^I(t)| + |a_{qp}^I| l_p^1 |e_p^K(t)| \\
&+ |b_{qp}^R| l_p^2 |e_p^J(t-\tau_1(t))| + |b_{qp}^J| l_p^2 |e_p^R(t-\tau_1(t))| + |b_{qp}^K| l_p^2 |e_p^I(t-\tau_1(t))|
\end{aligned}
$$

$$
\begin{aligned}
&+|b^I_{qp}|l^2_p|e^K_p(t-\tau_1(t))|+|d^R_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^J_p(s)|ds\\
&+|d^J_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^R_p(s)|ds+|d^K_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^I_p(s)|ds\\
&+|d^I_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^K_p(s)|ds\Bigg)-\sum_{p=1}^{n}\xi^J_{2p}|e^J_p(t-\tau_1(t))|\\
&-\sum_{p=1}^{n}\xi^J_{3p}\int_{t-\tau_2(t)}^{t}|e^J_p(s)|ds-\sum_{p=1}^{n}\big(\xi_{4p}|e^J_p(t)|^\alpha+\xi_{5p}|e^J_p(t)|^\beta\big).\\
\dot V_4(t)\le&-\sum_{p=1}^{n}\Bigg(c_p-\sum_{q=1}^{n}|a^R_{qp}|l^1_p+\xi_{1p}\Bigg)|e^K_p(t)|+\sum_{p=1}^{n}\sum_{q=1}^{n}\Bigg(|a^I_{qp}|l^1_p|e^J_p(t)|\\
&+|a^K_{qp}|l^1_p|e^R_p(t)|+|a^J_{qp}|l^1_p|e^I_p(t)|+|b^R_{qp}|l^2_p|e^K_p(t-\tau_1(t))|\\
&+|b^I_{qp}|l^2_p|e^J_p(t-\tau_1(t))|+|b^K_{qp}|l^2_p|e^R_p(t-\tau_1(t))|\\
&+|b^J_{qp}|l^2_p|e^I_p(t-\tau_1(t))|+|d^R_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^K_p(s)|ds\\
&+|d^I_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^J_p(s)|ds+|d^K_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^R_p(s)|ds\\
&+|d^J_{qp}|l^3_p\int_{t-\tau_2(t)}^{t}|e^I_p(s)|ds\Bigg)-\sum_{p=1}^{n}\xi^K_{2p}|e^K_p(t-\tau_1(t))|\\
&-\sum_{p=1}^{n}\xi^K_{3p}\int_{t-\tau_2(t)}^{t}|e^K_p(s)|ds-\sum_{p=1}^{n}\big(\xi_{4p}|e^K_p(t)|^\alpha+\xi_{5p}|e^K_p(t)|^\beta\big). \qquad (2.9)
\end{aligned}
$$

Merging these four inequalities, we have

$$
\begin{aligned}
\dot V(t)=&\dot V_1(t)+\dot V_2(t)+\dot V_3(t)+\dot V_4(t)\\
\le&-\sum_{p=1}^{n}\Bigg(c_p+\xi_{1p}-\sum_{q=1}^{n}\big(|a^R_{qp}|+|a^I_{qp}|+|a^J_{qp}|+|a^K_{qp}|\big)l^1_p\Bigg)|e^R_p(t)|\\
&-\sum_{p=1}^{n}\Bigg(\xi^R_{2p}-\sum_{q=1}^{n}\big(|b^R_{qp}|+|b^I_{qp}|+|b^J_{qp}|+|b^K_{qp}|\big)l^2_p\Bigg)|e^R_p(t-\tau_1(t))|\\
&-\sum_{p=1}^{n}\Bigg(\xi^R_{3p}-\sum_{q=1}^{n}\big(|d^R_{qp}|+|d^I_{qp}|+|d^J_{qp}|+|d^K_{qp}|\big)l^3_p\Bigg)\int_{t-\tau_2(t)}^{t}|e^R_p(s)|ds\\
&-\sum_{p=1}^{n}\Bigg(c_p+\xi_{1p}-\sum_{q=1}^{n}\big(|a^R_{qp}|+|a^I_{qp}|+|a^J_{qp}|+|a^K_{qp}|\big)l^1_p\Bigg)|e^I_p(t)|
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{p=1}^{n} \left(\xi_{2p}^{I} - \sum_{q=1}^{n} \left(|b_{qp}^{R}| + |b_{qp}^{I}| + |b_{qp}^{J}| + |b_{qp}^{K}|\right) l_p^2 \right) |e_p^I(t-\tau_1(t))| \\
&- \sum_{p=1}^{n} \left(\xi_{3p}^{I} - \sum_{q=1}^{n} \left(|d_{qp}^{R}| + |d_{qp}^{I}| + |d_{qp}^{J}| + |d_{qp}^{K}|\right) l_p^3 \right) \int_{t-\tau_2(t)}^{t} |e_p^I(s)| ds \\
&- \sum_{p=1}^{n} \left(c_p + \xi_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^{R}| + |a_{qp}^{I}| + |a_{qp}^{J}| + |a_{qp}^{K}|\right) l_p^1 \right) |e_p^J(t)| \\
&- \sum_{p=1}^{n} \left(\xi_{2p}^{J} - \sum_{q=1}^{n} \left(|b_{qp}^{R}| + |b_{qp}^{I}| + |b_{qp}^{J}| + |b_{qp}^{K}|\right) l_p^2 \right) |e_p^J(t-\tau_1(t))| \\
&- \sum_{p=1}^{n} \left(\xi_{3p}^{J} - \sum_{q=1}^{n} \left(|d_{qp}^{R}| + |d_{qp}^{I}| + |d_{qp}^{J}| + |d_{qp}^{K}|\right) l_p^3 \right) \int_{t-\tau_2(t)}^{t} |e_p^J(s)| ds \\
&- \sum_{p=1}^{n} \left(c_p + \xi_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^{R}| + |a_{qp}^{I}| + |a_{qp}^{J}| + |a_{qp}^{K}|\right) l_p^1 \right) |e_p^K(t)| \\
&- \sum_{p=1}^{n} \left(\xi_{2p}^{K} - \sum_{q=1}^{n} \left(|b_{qp}^{R}| + |b_{qp}^{I}| + |b_{qp}^{J}| + |b_{qp}^{K}|\right) l_p^2 \right) |e_p^K(t-\tau_1(t))| \\
&- \sum_{p=1}^{n} \left(\xi_{3p}^{K} - \sum_{q=1}^{n} \left(|d_{qp}^{R}| + |d_{qp}^{I}| + |d_{qp}^{J}| + |d_{qp}^{K}|\right) l_p^3 \right) \int_{t-\tau_2(t)}^{t} |e_p^K(s)| ds \\
&- \sum_{p=1}^{n} \xi_{4p} \left(|e_p^R(t)|^{\alpha} + |e_p^I(t)|^{\alpha} + |e_p^J(t)|^{\alpha} + |e_p^K(t)|^{\alpha} \right) \\
&- \sum_{p=1}^{n} \xi_{5p} \left(|e_p^R(t)|^{\beta} + |e_p^I(t)|^{\beta} + |e_p^J(t)|^{\beta} + |e_p^K(t)|^{\beta} \right). \qquad (2.10)
\end{aligned}
$$

By using Lemma 2.2.1 and Lemma 2.2.3, we obtain

$$
\begin{aligned}
\dot{V}(t) \leq& - \min_p \left(c_p + \xi_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^{R}| + |a_{qp}^{I}| + |a_{qp}^{J}| + |a_{qp}^{K}|\right) l_p^1 \right) V(t) \\
&- \sum_{p=1}^{n} \xi_{4p} \left(|e_p^R(t)|^{\alpha} + |e_p^I(t)|^{\alpha} + |e_p^J(t)|^{\alpha} + |e_p^K(t)|^{\alpha} \right) \\
&- \sum_{p=1}^{n} \xi_{5p} \left(|e_p^R(t)|^{\beta} + |e_p^I(t)|^{\beta} + |e_p^J(t)|^{\beta} + |e_p^K(t)|^{\beta} \right) \\
\leq& - \min_p \left(c_p + \xi_{1p} - \sum_{q=1}^{n} \left(|a_{qp}^{R}| + |a_{qp}^{I}| + |a_{qp}^{J}| + |a_{qp}^{K}|\right) l_p^1 \right) V(t) \\
&- \sum_{p=1}^{n} \xi_{4p} \left(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)| \right)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{p=1}^{n} 4^{1-\beta}\xi_{5p}\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big)^{\beta}\\
\leq &-\min_p\left(c_p+\xi_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^R|+|a_{qp}^I|+|a_{qp}^J|+|a_{qp}^K|\big)l_p^1\right)V(t)\\
&-\min_p(\xi_{4p})\left(\sum_{p=1}^{n}\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big)^{\alpha}\right)\\
&-\min_p(\xi_{5p})(4n)^{1-\beta}\left(\sum_{p=1}^{n}\big(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|\big)^{\beta}\right)\\
= &-\min_p(\xi_{4p})V^{\alpha}(t)-\min_p(\xi_{5p})(4n)^{1-\beta}V^{\beta}(t)\\
&-\min_p\left(c_p+\xi_{1p}-\sum_{q=1}^{n}\big(|a_{qp}^R|+|a_{qp}^I|+|a_{qp}^J|+|a_{qp}^K|\big)l_p^1\right)V(t). \qquad (2.11)
\end{aligned}
$$

□

Corollary 2.3.1. *The system* (2.5) *with the Assumption 2.2.2 and the controllers* (2.6) *achieves fixed time synchronization with the same sufficient conditions as in Theorem 2.3.1. The settling time obtained in this case is given by*

$$T_{set}^2=\frac{1}{\min_p(\xi_{4p})(1-\alpha)}+\frac{1}{\min_p(\xi_{5p})(4n)^{1-\beta}(\beta-1)}. \qquad (2.12)$$

Proof. Define the Lyapunov function as

$$V(t)=\sum_{p=1}^{n}(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|).$$

Calculating the Dini derivative of $V(t)$ along the trajectories of a given system as calculated in Theorem 2.3.1 in equation (2.7), and using Lemma 2.2.2 and Lemma 2.2.3, we obtain

$$\dot V(t)\leq-\min_p(\xi_{4p})V^{\alpha}(t)-\min_p(\xi_{5p})(4n)^{1-\beta}V^{\beta}(t),$$

and the corresponding settling time is given by expression (2.12). □

Remark 2.3.1. *Our considered model is more general and complex than the existing models to achieve the fixed-time synchronization of QVNNs. By taking the connection weights matrix $D = 0$ in the model (2.1), we will get the model used in [92] without distributed delay. Again, by taking the connection weight matrices $B = 0$ and $D = 0$ in (2.1), we get the model of [91] without delay term. However, the delay is inescapable in the actual implementation of neural networks.*

2.4 Numerical Example

In this section, a drive is made to validate the efficiency and effectiveness of the proposed method while applying it during the synchronization of two identical QVNNs with mixed time delays.

Example 2.4.1. *Consider QVNNs given in (2.1) for $n = 2$ as drive system as*

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bg(z(t - \tau_1(t))) + D\int_{t-\tau_2(t)}^{t} h(z(s))ds + I(t),$$

and a similar equation for the corresponding response system. For this, the parametric values are given below.

where $z = z^R + z^I i + z^J j + z^K k$,

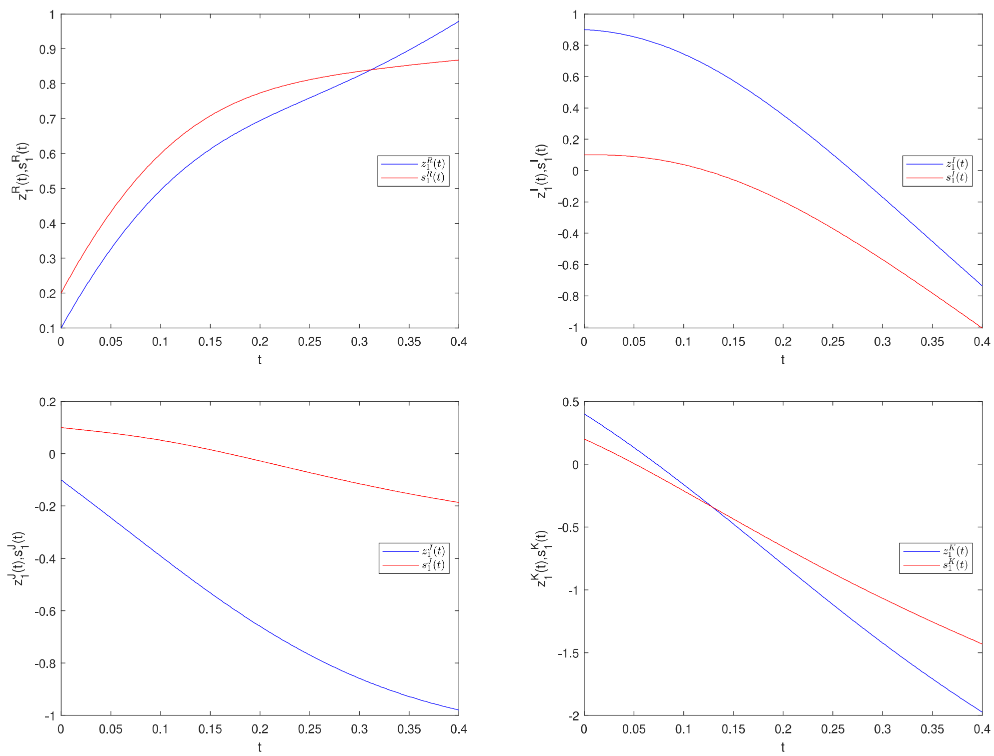

FIGURE 2.1: Plots of trajectories of $z_1^R(t)$ and $s_1^R(t)$, $z_1^I(t)$ and $s_1^I(t)$, $z_1^J(t)$ and $s_1^J(t)$, $z_1^K(t)$ and $s_1^K(t)$ for the drive-response systems (2.3) and (2.4) without controllers.

$$A = \begin{pmatrix} 1 + 2.2\bar{i} + 2\bar{j} - 1\bar{k} & -2 - 3.5\bar{i} + 2.5\bar{j} - 3\bar{k} \\ -2 + 1.5\bar{i} + 1\bar{j} + 1.5\bar{k} & 0.5 - 1.5\bar{i} + 0.5\bar{j} - 1\bar{k} \end{pmatrix},$$

$$B = \begin{pmatrix} 0.5 - 1.5\bar{i} + 2\bar{j} + -2\bar{k} & 1 + 1\bar{i} - 2\bar{j} - 1.5\bar{k} \\ 3 - \bar{i} + 1\bar{j} - 2\bar{k} & 0.5 - \bar{i} + 3\bar{j} - 2\bar{k} \end{pmatrix},$$

$$D = \begin{pmatrix} 0.3 + 2\bar{i} + 1\bar{j} + .5\bar{k} & -1 - 0.3\bar{i} + 2\bar{j} - 1\bar{k} \\ -1 - 0.2\bar{i} + 3\bar{j} - 1\bar{k} & 1 + 0.5\bar{i} - 1\bar{j} + 1.5\bar{k} \end{pmatrix},$$

$C = diag[1.5,\ 1.4],\ I = (1.2 - 1.2\bar{i} + 1.3\bar{j} + 1.4\bar{k},\ 1.4 + 1.3\bar{i} + 1.4\bar{j} - 1.3\bar{k})^T,$

$\tau_1(t) = \tau_2(t) = cos^2(t),$

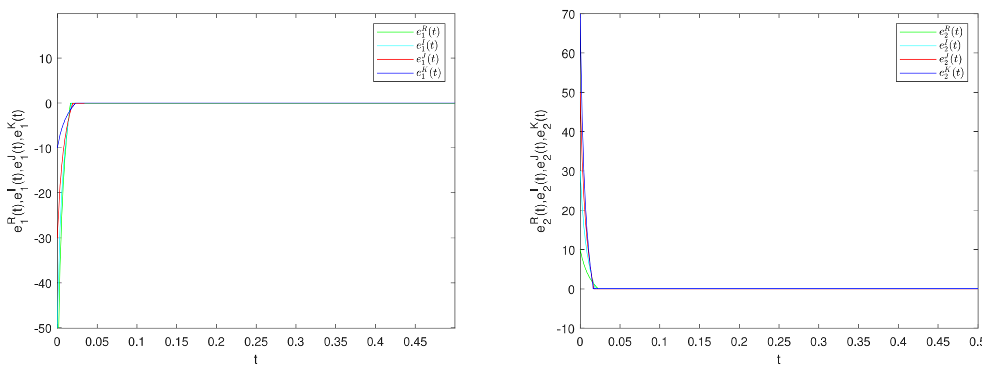

FIGURE 2.2: Plots of the trajectories of $e_1^R(t)$, $e_1^I(t)$, $e_1^J(t)$, $e_1^K(t)$ and $e_2^R(t)$, $e_2^I(t)$, $e_2^J(t)$, $e_2^K(t)$ error systems (2.5).

and let us take the activation functions as

$$f_j(z_j^\gamma) = tanh(z_j^\gamma),\ g_j(z_j^\gamma) = tanh(z_j^\gamma),\ h_j(z_j^\gamma) = tanh(z_j^\gamma),$$

for $j = 1,\ 2$ *and* $\gamma = R,\ I,\ J,\ K.$

Let us consider the initial conditions as

$z_1(0) = 80 + 70\bar{i} + 60\bar{j} + 50\bar{k},\ z_2(0) = 40 + 30\bar{i} + 20\bar{j} + 10\bar{k},\ s_1(0) = 10 + 20\bar{i} + 30\bar{j} + 40\bar{k},\ s_2(0) = 50 + 60\bar{i} + 70\bar{j} + 80\bar{k}.$

For this case, the trajectories of the system without controllers are shown in Figure 2.1, which clearly shows its non-synchronization. Again, Assumption 2.2.2 with the above-considered activation functions holds for $l_p^1 = 1 = l_p^2 = l_p^3$*. Now, considering* $\xi_{11} = 12,\ \xi_{12} = 14,\ \alpha = 0.5,\ \beta = 1.5,\ \xi_{21} = 20,\ \xi_{22} = 20,\ \xi_{31} = 20,\ \xi_{32} = 20,\ \xi_{4p} = 10,\ \xi_{5p} = 20$ *for* $p = 1,\ \ 2,$ *the controller functions become*

$$\begin{aligned}
U_1^\gamma(t) =& -12e_1^\gamma(t) - sign(e_1^\gamma(t))[20e_1^\gamma(t - cos^2(t)) + 20\int_{t-cos^2(t)}^{t} |e_1^\gamma(s)|ds \\
&+ 10|e_1^\gamma(t)|^{0.5} + 20|e_1^\gamma(t)|^{1.5}], \\
U_2^\gamma(t) =& -14e_2^\gamma(t) - sign(e_2^\gamma(t))[20e_2^\gamma(t - cos^2(t)) + 20\int_{t-cos^2(t)}^{t} |e_2^\gamma(s)|ds
\end{aligned}$$

$$+ 10|e_2^{\gamma}(t)|^{0.5} + 20|e_2^{\gamma}(t)|^{1.5}].$$

The synchronization of the trajectories of the systems with control functions is depicted in Figure 2.2. The settling times are calculated as $T^1_{set} = 0.4464$ *and* $T^2_{set} = 0.6656$*. Therefore, the settling time obtained by Lemma 2.2.1 gives a more accurate result than that of Lemma 2.2.2.*

Remark 2.4.1. *The settling time using Lemma 2.2.1 is less than that obtained using Lemma 2. The two settling time expressions are generally compared in [92]. Lemma 2.2.2 is used to prove synchronization results in [91, 93], which gives more conservative results.*

2.5 Conclusion

In this chapter, a drive is made to investigate the fixed time synchronization of QVNNs with mixed time-varying delays. To overcome the non-commutativity of quaternions, QVNNs have been decomposed into four equivalent RVNNs, which have supported the opening of a good scope of research in QVNNs as in RVNNs. To achieve the desired synchronization, a set of sufficient conditions is derived through designing a new controller and a proper choice of Lyapunov function. Two different lemmas have been used to get two different settling time expressions. The theoretical results are validated through a given numerical example.

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