CHAPTER 3

FIXED-TIME STABILITY THEOREM AND ITS APPLICATION TO NEURAL NETWORKS: AN ECONOMICAL CONTROL MECHANISM [1]

3.1 Introduction

In recent years, extensive research has been focused on exploring the dynamical behaviour of neural networks due to their various scientific and engineering applications that include associative memory, pattern recognition and image processing [115, 116, 117]. Among the various network models, Cohen-Grossberg neural networks (CGNNs) have emerged as a general network model that can be easily transformed into different network models like Hopfield neural networks [118], cellular neural networks [119, 120] and recurrent neural networks [121]. The CGNNs were first proposed by Cohen and Grossberg in 1983 [13], and have since found widespread applications like pattern recognition, parallel computation, and signal and image processing. It is important to keep in mind that time delay often arises in numerous real-world situations, which can potentially introduce instability in neural networks. Therefore, incorporating time delay effects into CGNNs has an intriguing research direction. Notably, many interesting results have been found in [48, 67, 85, 122, 123] and the associated references cited

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therein.

In 1986, Babcock and Westervelt [124] proposed a model of a neural network known as inertial neural networks by including an inductor into the neural circuit, based on a Hopfield neural network with the second-order derivative of the state. If the inertial nature of the neurons can be incorporated into their connection as shown in [124], the dynamics could be more complex than the conventional first-order system. The addition of the inertial term makes the chaotic search for memories in neural networks more apparent and demonstrates more complex behaviour such as bifurcation [125] and chaos [125, 126]. Based on the chaotic solution of inertial neural networks, the application in image encryption was put forward in [127]. Hence, the study of the dynamical behaviour and control design of inertial neural networks is found to be of great importance. Consequently, there is a significant research interest in inertial Cohen-Grossberg neural networks (ICGNNs) defined as second-order differential systems. The ICGNNs with delay have been discussed widely by researchers [54, 55, 128]. In [128], the authors have investigated the exponential stability of ICGNNs with delays; whereas, the authors in [54] have explored the exponential synchronization problem for ICGNNs with time delay. Moreover, the asymptotic stability of Markovian jump ICGNNs has been discussed in [55]. However, most of the results in these studies are focused on the stability and synchronization of ICGNNs based on the classical Lyapunov approach, and these properties are typically analyzed and established over an infinite time. Due to factors such as limited machine lifespan and practical application requirements like communication security, power grid, and voltage regulation, achieving stability and synchronization within a finite-time/fixed-time becomes necessary. This concept is known as finite-time/fixed-time stability or synchronization. Finite-time/fixed-time stability or synchronization indeed possesses several desirable properties, including limited convergence time, improved robustness, and enhanced disturbance rejection. Numerous results related to the finite-time/fixed-time technique have been put forth to explore the stability and synchronization of neural networks [65, 67, 82].

This chapter focuses on achieving fixed-time synchronization of ICGNNs in the presence of time-varying delays and desynchronizing impulsive effects. Firstly, a new lemma that ensures FxTS in the case of destabilizing impulses in the present context is investigated. This lemma is designed to involve fewer number of parameters as compared to Theorem 2.1.3.2 given in subchapter 2.1 and aims to provide a less conservative stability analysis. Furthermore, a unified controller is developed to achieve fixed-time synchronization of ICGNNs in the presence of desynchronizing impulsive effects and estimate the settling-time based on the proposed new lemma. Also, a sufficient condition $\tau_a > \frac{\ln \gamma}{a}$ is found for desynchronizing impulses and is established to achieve fixed-time synchronization of ICGNNs.

3.2 Problem Formulation and Preliminaries

The following ICGNNs are considered with delays and impulsive effects defined as the drive system:

$$\begin{cases} \ddot{x}_p(t) = -\beta_p \dot{x}_p(t) - \alpha_p(x_p(t))(h_p(x_p(t)) - \sum_{q=1}^{m} a_{pq} f_q(x_q(t)) \\ \qquad - \sum_{q=1}^{m} b_{pq} f_q(x_q(t - \tau_q(t))) + J_p(t)), & t \neq t_l, \\ x_p(t_l) = \mu x_p(t_l^-),\ \dot{x}_p(t_l) = \mu \dot{x}_p(t_l^-), l \in \mathbb{N}, \end{cases} \tag{3.2.1}$$

where $p, q = 1, 2, \ldots, m$, $x_p(t) \in \mathbb{R}$ is the pth neuron state at time t; $\ddot{x}_p(t)$ is said to be an inertial term of system (3.2.1); $\alpha_p(.)$ represents the neuron amplification function; $h_p(.)$ denotes an appropriately behaved function; $f_q(.)$ represents the activation function; $\tau_q(t)$ is the time-varying delay satisfies $0 \leq \tau_q(t) \leq \tau$; β_p is a positive constant; $J_p(t)$ denotes the external input; $A \triangleq (a_{pq})_{m\times m}$ and $B \triangleq (b_{pq})_{m\times m}$ be the connection weights. The impulsive sequence $\{t_1, t_2, \ldots, t_l, \ldots\}$ denotes strictly increasing moments such that

$\lim_{t\to\infty} t_l = \infty$ and $\mu \in \mathbb{R}^+$ be the impulsive strength. Assume that the state $x_p(t)$ is right continuous at $t = t_l$, i.e., $x_p(t_l^+) = \lim_{t\to t_l+0} x_p(t) = x_p(t_l)$ and $x_p(t_l^-) = \lim_{t\to t_l-0} x_p(t)$. Thus, the system's solution (3.2.1) is piecewise right continuous at $t = t_l$ for all $l \in \mathbb{N}$. The initial conditions of the system (3.2.1) are taken as $x_p(s) = \phi(s)$, $\dot{x}_p(s) = \psi(s)$, s$\in [-\tau, 0]$, where $\phi(s)$ and $\psi(s)$ are continuous functions defined on $[-\tau, 0]$.

Introducing the following variable transformation

$$y_p(t) = \dot{x}_p(t) + \epsilon_p x_p(t), \;\; p = 1, 2, \ldots, m, \tag{3.2.2}$$

where ϵ_p is the given positive scalar in $\mathbb{R}$, the ICGNNs (3.2.1) can be written as

$$\begin{cases} \dot{x}_p(t) = -\epsilon_p x_p(t) + y_p(t), & t \neq t_l, \\ \dot{y}_p(t) = -(\beta_p - \epsilon_p) y_p(t) - (\epsilon_p^2 - \beta_p \epsilon_p) x_p(t) & \\ \qquad -\alpha_p(x_p(t))(h_p(x_p(t)) - \sum_{q=1}^m a_{pq} f_q(x_q(t)) & \\ \qquad - \sum_{q=1}^m b_{pq} f_q(x_q(t - \tau_q(t))) + J_p(t)), & t \neq t_l, \\ x_p(t_l) = \mu x_p(t_l^-), & \\ y_p(t_l) = \eta x_p(t_l^-), l \in \mathbb{N}, & \end{cases} \tag{3.2.3}$$

where $\eta \in \mathbb{R}^+$ be the impulsive strength. The initial conditions are $x_p(s) = \phi(s), y_p(s) = \psi(s) + \epsilon_p \phi(s)$, $s \in [-\tau, 0]$. Furthermore, the corresponding response system without controller is defined as follows:

$$\begin{cases} \ddot{z}_p(t) = -\beta_p \dot{z}_p(t) - \alpha_p(z_p(t))(h_p(z_p(t)) & \\ \qquad - \sum_{q=1}^m a_{pq} f_q(z_q(t)) - \sum_{q=1}^m b_{pq} f_q(z_q(t - \tau_q(t))) + J_p(t)), & t \neq t_l, \\ z_p(t_l) = \mu z_p(t_l^-), \; \dot{z}_p(t_l) = \mu \dot{z}_p(t_l^-), l \in \mathbb{N}. & \end{cases} \tag{3.2.4}$$

The initial conditions are defined as $z_p(s) = \hat{\phi}(s), \dot{z}_p(s) = \hat{\psi}(s), s \in [-\tau, 0]$, where $\hat{\phi}(s)$ and $\hat{\psi}(s)$ are continuous functions defined on $[-\tau, 0]$. By introducing similar variable substitution as defined above, i.e., by taking $w_p(t) = \dot{z}_p(t) + \epsilon_p z_p(t)$, the response system is depicted as

$$\begin{cases} \dot{z}_p(t) = -\epsilon_p z_p(t) + w_p(t) + U_p, & t \neq t_l, \\ \dot{w}_p(t) = -(\beta_p - \epsilon_p)w_p(t) - (\epsilon_p^2 - \beta_p\epsilon_p)z_p(t) \\ \qquad -\alpha_p(z_p(t))(h_p(z_p(t)) - \sum_{q=1}^{m} a_{pq}f_q(z_q(t)) \\ \qquad - \sum_{q=1}^{m} b_{pq}f_q(z_q(t-\tau_q(t))) + J_p(t)) + \widetilde{U}_p, & t \neq t_l, \\ z_p(t_l) = \mu z_p(t_l^-), \\ w_p(t_l) = \eta w_p(t_l^-), l \in \mathbb{N}, \end{cases} \tag{3.2.5}$$

where U_p and $\widetilde{U}_p$ be the controllers which will be designed later on. The initial conditions are given by $z_p(s) = \hat{\phi}(s), w_p(s) = \hat{\psi}(s) + \epsilon_p\hat{\phi}(s), s \in [-\tau, 0]$.

Let us define the error system as $e_p(t) = z_p(t) - x_p(t)$ and $\hat{e}_p(t) = w_p(t) - y_p(t)$, so that the corresponding error system of (3.2.3) and (3.2.5) can be derived as

$$\begin{cases} \dot{e}_p(t) = -\epsilon_p e_p(t) + \hat{e}_p(t) + U_p, & t \neq t_l, \\ \dot{\hat{e}}_p(t) = -(\beta_p - \epsilon_p)\hat{e}_p(t) - (\epsilon_p^2 - \beta_p\epsilon_p)e_p(t) - (\alpha_p(z_p(t))h_p(z_p(t)) \\ \qquad -\alpha_p(x_p(t))h_p(x_p(t))) + \alpha_p(z_p(t))\sum_{q=1}^{m} a_{pq}(f_q(z_q(t)) - f_q(x_q(t))) \\ \qquad +(\alpha_p(z_p(t)) - \alpha_p(x_p(t)))\sum_{q=1}^{m} a_{pq}f_q(x_q(t)) \\ \qquad +(\alpha_p(z_p(t)) - \alpha_p(x_p(t)))\sum_{q=1}^{m} b_{pq}f_q(x_q(t-\tau_q(t))) \\ \qquad +\alpha_p(z_p(t))\sum_{q=1}^{m} b_{pq}(f_q(z_q(t-\tau_q(t))) - f_q(x_q(t-\tau_q(t)))) \\ \qquad -(\alpha_p(z_p(t)) - \alpha_p(x_p(t)))J_p(t) + \widetilde{U}_p, & t \neq t_l, \\ e_p(t_l) = \mu e_p(t_l^-), \\ \hat{e}_p(t_l) = \eta\hat{e}_p(t_l^-), l \in \mathbb{N}. \end{cases} \tag{3.2.6}$$

Now, for the main results of this chapter, some essential assumptions, lemmas and definitions are required for further analysis which are given below.

Assumption 3.2.1. The amplification function $\alpha_p(.)$ is differentiable and satisfies $\underline{\alpha}_p \leq \alpha_p(.) \leq \overline{\alpha}_p$ and $|\dot{\alpha}_p(.)| \leq \tilde{\alpha}_p$, where $\underline{\alpha}_p$, $\overline{\alpha}_p$ and $\tilde{\alpha}_p$ are some positive constants.

Assumption 3.2.2. Let $\Psi_p(.) \triangleq \alpha_p(.)h_p(.)$, where the function $\Psi_p(.)$ is differentiable and there exist positive constants $\overline{\Psi}_p$ and $\underline{\Psi}_p$ such that $\underline{\Psi}_p \leq \dot{\Psi}_p(.) \leq \overline{\Psi}_p$.

Assumption 3.2.3. The activation function $f_q(.)$ satisfies Lipschitz condition, and there exist positive Lipschitz constants l_q, m_q such that $|f_q(n_1) - f_q(n_2)| \leq l_q|n_1 - n_2|$, $|f_q(x_q)| \leq m_q$, where n_1, n_2, $x_q \in \mathbb{R}$, $q = 1, 2, \ldots, m$.

Definition 3.2.1. (See [129]) The response system (3.2.5) is called synchronized with the drive system (3.2.3) in a fixed-time, if for any initial value $\Phi(0)$, there exists a constant settling-time $T(\Phi(0)) \geq 0$ such that $\lim\limits_{t \to T(\Phi(0))} ||\Phi(t)|| = 0$, and $||\Phi(t)|| = 0$ for all $t \geq T(\Phi(0))$, where $\Phi(t) = (e_1(t), e_2(t), \ldots, e_p(t), \hat{e}_1(t), \hat{e}_2(t), \ldots, \hat{e}_p(t))^T$.

Definition 3.2.2. (See [129]) The response system (3.2.5) is called synchronized with the drive system (3.2.3) in a fixed-time, if it is synchronized in a finite-time and the settling-time function is bounded for any initial value $\Phi(0)$,i.e., there exists a $T_{max} > 0$ such that $T(\Phi(0)) \leq T_{max}$.

Lemma 3.2.1. (See [112]) Let $z_1, z_2, \ldots, z_k \geq 0$, $0 < \delta \leq 1$ and $\vartheta > 1$, then the following two inequalities hold:

$$\sum_{k=1}^{m} z_k^{\delta} \geq \left(\sum_{k=1}^{m} z_k\right)^{\delta}, \sum_{k=1}^{m} z_k^{\vartheta} \geq m^{1-\vartheta} \left(\sum_{k=1}^{m} z_k\right)^{\vartheta}.$$

Consider the following nonlinear impulsive system

$$\begin{cases} \dot{u}(t) = F(u(t)), & t \neq t_l, \\ \Delta u(t_l) = G(u(t_l^-)), & l \in \mathbb{N}, \end{cases} \tag{3.2.7}$$

where $u(t) \in \mathbb{R}^n$ be the state of system (3.2.7). Functions $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ are continuous with $F(0) = 0$ and $G(0) = 0$.

Definition 3.2.3. (See [50]) A function $V(u(t))$ defined on $\mathbb{R}^n$ is called to belong to class υ_0, if it satisfies

(1) V is continuous on the interval $[t_{l-1}, t_l)$, and for $u \in \mathbb{R}^n$, $l \in \mathbb{N}$ and $V(u(t_l^-)) = \lim\limits_{t \to t_l^-} V(u(t))$

(2) V is positive definite and radially unbounded.

(3) V is locally Lipschitz with respect to u and $V(0)$.

Lemma 3.2.2. For system (3.2.7), if there exists a function $V(u(t)) \in \upsilon_0$, $a, b > 0$, $1 \leq \beta < 2$ and $\gamma > 1$ such that

$$\begin{cases} \dot{V}(u(t)) \leq -aV(u(t)) - b\left(V(u(t))\right)^{\beta + \mathrm{sign}(V(u(t)) - 1)}, & t \neq t_l, \\ V(u(t_l)) \leq \gamma V(u(t_l^-)), l \in \mathbb{N}, \end{cases} \tag{3.2.8}$$

then the system (3.2.7) is FxTS if

$$\tau_a > \frac{\ln \gamma}{a}. \tag{3.2.9}$$

The settling-time function is estimated by $T = \frac{\ln\left[1 + \frac{\left(a - \frac{\ln \gamma}{\tau_a}\right)}{b\gamma^{-\beta N_0}}\right]}{\left(a - \frac{\ln \gamma}{\tau_a}\right)\beta} + \frac{\ln\left[\frac{b\gamma^{-(2-\beta)N_0}}{\gamma^{(2-\beta)N_0}\left(a - \frac{\ln \gamma}{\tau_a}\right) + b\gamma^{-(2-\beta)N_0}}\right]}{\left(\frac{\ln \gamma}{\tau_a} - a\right)(2-\beta)}$, where N_0 is a positive integer.

Proof. Consider the comparison system

$$\begin{cases} \dot{W}(t) = \begin{cases} -aW(t) - b\left(W(t)\right)^{\beta-1}, & 0 \le W(t) < 1, t \ne t_l, \\ -aW(t) - b\left(W(t)\right)^{\beta+\mathrm{sign}(W(t)-1)}, & W(t) \ge 1, t \ne t_l, \end{cases} \\ W(t_l) = \gamma W(t_l^-), t = t_l, \\ W(0) = W_0 \ge V(u(0)). \end{cases} \tag{3.2.10}$$

When compared with Eq. (3.2.10) and Eq. (3.2.8), we have $0 \le V(t) \le W(t)$. Then, if there exists a $T > 0$ such that $\lim\limits_{t\to T} W(t) = 0$ and $W(t) \equiv 0$, $\forall t \ge T$, we have $\lim\limits_{t\to T} V(t) = 0$ and $V(t) \equiv 0$, $\forall t \ge T$. To establish the FxTS for the system (3.2.7), we can proceed to the corresponding problem of system (3.2.8).

There are two cases that need to be discussed as follows:

Case 1: $W(t) \ge 1$.

Take $Q(t) = \left(W(t)\right)^{-\beta}$, then we have $Q(t)$ approaches 0 is similar to the case when $W(t)$ approaches $+\infty$ and $Q(t)$ approaches 1 is similar to the case when $W(t)$ approaches 1. Hence, we can get

$$\begin{cases} \dot{Q}(t) = b\beta + a\beta Q(t), & 0 < Q(t) \le 1, t \ne t_l, \\ Q(t_l) = \gamma_1 Q(t_l^-), t = t_l, \\ Q(0) = \left(W(0)\right)^{-\beta}, \end{cases} \tag{3.2.11}$$

where $\gamma_1 = \gamma^{-\beta} \in (0, 1)$. Therefore, we can see that $W(t)$ approaches 1 is equivalent to $Q(t)$ approaches 1.

Case 2: $0 < W(t) < 1$.

Take $Q(t) = \left(W(t)\right)^{2-\beta}$, we can get $Q(t)$ approaches 0 is similar to $W(t)$ approaches 0 and $Q(t)$ approaches 1 is similar to $W(t)$ approaches 1. Then $Q(t)$ becomes:

$$\begin{cases} \dot{Q}(t) = -b(2-\beta) - a(2-\beta)Q(t), \quad 0 < Q(t) < 1, t \neq t_l, \\ Q(t_l) = \gamma_2 Q(t_l^-), t = t_l, \\ Q(0) = (W(0))^{2-\beta}, \end{cases} \tag{3.2.12}$$

where $\gamma_2 = \gamma^{2-\beta} \in (1, \infty)$. Therefore, it can be follows that $W(t)$ approaches 0 is equivalent to $Q(t)$ approaches 0.

In view of above analysis, the FxTS of Eq. (3.2.10) have been divided into two cases: (i) the solution of Eq. (3.2.11) reaches 1 at fixed-time T_1 and (ii) the solution of Eq. (3.2.12) reaches 0 from 1 at fixed-time T_2. The FxTS of Eq. (3.2.10) is similar to Eq. (3.2.12) when Eq. (3.2.10) has an initial value smaller than 1. Therefore, the FxTS of Eq. (3.2.10) is guaranteed within the settling-time function $T = T_1 + T_2$. We consider two cases (i) and (ii) for impulsive strength $\gamma > 1$.

By utilizing the formula of variation of parameters, when $0 < Q(t) < 1$,the solution of Eq. (3.2.11) becomes

$$Q(t) = e^{a\beta t}\gamma_1^{N_\zeta(t,0)}Q(0) + b\beta \int_0^t e^{a\beta(t-s)}\gamma_1^{N_\zeta(t,s)}ds. \tag{3.2.13}$$

From Definition 2.1.2.4 and Eq. (3.2.13) with $Q(0) < 1$, we find that $Q(t)$ is increasing and $\lim_{t\to\infty} Q(t) = \infty$, when $0 < \gamma_1 < 1$. There exists $T_1 > 0$, such that $\lim_{t\to T_1} Q(t) = 1$ and $0 < Q(t) < 1 \ \forall \, 0 < t < T_1$, that is

$$e^{a\beta t}\left[\gamma_1^{N_\zeta(t,0)}Q(0) + b\beta \int_0^t e^{-a\beta s}\gamma_1^{N_\zeta(t,s)}ds\right] = 1.$$

Furthermore, we have

$$\begin{aligned}
b\beta e^{a\beta t}\int_0^t e^{-a\beta s}\gamma_1^{N_\zeta(t,s)}ds &\leq 1,\\
i.e.,\ b\beta e^{a\beta t}\int_0^t e^{-a\beta s}\gamma_1^{\frac{t-s}{\tau_a}+N_0}ds &\leq 1,
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \frac{e^{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]t}}{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]} &\leq \frac{1}{\gamma_1^{N_0}b\beta}+\frac{1}{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]},\\
i.e.,\ \left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]t &\leq \ln\left[1+\frac{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]}{\gamma_1^{N_0}b\beta}\right],\\
i.e.,\ t &\leq \frac{\ln\left[1+\frac{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]}{\gamma_1^{N_0}b\beta}\right]}{\left[a\beta+\frac{\ln\gamma_1}{\tau_a}\right]}=T_1.
\end{aligned}$$

Then substituting $\gamma_1=\gamma^{-\beta}$, one can find

$$T_1=\frac{\ln\left[1+\frac{\left(a-\frac{\ln\gamma}{\tau_a}\right)}{b\gamma^{-\beta N_0}}\right]}{\left(a-\frac{\ln\gamma}{\tau_a}\right)\beta}. \tag{3.2.14}$$

Similarly, Eq. (3.2.12) becomes

$$Q(t)=e^{-a(2-\beta)t}\gamma_2^{N_\zeta(t,0)}Q(0)-b(2-\beta)\int_0^t e^{-a(2-\beta)(t-s)}\gamma_2^{N_\zeta(t,s)}ds. \tag{3.2.15}$$

From Definition 2.1.2.4, and using $1<\gamma_2<\infty$ and $Q(0)=1$, we have

$$\begin{aligned}
Q(t)&=e^{-a(2-\beta)t}\gamma_2^{N_\zeta(t,0)}Q(0)-b(2-\beta)\int_0^t e^{-a(2-\beta)(t-s)}\gamma_2^{N_\zeta(t,s)}ds\\
&\leq e^{-a(2-\beta)t}\gamma_2^{\frac{t}{\tau_a}+N_0}-b(2-\beta)\int_0^t e^{-a(2-\beta)(t-s)}\gamma_2^{\frac{t-s}{\tau_a}-N_0}ds\\
&=\left[\gamma_2^{N_0}+\frac{b(2-\beta)\gamma_2^{-N_0}}{\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)}\right]e^{-\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)t}-\frac{b(2-\beta)\gamma_2^{-N_0}}{\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)}.
\end{aligned} \tag{3.2.16}$$

Assume that $R(t) = \left[\gamma_2^{N_0} + \frac{b(2-\beta)\gamma_2^{-N_0}}{\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)}\right] e^{-\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)t} - \frac{b(2-\beta)\gamma_2^{-N_0}}{\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)}$, then we have $R(0) = \gamma_2^{N_0} > 0$, $R(+\infty) = -\frac{b(2-\beta)\gamma_2^{-N_0}}{\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)} < 0$, and $\dot{R}(t) < 0$. Therefore, based on the fundamental calculus, there exists a unique $T_2 > 0$ such that $R(T_2) = 0$, which means that $R(t)$ is decreasing. Therefore, $Q(t) \leq R(t)$ and $Q(t) > 0$ then $Q(t)$ tends to 0 as t tends to T_2. From Eq. (3.2.16), we have

$$T_2 = \frac{\ln\left[\frac{b(2-\beta)\gamma_2^{-N_0}}{\gamma_2^{N_0}\left(a(2-\beta)-\frac{\ln\gamma_2}{\tau_a}\right)+b(2-\beta)\gamma_2^{-N_0}}\right]}{\left(\frac{\ln\gamma_2}{\tau_a} - a(2-\beta)\right)}.$$

Substituting $\gamma_2 = \gamma^{2-\beta}$ in above inequality, we have

$$T_2 = \frac{\ln\left[\frac{b\gamma^{-(2-\beta)N_0}}{\gamma^{(2-\beta)N_0}\left(a-\frac{\ln\gamma}{\tau_a}\right)+b\gamma^{-(2-\beta)N_0}}\right]}{\left(\frac{\ln\gamma}{\tau_a} - a\right)(2-\beta)}. \tag{3.2.17}$$

From Eq. (3.2.14) and Eq. (3.2.17), $Q(t)$ converges to 1 in fixed-time T_1 in case (i) and tends to 0 from 1 in fixed-time T_2 in case (ii). Therefore, we may observe that $Q(t) = 0$ for all $t \geq T_1 + T_2$. We found that for any initial value $W_0 > 0$, a fixed-time $T_1 + T_2$ exists which is independent of $V_0 > 0$, such that $\lim_{t\to T_1+T_2} W(t) = 0$ and $W(t) \equiv 0$ for $t \geq T_1 + T_2$. Hence, the system (3.2.7) is FxTS and the settling-time function is defined by $T = T_1 + T_2 = \frac{\ln\left[1+\frac{\left(a-\frac{\ln\gamma}{\tau_a}\right)}{b\gamma^{-\beta N_0}}\right]}{\left(a-\frac{\ln\gamma}{\tau_a}\right)\beta} + \frac{\ln\left[\frac{b\gamma^{-(2-\beta)N_0}}{\gamma^{(2-\beta)N_0}\left(a-\frac{\ln\gamma}{\tau_a}\right)+b\gamma^{-(2-\beta)N_0}}\right]}{\left(\frac{\ln\gamma}{\tau_a}-a\right)(2-\beta)}$. Therefore the proof of Lemma 3.2.2 is completed.

Remark 3.2.1. A new Lyapunov inequality has been presented in Lemma 3.2.2, which is different from Theorem 2.1.3.2 as given in subchapter 2.1. The Lyapunov inequality provides a sufficient condition for the system (3.2.7) to be FxTS in the case of destabilizing impulses. A linear term is added to the Lyapunov inequality to counter the destabilizing impulses. To achieve FxTS under the effect of destabilizing impulses corresponding to the system (3.2.7), we have found the condition $a > \frac{\ln \gamma}{\tau_a}$ for $a > 0$.

Remark 3.2.2. In [97], authors have considered the Lyapunov inequality as $\dot{V}(u(t)) \leq -aV(u(t))^{\alpha} - b\left(V(u(t))\right)^{\beta + \operatorname{sign}(V(u(t))-1)}$ where a, $b > 0$, $\beta > 1$, $0 < \alpha < 1$, in which two nonlinear terms are involved. But in [100], the Lyapunov inequality is considered as $\dot{V}(u(t)) \leq -a\left(V(u(t))\right)^{\beta + \operatorname{sign}(V(u(t))-1)}$ where $a > 0$, $1 \leq \beta < 2$, in which only one nonlinear term are involved. In the above two Lyapunov inequalities, the authors have achieved the FxTS for the system (3.2.7) corresponding to stabilizing impulses. However, the FxTS for a system (3.2.7) with destabilizing impulses has not been explored in the aforementioned literatures. The present research is focused on filling this gap and investigating in this direction.

Remark 3.2.3. Lemma 3.2.2 shows that there is only one exponent of the Lyapunov inequality, which serves to explicitly explain the two cases, i.e., the case when $V > 1$ and the case when $0 < V < 1$. This suggests that the Lyapunov inequality's exponent is crucial to the Lyapunov inequality itself and can add the two terms, one of which is greater than one and another of which is between 0 and 1. Therefore, Lemma 3.2.2 is the extension of the existing FxTS theorems and lemmas containing two terms in Theorem 2.1.3.2 as given in subchapter 2.1 and [97].

Remark 3.2.4. To realize FxTS, two nonlinear terms are required one is $-aV(u(t))^{\alpha}$ where $a > 0$, $0 < \alpha < 1$ and the second one is $-bV(u(t))^{\beta}$ where $b > 0, \beta > 1$. The FxTS achieved with two nonlinear terms may consume more energy. As a result, it may become less useful in practical. On the contrary, this study takes into account one nonlinear term, which simplifies the condition and reduces the number of parameters.

3.3 Fixed-Time Synchronization

In this section, a sufficient condition is proposed to achieve the fixed-time synchronization for ICGNNs (3.2.1) with desynchronizing impulses. The following controller (3.2.8), based on the Lyapunov inequality, is taken into consideration to control the system:

$$\begin{cases} U_p = -k_1 e_p(t) - \lambda \mathrm{sign}(e_p(t))|e_p(t)|^{\beta+\mathrm{sign}(V(e_p(t))-1)}, \\ \widetilde{U}_p = -k_2 \hat{e}_p(t) - \lambda \mathrm{sign}(\hat{e}_p(t))|\hat{e}_p(t)|^{\beta+\mathrm{sign}(V(\hat{e}_p(t))-1)} - \mathrm{sign}(\hat{e}_p(t)) \sum_{q=1}^{m} \delta_q |e_q(t-\tau_q(t))|, \end{cases} \tag{3.3.1}$$

where λ is a positive number and $1 \leq \beta < 2$ be a positive number, the other control parameters k_1, k_2 and δ_q are positive constants to be found out later on.

Theorem 3.3.1. Based on Assumptions 3.2.1-3.2.3 and if the control gains satisfy

$$\begin{cases} k_1 \geq -\epsilon_p + \beta_p \epsilon_p - \epsilon_p^2 - \underline{\Psi}_p + \sum_{q=1}^{m} \overline{\alpha}_q |a_{qp}| l_p + \tilde{\alpha}_p \sum_{q=1}^{m} |a_{pq}| m_q \\ \qquad + \tilde{\alpha}_p \sum_{q=1}^{m} |b_{pq}| m_q + \tilde{\alpha}_p |J_p^{+}(t)|, \\ k_2 \geq \epsilon_p - \beta_p + 1, \\ \delta_q \geq \overline{\alpha}_p |b_{pq}| l_q, \end{cases} \tag{3.3.2}$$

where $p, q = 1, 2, \ldots, m$, then the drive system (3.2.3) and response system (3.2.5) can be fixed-time synchronized under the controller (3.3.1) if

$$\tau_a > \frac{\ln \varpi}{\Gamma_{\min}}. \tag{3.3.3}$$

The settling-time is evaluated by $T = \frac{\ln\left[1+\frac{\left(\Gamma_{\min}-\frac{\ln\varpi}{\tau_a}\right)}{\lambda(2m)^{-\beta}\varpi^{-\beta N_0}}\right]}{\left(\Gamma_{\min}-\frac{\ln\varpi}{\tau_a}\right)\beta} + \frac{\ln\left[\frac{\lambda\varpi^{-(2-\beta)N_0}}{\varpi^{(2-\beta)N_0}\left(\Gamma_{\min}-\frac{\ln\varpi}{\tau_a}\right)+\lambda\varpi^{-(2-\beta)N_0}}\right]}{\left(\frac{\ln\varpi}{\tau_a}-\Gamma_{\min}\right)(2-\beta)}$.

Proof. Consider the Lyapunov function as

$$V(t) = V_1(t) + V_2(t) = \sum_{p=1}^{m} |e_p(t)| + \sum_{p=1}^{m} |\hat{e}_p(t)|. \tag{3.3.4}$$

Since the function $\alpha_p(.)$ and $h_p(.)$ are differentiable, by utilizing the mean value theorem, we can get

$$\alpha_p(z_p(t)) - \alpha_p(x_p(t)) = \dot{\alpha}_p(\iota_p)e_p(t),$$
$$(\alpha_p(z_p(t))h_p(z_p(t)) - \alpha_p(x_p(t))h_p(x_p(t))) = \Psi_p(z_p(t)) - \Psi_p(x_p(t)) = \dot{\Psi}_p(\tilde{\iota}_p)e_p(t),$$

where ι_p and $\tilde{\iota}_p$ lie between $z_p(t)$ and $x_p(t)$. From Eq. (3.3.4), differentiating $V_1(t)$ along the system's solution (3.2.6) for $t \neq t_l$, we obtain

$$\begin{aligned} V_1(t) &= \sum_{p=1}^{m} \mathrm{sign}(e_p(t))\dot{e}_p(t) \\ &= \sum_{p=1}^{m} \mathrm{sign}(e_p(t))\{-\epsilon_p e_p(t) + \hat{e}_p(t) - k_1 e_p(t) \\ &\leq \sum_{p=1}^{m} \{(-\epsilon_p - k_1)|e_p(t)| + \hat{e}_p(t) \\ &- \lambda |e_p(t)|^{\beta + \mathrm{sign}(V(e_p(t)) - 1)}\}. \end{aligned} \tag{3.3.5}$$

Similarly, differentiating $V_2(t)$ along the system's solution (3.2.6) for $t \neq t_l$, we get

$$\begin{aligned} V_2(t) &= \sum_{p=1}^{m} \mathrm{sign}(\hat{e}_p(t))\dot{\hat{e}}_p(t) \\ &= \sum_{p=1}^{m} \mathrm{sign}(\hat{e}_p(t))\{-(\beta_p - \epsilon_p)\hat{e}_p(t) - (\epsilon_p^2 - \beta_p \epsilon_p)e_p(t) \\ &- \dot{\Psi}_p(\tilde{\iota}_p)e_p(t) + \alpha_p(z_p(t)) \sum_{q=1}^{m} a_{pq}(f_q(z_q(t)) - f_q(x_q(t))) \end{aligned}$$

$$
\begin{aligned}
&+ \dot{\alpha}_p(\iota_p)e_p(t)\sum_{q=1}^{m} a_{pq}f_q(x_q(t)) \\
&+ \dot{\alpha}_p(\iota_p)e_p(t)\sum_{q=1}^{m} b_{pq}f_q(x_q(t-\tau_q(t))) \\
&+ \alpha_p(z_p(t))\sum_{q=1}^{m} b_{pq}(f_q(z_q(t-\tau_q(t))) - f_q(x_q(t-\tau_q(t)))) \\
&- \dot{\alpha}_p(\iota_p)e_p(t)J_p(t) - k_2\hat{e}_p(t) - \lambda\operatorname{sign}(\hat{e}_p(t))|\hat{e}_p(t)|^{\beta+\operatorname{sign}(V(\hat{e}_p(t))-1)} \\
&- \operatorname{sign}(\hat{e}_p(t))\sum_{q=1}^{m}\delta_q|e_q(t-\tau_q(t))|\},
\end{aligned}
$$

$$
\begin{aligned}
i.e., V_2(t) \leq \sum_{p=1}^{m}\{&-(\beta_p - \epsilon_p + k_2)|\hat{e}_p(t)| + (\beta_p\epsilon_p - \epsilon_p^2 - \underline{\Psi}_p)|e_p(t)|\} \\
&+ \overline{\alpha}_p\sum_{q=1}^{m}|a_{pq}|l_q|e_q(t)| + \tilde{\alpha}_p|e_p(t)|\sum_{q=1}^{m}|a_{pq}|m_q + \tilde{\alpha}_p|e_p(t)|\sum_{q=1}^{m}|b_{pq}|m_q \\
&+ \tilde{\alpha}_p|e_p(t)||J_p^{+}(t)| + \overline{\alpha}_p\sum_{q=1}^{m}|b_{pq}|l_q|e_q(t-\tau_q(t))| - \lambda|\hat{e}_p(t)|^{\beta+\operatorname{sign}(V(\hat{e}_p(t))-1)} \\
&- \sum_{q=1}^{m}\delta_q|e_q(t-\tau_q(t))|\}. \qquad (3.3.6)
\end{aligned}
$$

From Eq. (3.3.5) and Eq. (3.3.6), we have

$$
\begin{aligned}
\dot{V}(t) \leq \sum_{p=1}^{m}(&-\epsilon_p + \beta_p\epsilon_p - \epsilon_p^2 - \underline{\Psi}_p - k_1 + \sum_{q=1}^{m}\overline{\alpha}_q|a_{qp}|l_p + \tilde{\alpha}_p\sum_{q=1}^{m}|a_{pq}|m_q \\
&+ \tilde{\alpha}_p\sum_{q=1}^{m}|b_{pq}|m_q + \tilde{\alpha}_p|J_p^{+}(t)|)|e_p(t)| + \sum_{p=1}^{m}(\epsilon_p - \beta_p - k_2 + 1)|\hat{e}_p(t)| \\
&+ \sum_{p=1}^{m}\sum_{q=1}^{m}(\overline{\alpha}_p|b_{pq}|l_q - \delta_q)|e_q(t-\tau_q(t))| - \lambda|e_p(t)|^{\beta+\operatorname{sign}(V(e_p(t))-1)} - \lambda|\hat{e}_p(t)|^{\beta+\operatorname{sign}(V(\hat{e}_p(t))-1)}.
\end{aligned}
$$

If $k_1 \geq -\epsilon_p + \beta_p\epsilon_p - \epsilon_p^2 - \underline{\Psi}_p + \sum_{q=1}^{m}\overline{\alpha}_q|a_{qp}|l_p + \tilde{\alpha}_p\sum_{q=1}^{m}|a_{pq}|m_q + \tilde{\alpha}_p\sum_{q=1}^{m}|b_{pq}|m_q +$

$\tilde{\alpha}_p|J_p^+(t)|$, $k_2 \geq \epsilon_p - \beta_p + 1$, $\delta_q \geq \overline{\alpha}_p|b_{pq}|l_q$, p, $q = 1, 2, \ldots, m$, one can obtain that

$$\dot{V}(t) \leq -\Gamma_{\min}\sum_{p=1}^{m}(|e_p(t)|+|\hat{e}_p(t)|)-\lambda\sum_{p=1}^{m}(|e_p(t)|^{\beta+\text{sign}(V(e_p(t))-1)}+|\hat{e}_p(t)|^{\beta+\text{sign}(V(\hat{e}_p(t))-1)}), \tag{3.3.7}$$

where $\Gamma_1 = \epsilon_p - \beta_p\epsilon_p + \epsilon_p^2 + \underline{\Psi}_p + k_1 - \overline{\alpha}_p\sum_{q=1}^{m}\overline{\alpha}_q|a_{qp}|l_p - \tilde{\alpha}_p\sum_{q=1}^{m}|a_{pq}|m_q - \tilde{\alpha}_p\sum_{q=1}^{m}|b_{pq}|m_q - \tilde{\alpha}_p|J_p^+(t)| > 0$, $\Gamma_2 = -\epsilon_p + \beta_p + k_2 - 1 > 0$ and $\Gamma_{\min} = \min\{\min\Gamma_1, \min\Gamma_2\}$. When $V(t) \geq 1$, $\beta + \text{sign}(V(t) - 1) = \beta + 1 > 1$, then following inequality holds according to Lemma 3.2.1

$$\begin{aligned}\sum_{p=1}^{m}|e_p(t)|^{\beta+1} + \sum_{p=1}^{m}|\hat{e}_p(t)|^{\beta+1} &\geq m^{-\beta}\left[\left(\sum_{p=1}^{m}|e_p(t)|\right)^{\beta+1} + \left(\sum_{p=1}^{m}|\hat{e}_p(t)|\right)^{\beta+1}\right] \\ &\geq (2m)^{-\beta}\left(\sum_{p=1}^{m}|e_p(t)| + \sum_{p=1}^{m}|\hat{e}_p(t)|\right)^{\beta+1}.\end{aligned} \tag{3.3.8}$$

When $0 < V(t) < 1$, $0 < \beta + \text{sign}(V(t) - 1) = \beta - 1 < 1$, then it follows from Lemma 3.2.1 that

$$\begin{aligned}\sum_{p=1}^{m}|e_p(t)|^{\beta-1} + \sum_{p=1}^{m}|\hat{e}_p(t)|^{\beta-1} &\geq \left(\sum_{p=1}^{m}|e_p(t)|\right)^{\beta-1} + \left(\sum_{p=1}^{m}|\hat{e}_p(t)|\right)^{\beta-1} \\ &\geq \left(\sum_{p=1}^{m}|e_p(t)| + \sum_{p=1}^{m}|\hat{e}_p(t)|\right)^{\beta-1}.\end{aligned} \tag{3.3.9}$$

From inequalities (3.3.7), (3.3.8) and (3.3.9), we find that when $t \neq t_l$,

$$\dot{V}(t) \leq \begin{cases} -\Gamma_{\min}V(t) - \lambda(2m)^{-\beta}V(t)^{\beta+1}, & V(t) \geq 1, \\ -\Gamma_{\min}V(t) - \lambda V(t)^{\beta-1}, & 0 < V(t) < 1. \end{cases} \tag{3.3.10}$$

Additionally, when $t = t_l$ we have

$$\begin{aligned} V(t_l) &= V_1(t_l) + V_2(t_l) \\ &= \sum_{p=1}^{m} |e_p(t_l)| + \sum_{p=1}^{m} |\hat{e}_p(t_l)| \\ &= \sum_{p=1}^{m} \mu |e_p(t_l^-)| + \sum_{p=1}^{m} \eta |\hat{e}_p(t_l^-)| \\ &\leq \max\{\mu, \eta\} V(t_l^-) = \varpi V(t_l^-), \end{aligned} \tag{3.3.11}$$

where $\varpi = \max\{\mu, \eta\}$ and we have $\varpi > 1$. Hence, based on Lemma 3.2.2, the drive system (3.2.3) and response system (3.2.5) can be fixed-time synchronized and the settling-time T is estimated by

$$T = \frac{\ln\left[1 + \frac{\left(\Gamma_{\min} - \frac{\ln\varpi}{\tau_a}\right)}{\lambda(2m)^{-\beta}\varpi^{-\beta N_0}}\right]}{\left(\Gamma_{\min} - \frac{\ln\varpi}{\tau_a}\right)\beta} + \frac{\ln\left[\frac{\lambda\varpi^{-(2-\beta)N_0}}{\varpi^{(2-\beta)N_0}\left(\Gamma_{\min} - \frac{\ln\varpi}{\tau_a}\right) + \lambda\varpi^{-(2-\beta)N_0}}\right]}{\left(\frac{\ln\varpi}{\tau_a} - \Gamma_{\min}\right)(2-\beta)}. \tag{3.3.12}$$

Therefore, the proof of the present theorem is completed.

Remark 3.3.1. Theorem 3.3.1 introduces a new sufficient condition for fixed-time synchronization in ICGNNs with desynchronizing impulses. The estimated settling-time is dependent on the impulsive sequence class and the continuous-time subsystem parameters with the condition $\tau_a > \frac{\ln\varpi}{\Gamma_{\min}}$. Furthermore, the condition is needed to ensure fixed-time synchronization for the drive system (3.2.3) and response system (3.2.5). However, if the condition $\tau_a > \frac{\ln\varpi}{\Gamma_{\min}}$ is not satisfied, then the impulsive perturbation may destroy the stability of the system which is demonstrated in the numerical section.

Remark 3.3.2. As discussed in subchapter 4.1 and also in the article by [84], the designed controller for achieving fixed-time synchronization typically comprises three terms, namely $u_p(t) = -\eta \text{sign}(e_p(t)) - \mu \text{sign}(e_p(t))|e_p(t)|^{\alpha} - \theta \text{sign}(e_p(t))|e_p(t)|^{\beta}$, where

η, μ, $\theta > 0$, $0 < \alpha < 1$ and $\beta > 1$. The last two terms aim to facilitate fixed-time synchronization and the settling-time estimation. The term β can help to reach the state to 1 and the term α can help to turn state 1 to 0. However, these terms with α and β are too conservative. It would be more advantageous to introduce a new condition on the Lyapunov inequality, which combines these two processes into a unified framework. Hence, Lemma 3.2.2 of the present study gives a superior approach. Theorem 3.3.1 provides a new sufficient condition for achieving fixed-time synchronization in ICGNNs subject to desynchronizing impulses.

3.4 Numerical Simulations and Discussions

This section presents two numerical examples which are provided to illustrate the effectiveness of the proposed theoretical results.

Example 3.4.1. Consider the following nonlinear impulsive system

$$\begin{cases} \dot{u}_p(t) = -a_p u_p(t) - b_p \text{sign}(u_p(t))|u_p(t)|^{\beta+\text{sign}(|u_p(t)|-1)}, \quad t \neq t_l, t \geq 0, \\ u_p(t_l) = \gamma_p u_p(t_l^-), l \in \mathbb{N}, p = 1,2,3, \end{cases} \tag{3.4.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t)) \in \mathbb{R}^n$, $1 \leq \beta < 2$, $a_p > 0$, $b_p > 0$, $\gamma_p \in (1, \infty)$. Constructing the Lyapunov function $V(u(t)) = \sum_{p=1}^{3} |u_p(t)|$. After the simple calculation, we can get the condition of Lemma 3.2.2 for $u(t) \in \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{cases} \dot{V}(u(t)) \leq -aV(u(t)) - b(V(u(t)))^{\beta+\text{sign}(V(u(t))-1)}, \quad t \neq t_l, t \geq 0, \\ V(u(t_l)) \leq \gamma V(u(t_l^-)), l \in \mathbb{N}, \end{cases}$$

where $b = 3^{-\beta} \min\{b_p,\ p = 1,2,3\}$, $a = \min\{a_p, p = 1,2,3\}, \gamma = \max\{\gamma_p, p = 1,2,3\}$. Let us choose $a_1 = 0.3$, $a_2 = 0.5$ $a_3 = 0.7$, $b_1 = b_2 = 2.4$, $b_3 = 2.3$, $\gamma_1 = 1.1$, $\gamma_2 = 1.1$ and $\gamma_3 = 1.2$ and $\beta = 1.3$. Considering the impulse sequences $\{t_l\}_{l\in\mathbb{N}}$ from [106] which is defined by

$$t_l - t_{l-1} = \begin{cases} \theta, & \text{if } \operatorname{mod}(l, N_0) \neq 0, \\ N_0(\tau_a - \theta) + \theta, & \text{if } \operatorname{mod}(l, N_0) = 0, \end{cases} \tag{3.4.2}$$

where θ and N_0 are positive numbers satisfying $\theta \leq \tau_a$ and $N_0 \in \mathbb{N}$, then τ_a and N_0 are the average impulsive interval and elasticity number of $\{t_l\}_{l\in\mathbb{N}}$. Choose an impulsive sequence $N_0 = 2$, $\theta = 0.5$ and the strength of impulses becomes $\gamma = 1.2$. Then we have $\frac{\ln\gamma}{a} = 0.60$ and choose $\tau_a = 0.8$ such that $\tau_a > \frac{\ln\gamma}{a}$, therefore, sufficient condition of Lemma 3.2.2 is satiefied. Hence, FxTS of the system (3.4.1) is guaranteed within the estimated settling-time as given by $T = 7.59$.

Remark 3.4.1. The key contribution is the establishment of condition $\tau_a > \frac{\ln\gamma}{a}$. This condition ensures that the impulsive system will achieve FxTS when the perturbation occurs at the impulsive moments separated by at least $\frac{\ln\gamma}{\tau_a}$. However, if the condition is not satisfied, then the system may lose stability due to the presence of disturbance at the impulsive moments. To support this claim, the numerical verification is provided in the form of Figure 3.4.1(a) and Figure 3.4.1(b).

Example 3.4.2. Consider the parameters for the model of system (3.2.1) as

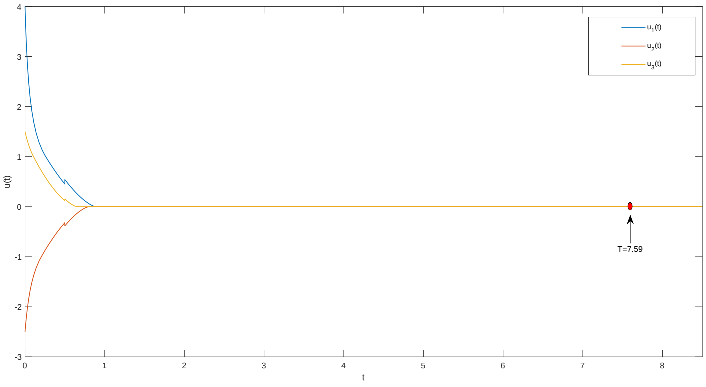
$$\beta_1 = \beta_2 = 1.3,\ \epsilon_1 = \epsilon_2 = 1.1,\ \alpha_p(x_p) = 0.3 + \frac{0.1}{1 + x_p^2},\ \alpha_p(y_p) = 0.3 + \frac{0.1}{1 + y_p^2},$$

$$h_p(x_p) = 0.4x_p,\ h_p(y_p) = 0.4y_p,\ \tau_q(t) = 0.5 + 0.25\cos(t),\ J_p(t) = 0.5\sin(t).$$

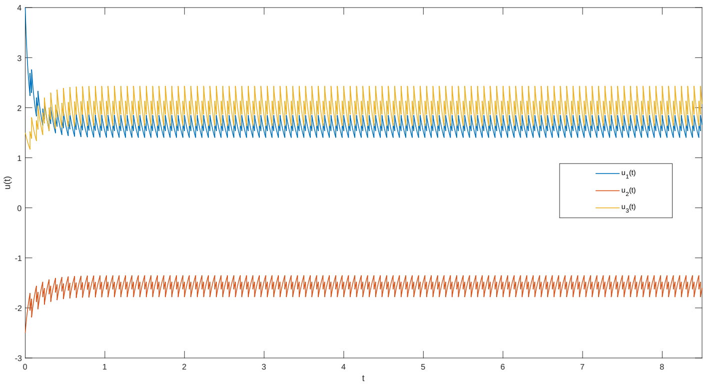
$$A = \begin{bmatrix} 1.5 & 2.5 \\ 1 & 1.3 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1.3 \\ 1.5 & 1 \end{bmatrix},$$

The activation function can be defined as $f_q(.) = \tanh(.)$ $(q = 1, 2)$. Evidently $J_1^+ = 0.5$, $J_2^+ = 0.5$, $l_q = 1$, $m_q = 1$, $0.3 \leq \alpha_p(.) \leq 0.4$, $\underline{\alpha}_p = 0.3$, $\overline{\alpha}_p = 0.4$, $\Psi_p(.) = \alpha_p(.)h_p(.)$,

$0.12 \leq \dot{\Psi}_p(.) \leq 0.16$, $\underline{\Psi}_p = 0.12$, $\overline{\Psi}_p = 0.16$, $|\dot{\alpha}_p(.)| \leq 0.05$, $\tilde{\alpha}_p = 0.05$. The time evolution and the phase trajectory of the system (3.2.3) with initial conditions $x_1(0) = 2.5, x_2(0) = -2.5, y_1(0) = 1.5, y_2(0) = -1.5$ and without impulsive effects are respectively shown through Figure 3.4.2 and Figure 3.4.3.

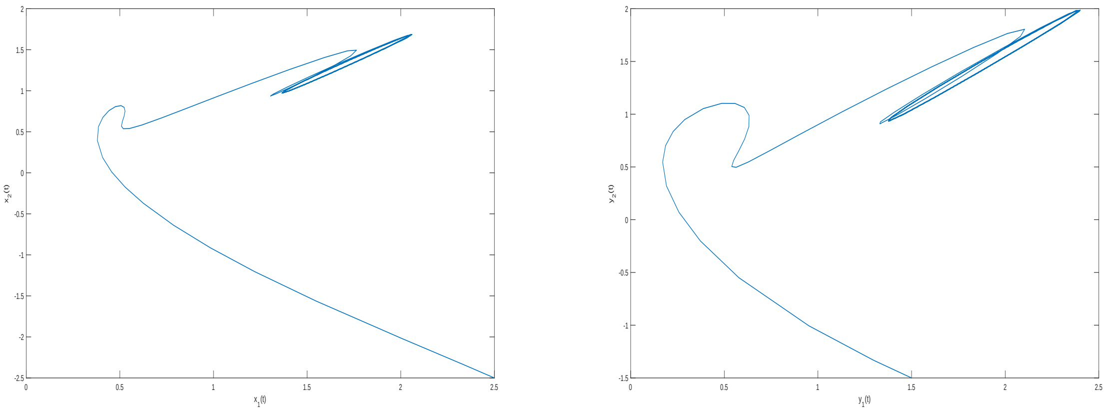


3.1 (a)



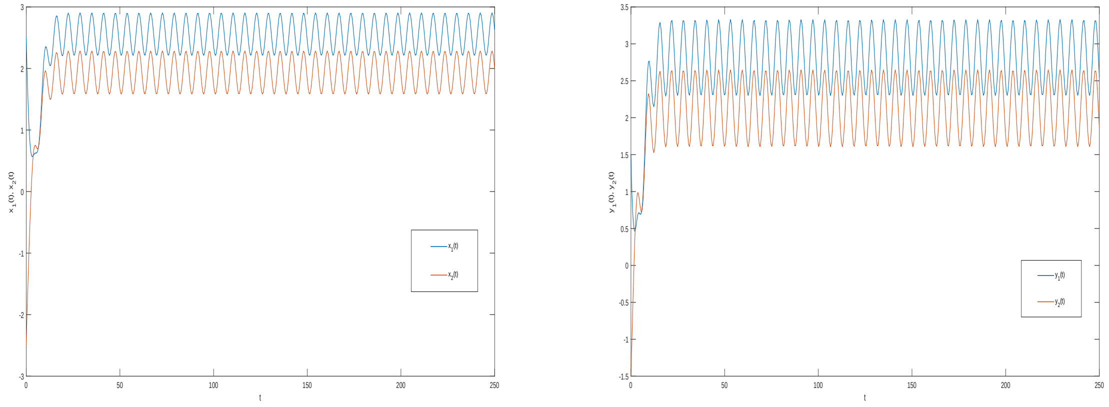
3.1 (b)

Figure 3.4.1: (a) The state trajectories to achieved fixed-time stability of the system (3.4.1) with the sufficient condition $\tau_a = 0.80 > 0.60$ (b) The state trajectories of the system (3.4.1) with the sufficient condition $\tau_a = 0.06 < 0.6$.



(a) (b)

Figure 3.4.2: The phase plot of inertial Cohen-Grossberg neural networks (3.2.3) without impulsive effects.



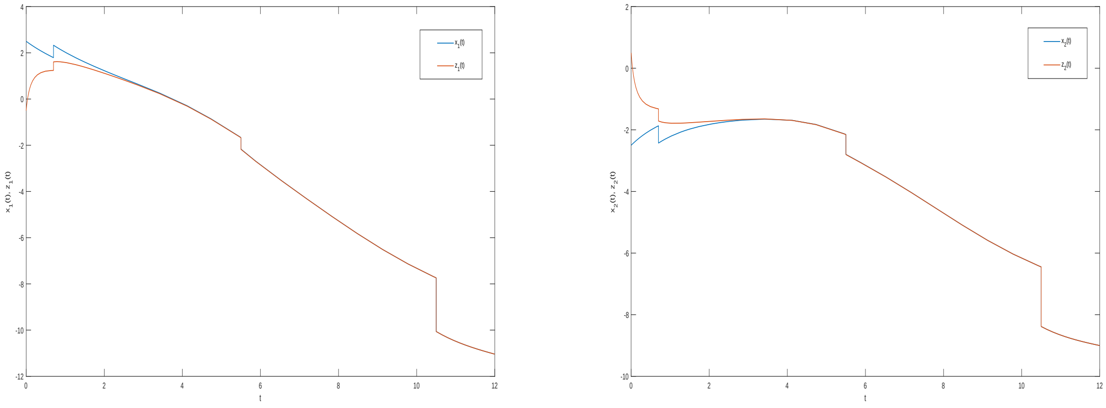
(a) (b)

Figure 3.4.3: Time evolution of state trajectories of inertial Cohen-Grossberg neural networks (3.2.3) without impulsive effects.

By simple computations, the control gains given in Theorem 3.3.1 should satisfy

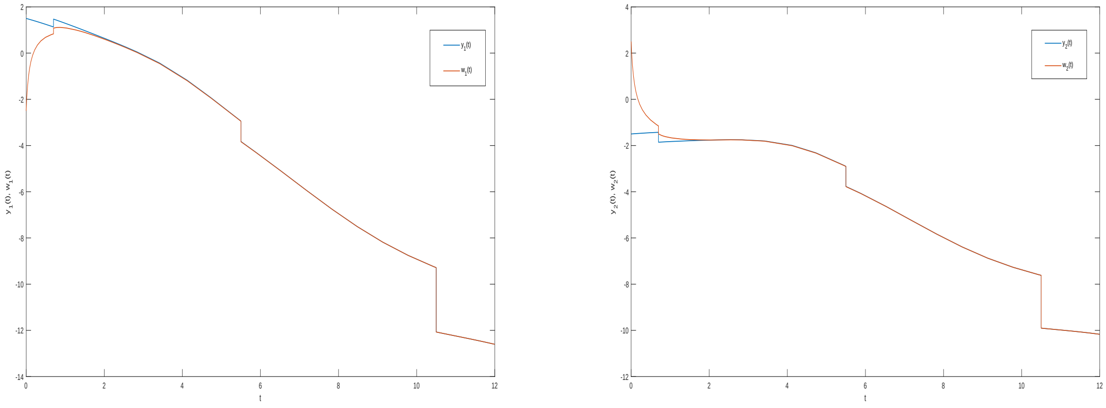
$$
\begin{aligned}
k_1 \geq & -\epsilon_1 + \beta_1\epsilon_1 - \epsilon_1^2 - \underline{\Psi}_1 + \sum_{q=1}^{2} \overline{\alpha}_q |a_{q1}| l_1 + \tilde{\alpha}_1 \sum_{q=1}^{2} |a_{1q}| m_q \\
& + \tilde{\alpha}_1 \sum_{q=1}^{2} |b_{1q}| m_q + \tilde{\alpha}_1 |J_1^{+}(t)| = 0.34,
\end{aligned}
$$

$$
\begin{aligned}
k_1 \geq & -\epsilon_2 + \beta_2\epsilon_2 - \epsilon_2^2 - \underline{\Psi}_2 + \sum_{q=1}^{2} \overline{\alpha}_q |a_{q2}| l_2 + \tilde{\alpha}_2 \sum_{q=1}^{m} |a_{2q}| m_q \\
& + \tilde{\alpha}_2 \sum_{q=1}^{m} |b_{2q}| m_q + \tilde{\alpha}_2 |J_2^{+}(t)| = 0.785,
\end{aligned}
$$



(a) (b)

Figure 3.4.4: The time evolution of drive system (3.2.3) and response system (3.2.5) with controller (3.3.1).



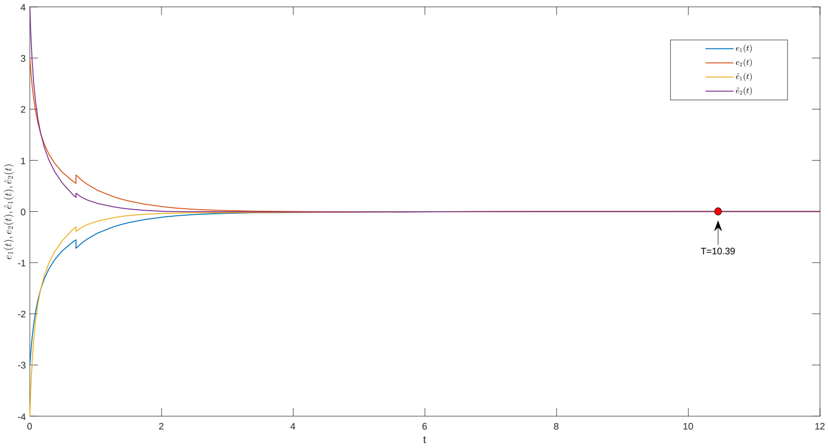
(a) (b)

Figure 3.4.5: The time evolution of drive system (3.2.3) and response system (3.2.5) with controller (3.3.1).

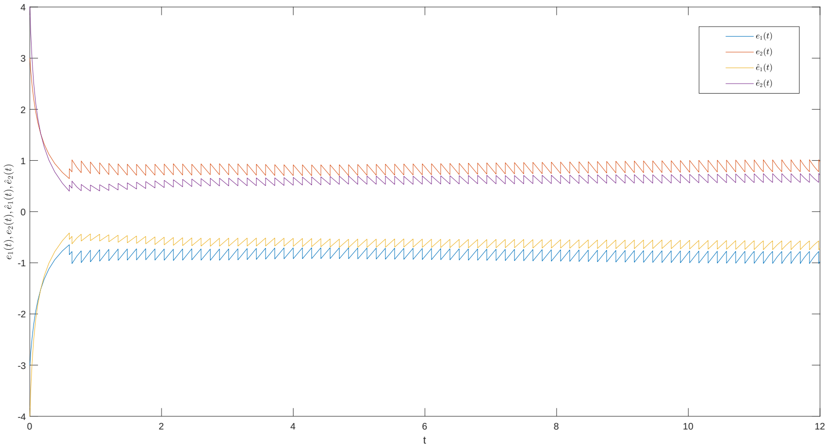
$k_2 \geq \epsilon_1 - \beta_1 + 1 = 0.8$, $k_2 \geq \epsilon_2 - \beta_2 + 1 = 0.8$,

$\delta_1 \overline{\alpha}_1 |b_{11}| l_1 = 0.4$, $\delta_1 \geq \overline{\alpha}_1 |b_{12}| l_2 = 0.52$, $\delta_2 \geq \overline{\alpha}_2 |b_{21}| l_1 = 0.6$, $\delta_2 \geq \overline{\alpha}_2 |b_{22}| l_2 = 0.4$.

Then the control gains of controller (3.3.1) are chosen such that $k_1 = 0.5$, $k_2 = 0.9$, $\delta_1 = 0.55$, $\delta_2 = 0.65$. Therefore, all the conditions (3.3.2) given in Theorem 3.3.1 are satisfied. Further the other parameters are chosen as $\lambda = 1.2$, $\beta = 1.5$ so that we get $\Gamma_{\min} = 0.11$. The impulsive strength is taken as $\mu = 1.1$ and $\eta = 1.3$, then we have $\varpi = 1.3$. Now, calculating $\frac{\ln \varpi}{\Gamma_{\min}} = 2.39$ and choosing $\tau_a = 2.5$, $N_0 = 2$ and $\theta = 0.2$ from the impulsive sequence $\{t_l\}_{l \in \mathbb{N}}$ which is taken from [106]. This implies that the sufficient condition $\tau_a > \frac{\ln \varpi}{\Gamma_{\min}}$ in Theorem 3.3.1 holds good and the drive system (3.2.3) and response system (3.2.5) with impulsive effects under the designed controller (3.3.1) is fixed-time synchronization and the estimated settling-time is bounded by $T = 10.39$. The evolutions of state trajectories of the drive system (3.2.3) and response system (3.2.5) under the controller (3.3.1) are depicted in Figure 3.4.4 and Figure 3.4.5. Further, the evolutions of the synchronization error state (3.2.6) under the controller (3.3.1) are described in Figure 3.4.6(a), which indicates that the synchronization can be achieved in fixed time. If the sufficient condition $\tau_a > \frac{\ln \varpi}{\Gamma_{\min}}$ does not hold, then we find τ_a for that the system (3.2.6) may not be synchronized in fixed-time, which is demonstrated in Figure 3.4.6(b) for $\tau_a = 0.09$, $N_0 = 2$ and $\theta = 0.1$. Therefore, the effectiveness of proposed sufficient condition for achieving fixed-time synchronization of the ICGNNs with desynchronizing impulses is numerically verified.



3.6 (a)



3.6 (b)

Figure 3.4.6: (a) The time evolution of error system (3.2.6) with controller (3.3.1) and $\tau_a = 2.5 > 2.39$ (b) The time evolution of error system (3.2.6) with controller (3.3.1) and $\tau_a = 0.09 < 2.39$.

3.5 Conclusion

This chapter investigates the fixed-time synchronization of ICGNNs under the effects of desynchronizing impulses. To address this problem, a new lemma is presented, demonstrating that the impulsive system is FxTS. A novel Lyapunov inequality has been employed to establish a sufficient condition for achieving FxTS to the systems subject to destabilizing impulses. In this scientific contribution, the results are derived through the use of the comparison principle and the concept of average impulsive interval. From the sufficient condition, it can be obtained that if the impulsive disturbances to the system activate at the impulsive times that are separated by a certain length, then FxTS of impulsive system is possible to obtain. Based on this lemma, a unified controller is designed, and sufficient conditions are established to achieve the fixed-time synchronization of ICGNNs with desynchronizing impulses. Moreover, this chapter establishes the conditions required for achieving fixed-time synchronization in the presence of desynchronizing impulses. The settling-time function is shown to be influenced by the parameters of the impulsive sequences. Finally, the effectiveness of the proposed theoretical results are illustrated through the presentation of two numerical examples. Predefined-time synchronization, as well as the impact of randomly distributed impulses and stochastic impulse strength on fixed-time synchronization will be carried out in future.