Chapter 2

Numerical Approximation of Fractional Variational Problems with Several Dependent Variables Using Jacobi Poly-Fractonomials

This chapter deals FVPs with several dependent variables. In Section 2.1, we give the introduction on FVPs. In Section 2.2, we present the fractional variational problem with several dependent variables and the corresponding Euler-Lagrange equation. Using Jacobi poly-fractonomials as a basis, we develop a numerical scheme for FVPs with several dependent variables in Section 2.3. Section 2.4 presents the convergence of the presented method. Section 2.5 presents the error analysis of the proposed method. We discuss some examples and show the precision of our technique in Section 2.6. Section 2.7 concludes the chapter.

2.1 Introduction

Our focus over here is to present a method for solving the FVPs with several dependent variables using Jacobi poly-fractonomials. The advantage of using Jacobi poly-fractonomials as basis functions is that they provide exponential convergence when approximating fractional polynomial functions, making them much more efficient than standard polynomials [63]. Using Jacobi poly-fractonomials, Mao et al. [64] developed the Rayleigh-Ritz method with exponential order accuracy for FVPs defined in terms of the Riemann-Liouville fractional derivative. Motivated by these works, we present a new numerical scheme, using Jacobi poly-fractonomials as a basis for solving FVPs with several dependent variables in terms of Caputo fractional derivative and a set of homogeneous initial conditions. Further, we extend the scheme also for non homogeneous initial conditions.

2.2 The Fractional Variational Problem and Euler-Lagrange Equations

We study here the problems of fractional calculus of variations with several dependent variables, where the objective functional involves the left Caputo fractional derivative. We define the functional,

$$Min\ J[y_1, y_2, ..., y_n] = \int_0^L G\left(x, y_1(x), ..., y_n(x), {}_0^C D_x^{\alpha} y_1(x), ..., {}_0^C D_x^{\alpha} y_n(x)\right)\, dx \quad (2.1)$$

with initial conditions,

$$y_i(0) = 0, \quad (2.2)$$

where, $1 \leq i \leq n$ and $0 < \alpha < 1$.

In the above FVP, the Lagrangian function G is continuously differentiable, $y_i : [0, L] \longrightarrow \mathbb{R}$ is absolutely continuous on $[0, L]$ such that ${}_0^C D_x^\alpha y_i(x)$ exist and continuous on $[0, L]$, for each $i = 1, 2, ..., n$. If the initial conditions are not homogeneous ($y_i(0) \neq 0$), then using transformation $\overline{y_i}(x) = y_i(x) - y_i(0)$, we make the initial conditions homogeneous, and also ${}_0^C D_x^\alpha \overline{y_i}(x)$ is continuous on $[0, L]$. Now, the Euler-Lagrange condition [31, 54, 64, 61] for equations (2.1) and (2.2) can be obtained as,

$$\frac{\partial G}{\partial y_i} + {}_x^C D_L^\alpha \frac{\partial G}{\partial\, {}_0^C D_x^\alpha y_i} = 0 \qquad 1 \leq i \leq n, \tag{2.3}$$

and the transversality condition be describe as,

$$\left(\frac{\partial G}{\partial\, {}_0^C D_x^\alpha y_i}\right) {}_0^C D_x^{\alpha-1} \eta_i(x) = 0, \qquad x = L, \qquad 1 \leq i \leq n. \tag{2.4}$$

Here, ${}_x^C D_L^\alpha$ and $\dfrac{\partial G}{\partial\, {}_0^C D_x^\alpha y_i}$ denote the right Caputo derivative, and the functional G is partially differentiated with respect to the left Caputo derivative ${}_0^C D_x^\alpha y_i$ for $i = 1, 2, ..., n$, respectively. $\eta_i(x)$ is an arbitrary function, which fulfil the condition $\eta_i(0) = 0$ and ${}_0^C D_x^{\alpha-1}$ is to be read as integral of order $1 - \alpha$. The optimal solution of the problem (2.1)-(2.2) should satisfy the system (2.3) and boundary conditions (2.2).

Now, $y_i(x)$ at $x = L$ is not defined, it pursues from Eq. (2.4) that

$$\left(\frac{\partial G}{\partial\, {}_0^C D_x^\alpha y_i}\right)\Bigg|_{x=L} = 0. \tag{2.5}$$

Eq. (2.5) is the generalized boundary condition for FVP defined by Eqs. (2.1) and (2.2).

2.3 Jacobi Poly-fractonomial and the Numerical Scheme for Fractional Variational Problems

Here, we present a numerical technique for FVPs with several variables based on Jacobi poly-fractonomials [62]. Later, We also discuss the method for a nonzero initial conditions, i.e., $y_i(0) \neq 0$.

In 2013, Zayernouri et al. [62] proposed two types of fractional Sturm-Liouville problems, regular fractional Sturm—Liouville problems(RFSLP) as well as singular fractional Sturm–Liouville problems(SFSLP), and also determined corresponding eigenfunctions in fractional polynomial form, and named it as Jacobi poly-fractonomials. Jacobi poly-fractonomials are dense in $L^2_\omega[-1,1]$ and orthogonal with respect to the weight function $\omega(t) = (1-t)^{-\alpha}(1+t)^{-\alpha}$. The Jacobi poly-fractonomial's basis achieves exponential convergence in approximating fractional terms comparatively to other standard polynomial basis functions. In 2014, Zayernouri et al. [63, 102] proposed a spectral method using Jacobi Poly-fractonomials as basis function for fractional ODEs which lead to exponentially fast decay of the errors. Further, Mao et al. [64] developed an exponentially exact numerical method for FVPs, using the Jacobi ploy-fractonomials as a basis function.

In this chapter, we present the Ritz method using Jacobi poly-fractonomials for solving FVPs with several dependent variables. Now, we mention the eigenfunctions of the RFSLP-I [62]

$$ {}^{(1)}P_n^{\alpha}(t) = (1+t)^{\alpha} J_{n-1}^{-\alpha,\alpha}(t), \qquad t \in [-1,1],\ n = 1,2,..., \tag{2.6}$$

where $J_n^{-\alpha,\alpha}(t)$ are the the standard Jacobi polynomials.

Using the transformation $t = \left(\frac{2x}{L}\right) - 1$, the interval $[-1,1]$ becomes to $[0, L]$. So, the shifted Jacobi poly-fractonomials are given as

$$ {}^{(1)}\widetilde{P}_n^{\alpha}(x) = \left(\frac{2}{L}\right)^{\alpha} x^{\alpha} J_{n-1}^{-\alpha,\alpha}\left(\frac{2x}{L}-1\right), \qquad x \in [0, L], \qquad n = 1, 2, \ldots \quad . \tag{2.7} $$

The left Caputo fractional derivatives of ${}^{(1)}\widetilde{P}_n^{\alpha}(x)$ given by Eq.(2.7) are derived as[63]

$$ {}_{0}^{C}D_x^{\alpha}\left({}^{(1)}\widetilde{P}_n^{\alpha}(x)\right) = \left(\frac{2}{L}\right)^{\alpha} \frac{\Gamma(n+\alpha)}{\Gamma(n)} J_{n-1}^{0,0}\left(\frac{2x}{L}-1\right), \qquad x \in [0, L], \qquad n = 1, 2, \ldots, \tag{2.8} $$

where the $J_{n-1}^{0,0}(t)$ are the standard Legendre polynomial of order $(n-1)$.

Similarly, we can find the other shifted Jacobi poly-fractonomials of second kind

$$ {}^{(2)}\widetilde{P}_n^{\alpha}(x) = \left(\frac{2}{L}\right)^{\alpha} (L-x)^{\alpha} J_{n-1}^{\alpha,-\alpha}\left(\frac{2x}{L}-1\right), \qquad x \in [0, L], \qquad n = 1, 2, \ldots \quad . \tag{2.9} $$

Equation (2.9) is acquired by ${}^{(2)}\widetilde{P}_n^{\alpha}(x) = {}^{(2)}P_n^{\alpha}(t(x))$, where ${}^{(2)}P_n^{\alpha}(t)$ are the eigen-functions of RFSLP-II, also define as

$$ {}^{(2)}P_n^{\alpha}(t) = (1-t)^{\alpha} J_{n-1}^{\alpha,-\alpha}(t), \qquad t \in [-1, 1], \quad n = 1, 2, \ldots \quad . \tag{2.10} $$

Right Caputo fractional derivative of ${}^{(2)}\widetilde{P}_n^{\alpha}(x)$ are given as

$$ {}_{x}^{C}D_L^{\alpha}\left({}^{(2)}\widetilde{P}_n^{\alpha}(x)\right) = \left(\frac{2}{L}\right)^{\alpha} \frac{\Gamma(n+\alpha)}{\Gamma(n)} J_{n-1}^{0,0}\left(\frac{2x}{L}-1\right), \qquad x \in [0, L], \qquad n = 1, 2, \ldots, \tag{2.11} $$

which is similar to Eq.(2.8).

2.3.1 Solving the Fractional Variational Problems

We start with assumption that the FVPs are dependent upon the homogeneous conditions as given in Eq. (2.2). Initially, we estimate the solution $y_i(x)$ for $i =$

$1, 2, ..., n$ of the FVPs (2.1) and (2.2) using the Jacobi poly-fractonomials of the first type as,

$$y_i^*(x) \approx y_N^{*i}(x) = \sum_{j=1}^{N} c_j^{i\,(1)} \widetilde{P}_j^{\alpha}(x), \qquad 1 \leq i \leq n, \tag{2.12}$$

where c_j^i are unknown coefficients, which are to be determined. Here, $N \in \mathbb{N}$ is the number of basis function. The approximate solution y_N^{*i} should satisfy the homogeneous initial conditions. Now putting Eq. (2.12) into Eq. (2.1), we attain the following approximate equation

$$J(c_1^1, ..., c_N^1, ..., c_1^n, ..., c_N^n) = \int_0^L G\left(x, y_N^{*1}, ..., y_N^{*n}, {}^{C}{}_0D_x^{\alpha} y_N^{*1}, ..., {}^{C}{}_0D_x^{\alpha} y_N^{*n}\right) dx \quad . \tag{2.13}$$

Now, using the Rayleigh-Ritz method, we obtain the necessary conditions for an extremum,

$$\frac{\partial J}{\partial c_j^i} = 0, \quad 1 \leq j \leq N, \ 1 \leq i \leq n. \tag{2.14}$$

We can find the optimizing values of c_j^i, $1 \leq j \leq N$, $1 \leq i \leq n$ by solving the algebraic system (2.14), for equation (2.13) and henceforth we find the approximated function $y_N^{*i}(x)$ by Eq. (2.12), which approximate the solution $y_i^*(x)$ of the FVP (2.1) and (2.2).

If the condition is non-homogeneous, i.e., $y_i(0) \neq 0$, then we reconstruct the problem by using suitable transformation $\overline{y}_i(x) = y_i(x) - y_i(0)$ to make homogeneous initial condition. For the Caputo fractional derivative, we observe that, ${}_0^C D_x^{\alpha} \overline{y}_i(x) = {}_0^C D_x^{\alpha} y_i(x)$.

2.4 Convergence Analysis

Here, we discuss the convergence and fractional variational error of the method presented in Section 2.3 for the FVP (2.1) and (2.2). Now, we will show that the approximate solution leads to appropriate solution such that, $|J(y_N^{*1}, y_N^{*2}, ..., y_N^{*n}) - J(y_1^*, y_2^*, ..., y_n^*)| \to 0$, where, $y_N^{*i}(x)$ and $y_i^*(x)$ for $i = 1, 2, ..., n$ are the approximate solutions corresponding N basis functions and exact solutions which minimize the functional J respectively. Now, consider the functional spaces equipped along the norm $\|.\|$. Let, $[0, L] = \Theta$

$$\begin{aligned}
H_i(\Theta) =& \{y_i(x) \in C^1(\Theta) \mid y_i(0) = 0\}, \qquad i = 1, 2, ..., n \\
F_N^n[\Theta] =& H_1(\Theta) \cap \left\langle \{{}^{(1)}\widetilde{P}_j^{\alpha}\}_{j=1}^N \right\rangle \times ... \times H_n(\Theta) \cap \left\langle \{{}^{(1)}\widetilde{P}_j^{\alpha}\}_{j=1}^N \right\rangle \\
\|y_i\| =& \|y_i\|_\infty + \|y_i'\|_\infty, \quad i = 1, 2, ..., n \\
\|(y_1, ..., y_n)\| =& \sum_{i=1}^n \|y_i\| .
\end{aligned}$$

The function space $\left\langle \{{}^{(1)}\widetilde{P}_j^{\alpha}\}_{j=1}^N \right\rangle$ spanned by first type of Jacobi poly-fractonomials, and also the set $\{{}^{(1)}\widetilde{P}_j^{\alpha} : j = 1, 2, ...\}$ structures a basis on the Hilbert space $L_\omega^2[-1, 1]$. Zayernouri et al. [62] proved that the shifted Jacobi poly-fractonomial are dense in Hilbert space $H_i(\Theta)$. The Lemma is given as follows.

Lemma 2.1. [62] *For each $\widetilde{y} \in H(\Theta)$, there exist a sequence of the Jacobi poly-fractonomial function such that $\widetilde{y}_N = \sum_{j=1}^N c_j {}^{(1)}\widetilde{P}_j^{\alpha} \xrightarrow{\|\cdot\|} \widetilde{y}$ when $N \to +\infty$.*

Now, we will generalize Lemma 2.1 for the proposed approach.

Theorem 2.2. *For each $y_i^* \in H_i(\Theta)$ $i = 1, 2, ..., n$, there exist a sequence of the Jacobi poly-fractonomials such that, $y_N^{*i} = \sum_{j=1}^N c_j^i {}^{(1)}\widetilde{P}_j^{\alpha} \xrightarrow{\|\cdot\|} y_i^*$ when $N \to +\infty$.*

Proof. If $y_i^*(x) \in L_\omega^2[-1, 1]$ then clearly $g_i(x) = (1 + x)^{-\alpha} y_i^*(x) \in L_\omega^2[-1, 1]$, for $\alpha \in (0, 1)$[62]. Now we prove the theorem:

$$\left\| \sum_{j=1}^N c_j^i \, {}^{(1)}\widetilde{P}_j^{\alpha}(x) - y_i^*(x) \right\|_{L_\omega^2[-1,1]} = \left\| \sum_{j=1}^N c_j^i \, (1 + x)^{\alpha} J_{j-1}^{-\alpha,\alpha}(x) - y_i^*(x) \right\|_{L_\omega^2[-1,1]}$$

$$= \left\| (1+x)^{\alpha} \left(\sum_{j=1}^{N} c_j^i \, J_{j-1}^{-\alpha,\alpha}(x) - (1+x)^{-\alpha} y_i^*(x) \right) \right\|_{L^2_\omega[-1,1]}$$

$$= \left\| (1+x)^{\alpha} \left(\sum_{j=1}^{N} c_j^i \, J_{j-1}^{-\alpha,\alpha}(x) - g_i(x) \right) \right\|_{L^2_\omega[-1,1]}$$

(Using Cauchy-Schwartz inequality)

$$\leq \|(1+x)^{\alpha}\|_{L^2_\omega[-1,1]} \left\| \left(\sum_{j=1}^{N} c_j^i \, J_{j-1}^{-\alpha,\alpha}(x) - g_i(x) \right) \right\|_{L^2_\omega[-1,1]}$$

$$\leq C \left\| \left(\sum_{j=1}^{N} c_j^i \, J_{j-1}^{-\alpha,\alpha}(x) - g_i(x) \right) \right\|_{L^2_\omega[-1,1]} .$$

Hence,

$$\lim_{N\to\infty} \left\| \sum_{j=1}^{N} c_j^i \, {}^{(1)}\widetilde{P}_j^{\alpha}(x) - y_i^*(x) \right\|_{L^2_\omega[-1,1]} \leq \lim_{N\to\infty} C \left\| \left(\sum_{j=1}^{N} c_j^i \, J_{j-1}^{-\alpha,\alpha}(x) - g_i(x) \right) \right\|_{L^2_\omega[-1,1]} = 0, \tag{2.15}$$

by Weierstrass theorem. Therefore, $\sum_{j=1}^{N} c_j^i \; {}^{(1)}\widetilde{P}_j^{\alpha}(x) \xrightarrow{L^2_w[-1,1]} y_i^*(x)$, implying that $\{{}^{(1)}\widetilde{P}_j^{\alpha}(x), \quad j = 1, 2, ...\}$ is dense in the Hilbert space H_i and form a basis for L^2_ω. Since y_i and H_i are arbitrary, so it is true for all $i = 1, 2, ..., n$. □

Remark 2.3. Now, from the Eq. (2.15), $\lim_{N\to\infty} y_N^{*i} = \lim_{N\to\infty} \sum_{j=1}^{N} c_j^i \; {}^{(1)}\widetilde{P}_j^{\alpha}(x) = y_i^*$ shows that y_N^{*i} uniformly converges to y_i^* for each $i = 1, 2, ...n$.

Now, we demonstrate that the functional J is continuous on its domain, $D_n^1[\Theta] = \overbrace{C^1[\Theta] \times ... \times C^1[\Theta]}^{n \ times}$ in upcoming Lemma.

Lemma 2.4. [61] *The Functional J is continuous on the space $(D_n^1[\Theta], \|.\|)$.*

Proof. Let, $(y_1^*, ..., y_n^*) \in D_n^1[\Theta]$. Consider $d > 0$ and
$I = \Theta \times \prod_{k=0}^{1} \prod_{i=1}^{n} [-l_i^k - d, l_i^k + d]$,

where, $l_i^0 = \|y_i^*\|_\infty$, $l_i^1 = \|{}_0^C D_t^\alpha y_i^*\|_\infty$. For all $x \in \Theta$, we have

$Y^* = (x, y_1^*, ..., y_n^*, {}_0^C D_t^\alpha y_1^*, ..., {}_0^C D_t^\alpha y_n^*) \in I$.

Let, $\delta > 0$ and $\| (y_1, ..., y_n) - (y_1^*, ..., y_n^*) \| < \delta$; hence, we have $\|y_i - y_i^*\| < \delta$, $1 \leq i \leq n$ and

$$\begin{aligned}
\|{}_0^C D_x^\alpha y_i - {}_0^C D_x^\alpha y_i^*\|_\infty &= \| {}_0^C D_x^\alpha (y_i - y_i^*) \|_\infty \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} \left|[y_i(\tau) - y_i^*(\tau)]'\right| d\tau \\
&\leq \frac{\| (y_i - y_i^*)' \|_\infty x^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\leq c\delta, \qquad [64]
\end{aligned}$$

for the sufficiently small δ, we have $Y = \left(x, y_1, ..., y_n, {}_0^C D_x^\alpha y_1, ..., {}_0^C D_x^\alpha y_n\right) \in I$ for all $x \in \Theta$.

Since, I is compact set and the Lagrange function is continuously differentiable such that $G \in C^1(I)$, so using the continuity mapping theorem G is uniformly continuous on compact set I . So, if $\delta > 0$ be a sufficiently small, for any given $\epsilon > 0$, then $|Y - Y^*| < \delta$ implies that $|G(Y) - G(Y^*)| < \epsilon$, and we have $|J(y_1, ..., y_n) - J(y_1^*, ..., y_n^*)| = \left|\int_0^L [G(Y) - G(Y^*)]\, dx\right| < L\epsilon$. □

Theorem 2.5. [61, 64] *Let us assume that $y_i^*(x)$, $i = 1, 2, ..., n$ minimize the functional J on $H_i(\Theta)$, $1 \leq i \leq n$, and $y_N^{*i}(x)$, $1 \leq i \leq n$ minimize the functional J on $F_n^N(\Theta)$. So, it means $\lim_{N\to\infty} J[y_N^{*1}, ..., y_N^{*n}] = J[y_1^*, ..., y_n^*]$.*

Proof. For each $\epsilon > 0$, there exist $y_i^*(x) \in H_i(\Theta)$, $1 \leq i \leq n$ such that $J[y_1^*, ..., y_n^*] \leq J[y_1^*, ..., y_n^*] + \epsilon$, from Lemma 2.4. Now using the theorem 2.2 and Lemma 2.4, J is continuous on the domain $(D_n^1[\Theta], \|.\|)$, so we can say that there exist a $\delta > 0$ and integer $N_1 > 0$ such that when $N > N_1$, $\| (y_N^{*1}, ..., y_N^{*n}) - (y_1^*, ..., y_n^*) \| < \delta$, $|J(y_N^{*1}, ..., y_N^{*n}) - J(y_1^*, ..., y_n^*)| \leq \epsilon$, where $y_N^{*1}, ..., y_N^{*n} \in F_n^N(\Theta)$. So we can obtain the following result $J[y_1^*,...,y_n^*] \leq J[y_N^{*1},...,y_N^{*n}] \leq J[y_1^*,...,y_n^*]+\epsilon$

Since, $\epsilon > 0$ is arbitrary, $\lim\limits_{N\to\infty} J[y_N^{*1},...,y_N^{*n}] = J[y_1^*,...,y_n^*]$. □

Remark 2.6. Here, we discuss the convergence analysis for homogeneous case. If the initial condition is non-homogeneous such that $y_i(0) \neq 0 \quad 1 \leq i \leq n$, then we can reformulate the problem using suitable transformation to make condition homogeneous, as we discussed in Section 2.3.1.

2.5 Error Analysis

This section focuses on investigating the error analysis and evaluating an upper bound of the error for the discussed method. In Section 2.3, we have discussed that Jacobi poly-fractonomials are orthogonal with respect to the weight function $\omega(t) = (1-t)^{-\alpha}(1+t)^{-\alpha}$ in the interval $[-1,1]$. The orthogonality condition [62] is given by

$$\int_{-1}^{1} (1-t)^{-\alpha}(1+t)^{-\alpha}\, {}^{(1)}P_j^{\alpha}(t) {}^{(1)}P_k^{\alpha}(t)\, dt = \frac{2}{(2j-1)} \frac{\Gamma(j-\alpha)\Gamma(j+\alpha)}{(j-1)!\Gamma(j)}\delta_{jk}, \quad (2.16)$$

where, δ_{jk} is the Kronecker delta function.

The functions $y_i(x)$ for $i = 1,2,3,...,n$, are square integrable functions in $[0, L]$, may be expressed in terms of shifted Jacobi poly-fractonomials as,

$$y_i(x) = \sum_{j=1}^{\infty} c_j^i \, {}^{(1)}\widetilde{P}_j^{\alpha}(x),\ i = 1,2,...,n, \quad (2.17)$$

where the coefficients c_j^i are given by

$$c_j^i = \frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \int_0^L \left(2-\frac{2x}{L}\right)^{-\alpha} \left(\frac{2x}{L}\right)^{-\alpha} y_i(x) {}^{(1)}P_j^{\alpha}(x)\, dx. \quad (2.18)$$

Lemma 2.7. *For each continuous function, $y_i(x)$, $i = 1, 2, .., n$ defined on Θ with bound $|y_i(x)| \leq m_i$, $i = 1, 2, ..., n$, the coefficient of shifted Jacobi poly-fractonomials expansion (2.18) is bounded, and*

$$|c_j^i| \leq \left(\frac{2^{-\alpha}}{(1-\alpha)} \frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^{\frac{1}{2}} L\, m_i. \tag{2.19}$$

Proof. From Eq.(2.18), we have

$$\begin{aligned}
|c_j^i|^2 &= \left| \frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \int_0^L \left(2 - \frac{2x}{L}\right)^{-\alpha} \left(\frac{2x}{L}\right)^{-\alpha} y_i(x)^{(1)} P_j^{\alpha}(x)\, dx \right|^2 \\
&\leq \left(\frac{(2j-1)}{2} \frac{(j-1)!\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^2 \int_0^L |y_i(x)|^2\, dx \\
&\int_0^L \left| \left(2 - \frac{2x}{L}\right)^{-\alpha} \left(\frac{2x}{L}\right)^{-\alpha} \left(\frac{2x}{L}\right)^{\alpha} J_{j-1}^{-\alpha,\alpha}\left(\frac{2x}{L} - 1\right) \right|^2 dx \\
&\leq \left(\frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^2 L\, m_i^2 \int_0^L \left| \left(2 - \frac{2x}{L}\right)^{-2\alpha} \left(J_{j-1}^{-\alpha,\alpha}\left(\frac{2x}{L} - 1\right)\right)^2 \right| dx \\
&\leq \left(\frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^2 L\, m_i^2 \int_0^L \left| \left(2 - \frac{2x}{L}\right)^{-\alpha} \right| dx \\
&\int_0^L \left| \left(2 - \frac{2x}{L}\right)^{-\alpha} \left(J_{j-1}^{-\alpha,\alpha}\left(\frac{2x}{L} - 1\right)\right)^2 \right| dx.
\end{aligned}$$

Now, by using the properties of the shifted Jacobi poly-fractonomials and the orthogonality condition, we get

$$\begin{aligned}
|c_j^i|^2 &\leq \left(\frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^2 L\, m_i^2 \frac{2^{-\alpha} L}{(1-\alpha)} \frac{2}{(2j-1)} \frac{\Gamma(j-\alpha)\Gamma(j+\alpha)}{(j-1)!\,\Gamma(j)} \\
|c_j^i| &\leq \left(\frac{2^{-\alpha}}{(1-\alpha)} \frac{(2j-1)}{2} \frac{(j-1)!\,\Gamma(j)}{\Gamma(j-\alpha)\Gamma(j+\alpha)} \right)^{\frac{1}{2}} L\, m_i.
\end{aligned}$$

This proves the required relation (2.19). □

Theorem 2.8. *For $\alpha \in (0,1)$ and $y_i(x) \in L^2(\Theta)$ the error bound is given by*

$$\|y_i(x) - y_N^{*i}(x)\|_{L^2(\Theta)} \leq C_1 \|{}_0^C D_t^\alpha y_i(x)\|_{L^2(\Theta)}, \quad \forall\, i = 1, ..., n. \tag{2.20}$$

Proof. From [[94], page No. A1797], we have

$$\|{}_0^C D_t^\alpha y_i(x)\|_{L^2(\Theta)}^2 \geq \sum_{j=1}^{\infty} \left[|c_j^i| \frac{\Gamma(j+\alpha)}{\Gamma(j)} \frac{2^\alpha}{L^\alpha} \right]^2 \frac{2}{2j-1}. \tag{2.21}$$

Now, a combination of Eqs.(2.8), (2.12), (2.18) and (2.21) leads to

$$\begin{aligned}
\left\|y_i(x) - y_N^{*i}(x)\right\|_{L^2(\Theta)}^2 &= \left\|{}_0I_t^\alpha \, {}_0^C D_t^\alpha y_i(x) - {}_0 I_t^\alpha \, {}_0^C D_t^\alpha y_N^{*i}(x)\right\|_{L^2(\Theta)}^2 \\
&= \left\|{}_0I_t^\alpha \, {}_0^C D_t^\alpha \left(y_i(x) - y_N^{*i}(x)\right)\right\|_{L^2(\Theta)}^2 \\
&= \left\|{}_0I_t^\alpha \, \left({}_0^C D_t^\alpha (y_i(x) - y_N^{*i}(x))\right)\right\|_{L^2(\Theta)}^2 \\
&\leq \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \left\| {}_0^C D_t^\alpha \left(\sum_{j=1}^{\infty} c_j^i \, {}^{(1)}\widetilde{P}_j^\alpha(x) - \sum_{j=1}^{N} c_j^i \, {}^{(1)}\widetilde{P}_j^\alpha(x) \right) \right\|_{L^2(\Theta)}^2 \\
&= \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \left\| \sum_{j=N+1}^{\infty} c_j^i \, {}_0^C D_t^\alpha \left({}^{(1)}\widetilde{P}_j^\alpha(x) \right) \right\|_{L^2(\Theta)}^2 \\
&= \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \left\| \sum_{j=N+1}^{\infty} c_j^i \, \frac{2^\alpha}{L^\alpha} \frac{\Gamma(j+\alpha)}{\Gamma(j)} J_{j-1}(x) \right\|_{L^2(\Theta)}^2 \\
&\leq \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \left(\sum_{j=N+1}^{\infty} \int_{-1}^{1} \left| c_j^i \, \frac{2^\alpha}{L^\alpha} \frac{\Gamma(j+\alpha)}{\Gamma(j)} J_{j-1}(x) \right|^2 dx \right) \\
&= \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \left(\sum_{j=N+1}^{\infty} |c_j^i|^2 \left(\frac{2^\alpha}{L^\alpha} \frac{\Gamma(j+\alpha)}{\Gamma(j)} \right)^2 \frac{2}{2j-1} \right) \\
&\leq \frac{L^{2\alpha}}{(\Gamma(1+\alpha))^2} \|{}_0^C D_t^\alpha y_i(x)\|_{L^2(\Theta)}^2 \\
\left\|y_i(x) - y_N^{*i}(x)\right\|_{L^2(\Theta)} &\leq C_1 \left\|{}_0^C D_t^\alpha y_i(x)\right\|_{L^2(\Theta)}.
\end{aligned}$$

This completes the proof of the error bounds (2.20) of the approximation of the solution of $y_i(x)$, for each $i = 1, 2, 3, ..., n$. □

2.6 Illustrative Numerical Examples

To validate the proposed scheme, we discuss the simulation results for three numerical examples in this Section. In the first example, the FVP is defined in terms of the left hand Caputo fractional derivatives. The second example contains the right hand Caputo fractional derivatives in the functional. The constraint problem has been taken in example third.

Example 2.1. Minimize the functional [61]
$Min\ J[y_1, y_2] = \int\limits_0^1 \left({}_0^C D_t^{\frac{1}{2}} y_1 +_0^C D_t^{\frac{1}{2}} y_2 - g(t)\right)^2 dt,$
where, $g(t) = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{15\sqrt{\pi}t^2}{16}$, and the initial conditions, $y_1(0) = 1, \quad y_2(0) = 0.$

Here, $y_1^*(t) = 1 + t^2$ and $y_2^*(t) = t^{\frac{5}{2}}$ are the exact solutions as given in [61] . Using the method introduced in Section 2.3, we find an approximate solution for given problem.

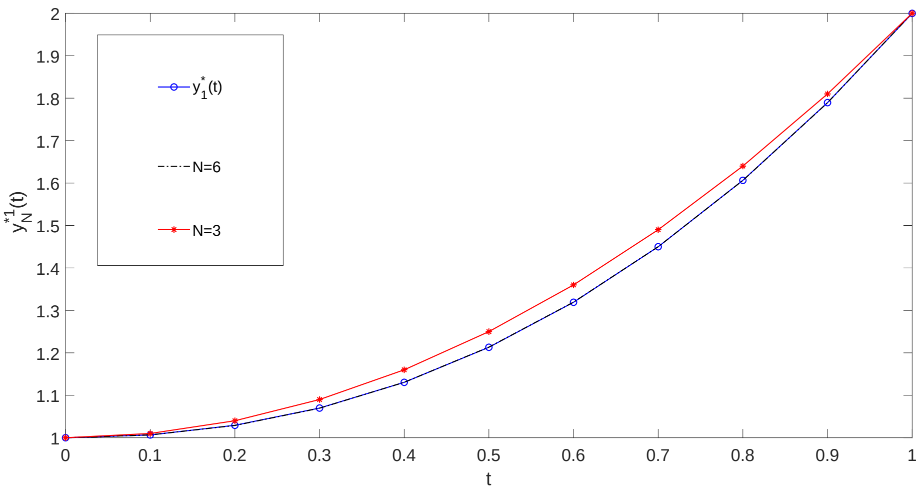

FIGURE 2.1: Exact solution and approximate solution of $y_1(t)$

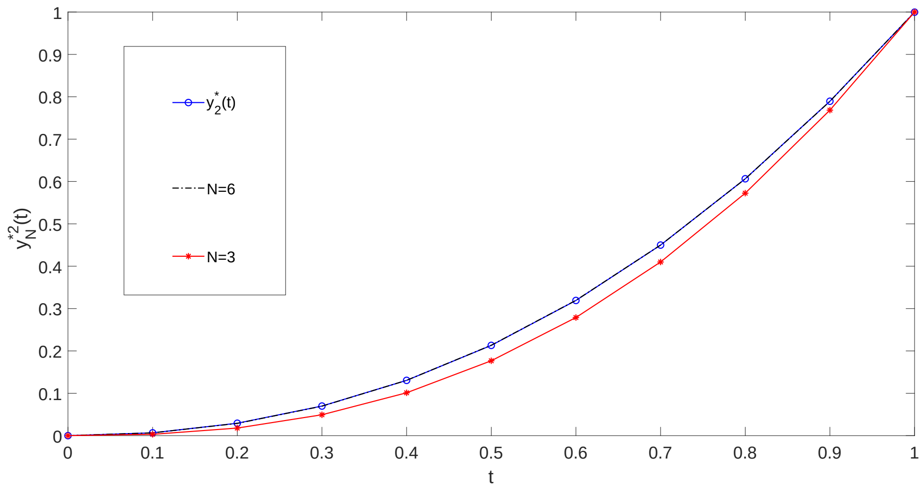

FIGURE 2.2: Exact solution and approximate solution of $y_2(t)$

In Fig. 2.1 and Fig. 2.2, the approximate results are plotted for various N with the exact solution, which show that the approximate solutions converge to the exact solutions.

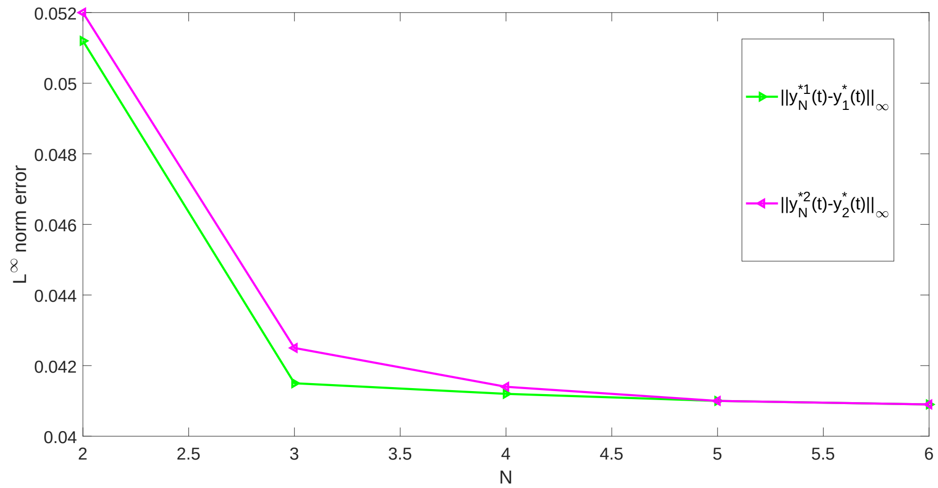

FIGURE 2.3: The L^∞ norm error against N

In Fig. 2.3, we plot the maximum absolute error functions $||y_N^{*1}(t) - y_1^*(t)||_\infty$ and $||y_N^{*2}(t) - y_2^*(t)||_\infty$ respectively. As shown in Fig. 2.3, the maximum error is 4.10×10^{-2}

for $N = 6$. In Fig. 2.4 , we plot the combined absolute error of approximated solutions and exact solutions. Here the maximum error reaches 1.81×10^{-4}.

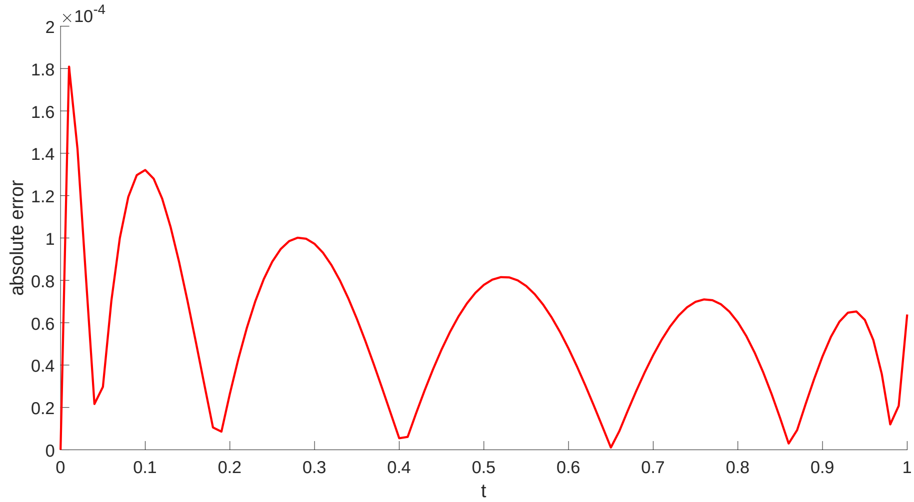

FIGURE 2.4: The absolute error for combined y_1 and y_2 with $N = 6$

Example 2.2. Consider the Fractional variational problem with right Caputo derivative

$Min\ J[y_1, y_2] = \frac{1}{2}\int\limits_0^1 \left({}_t^C D_1^\alpha y_1 +_t^C D_1^\alpha y_2 - f(t)\right)^2 dt,$

where, $f(t) = \frac{2}{\Gamma(2-\alpha)}(1-t)^{1-\alpha}$, subject to initial conditions, $y_1(1) = 1, \quad y_2(1) = 0.$

The exact solutions for the $\alpha = 1$ to the given problem are, $y_1^*(t) = 2 - t$ and $y_2^*(t) = 1 - t$.

Now, we find the approximate solutions using the method as discussed in Section 2.3, and compare with the exact solutions by plotting the curves.

In Fig. 2.5 and Fig. 2.6, we have plotted the approximate results versus the exact solutions for various N and $\alpha = 0.8$, and shown that the approximate solutions converge to the exact solutions. Fig. 2.7 and Fig. 2.8 exhibit the outcomes for various estimations in the wake of performing the union examination for each case. As α approaches to 1, the outcomes recuperate the analytical solution of the integer-order problem, and along these lines approve the numerical approximation. In Fig.

2.9, we plot the absolute error $|y_N^{*1}(t) - y_1^*(t)|$ and $|y_N^{*2}(t) - y_2^*(t)|$ for $\alpha = 0.9$ and $N = 6$ respectively. As shown in Fig. 2.9, the maximum error is 7.228×10^{-3} for $N = 6$.

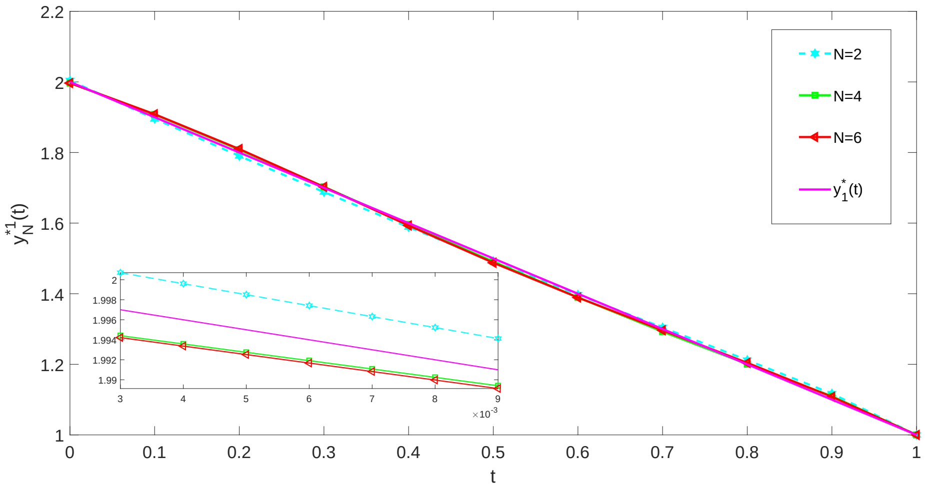

FIGURE 2.5: Comparison of the approximate solution $y_N^{*1}(t)$ for $\alpha = 0.8$, with different N and exact solution $y_1^*(t)$ for $\alpha = 1$

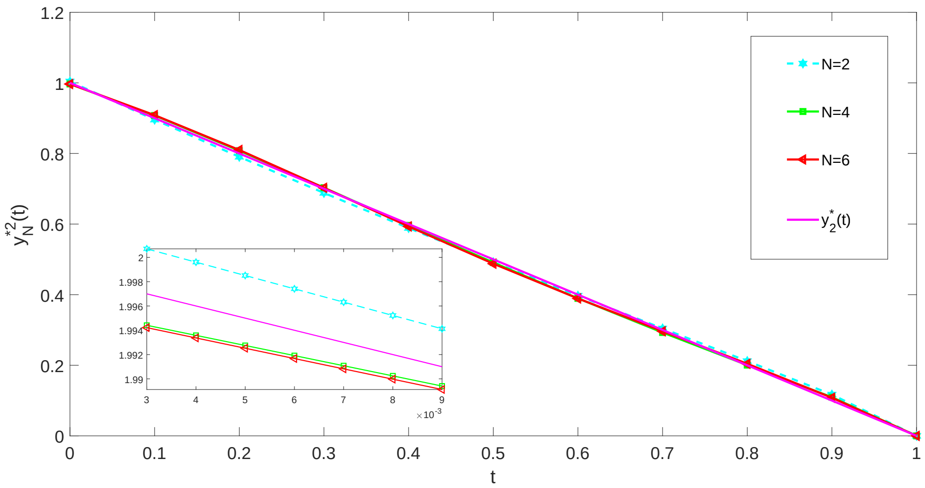

FIGURE 2.6: Comparison of the approximate solution $y_N^{*2}(t)$ for $\alpha = 0.8$, with different N and exact solution $y_2^*(t)$ for $\alpha = 1$

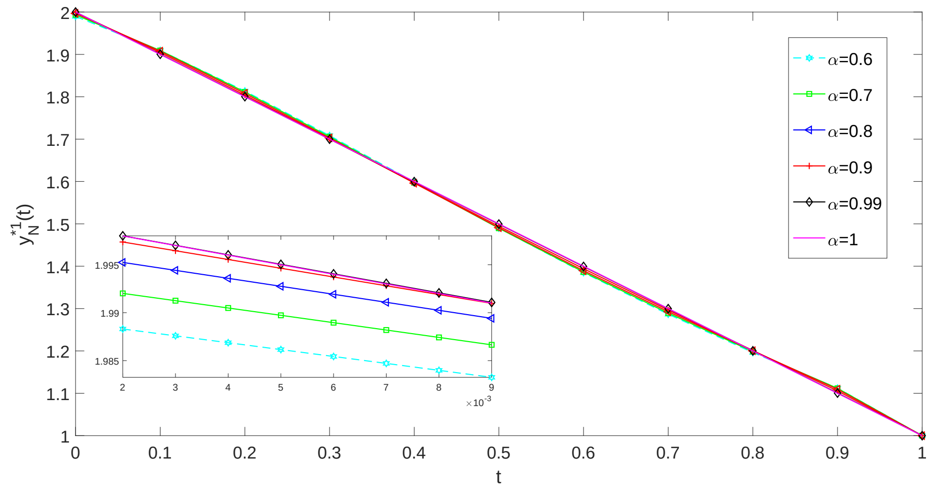

FIGURE 2.7: The function $y_N^{*1}(t)$ when $N = 4$ at different values of α

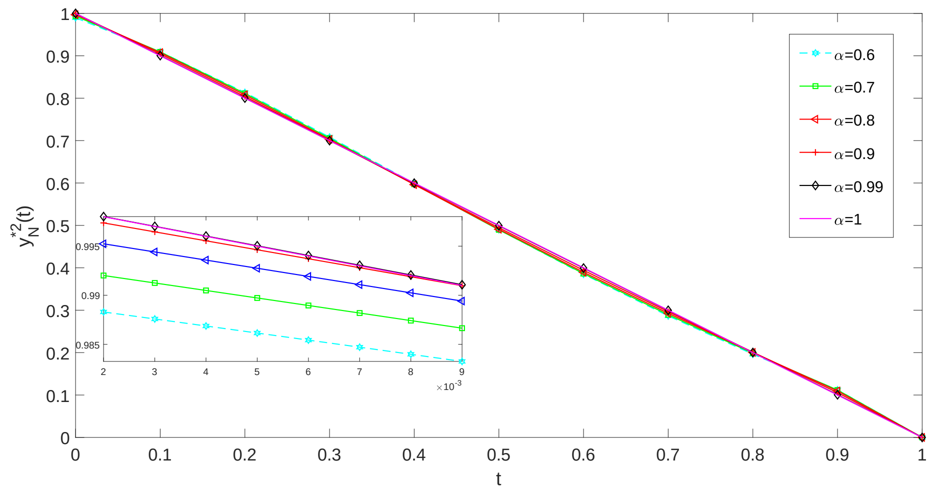

FIGURE 2.8: The function $y_N^{*2}(t)$ when $N = 4$ at different values of α

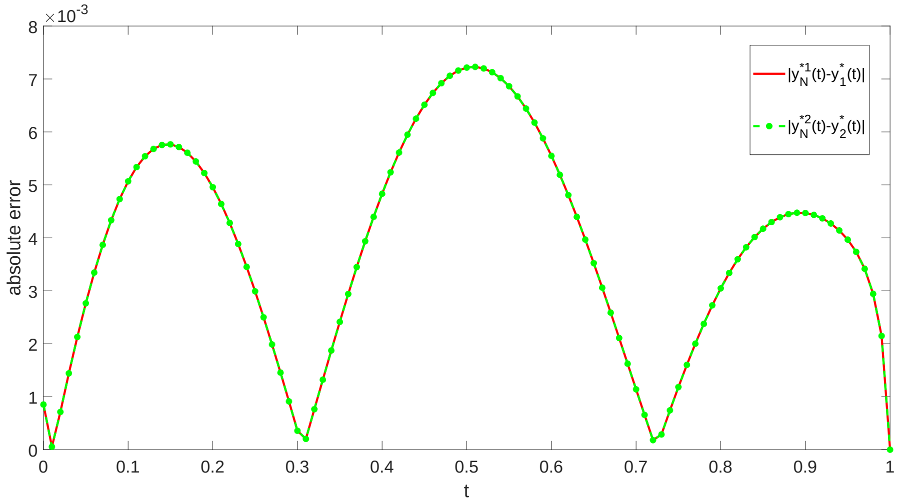

FIGURE 2.9: The absolute error for y_1 and y_2 with $N = 6$ and $\alpha = 0.9$

Example 2.3. Consider the following fractional variational problem [31].
$Min\ J[y] = \frac{1}{2}\int_0^1 [y_1^2 + y_2^2]\, dx$, such that, ${}_0^C D_x^\alpha y_1 = -y_1 + y_2$, and $y_1(0) = 1$.
For this problem, we have the exact solution, in the case of $\alpha = 1$ [31],
$y_1^*(x) = \cosh\left(\sqrt{2}x\right) + \beta \sinh\left(\sqrt{2}x\right)$, $y_2^*(x) = (1+\sqrt{2}\beta)\cosh\left(\sqrt{2}x\right) + (\sqrt{2}+\beta)\sinh\left(\sqrt{2}x\right)$,
where,
$\beta = -\frac{\cosh\left(\sqrt{2}\right)+\sqrt{2}\sinh\left(\sqrt{2}\right)}{\sinh\left(\sqrt{2}\right)+\sqrt{2}\cosh\left(\sqrt{2}\right)} \approx -0.98.$

Now, we find the approximate solution of the stated problem using mentioned method in Section 2.3, and discuss the solutions for different values of $\alpha \in (0, 1)$. Fig. 2.10 and Fig. 2.11 demonstrate that the convergence result of $y_N^{*1}(x)$ and $y_N^{*2}(x)$ for various N with $\alpha = 0.9$. In Fig. 2.12 and Fig. 2.13, we plotted the curves for approximate solutions at different values of α, and observe that when α tends to 1, the numerical solutions approach to the exact solutions. In Table 2.1, we discuss the absolute error function $|y_N^{*1}(t) - y_1^*(t)|$ and $|y_N^{*2}(t) - y_2^*(t)|$ for $\alpha = 0.9$ respectively.

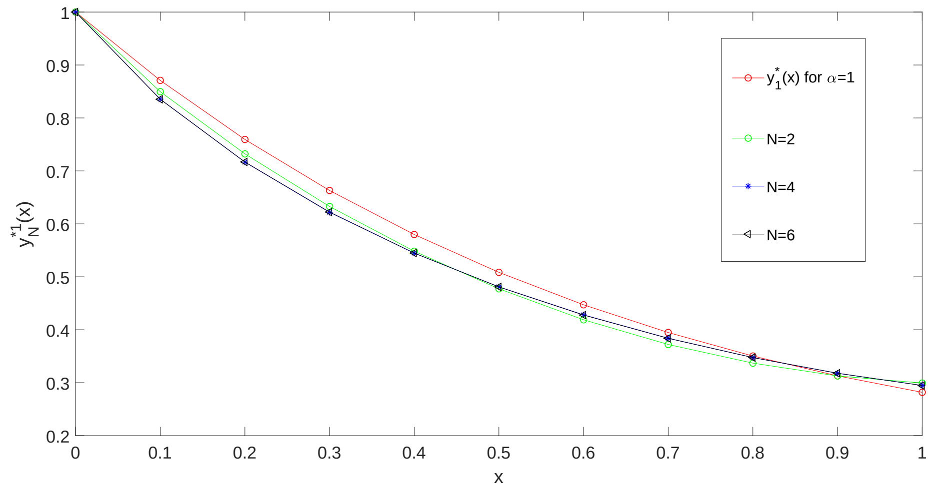

FIGURE 2.10: Comparison of the approximate solution $y_N^{*1}(x)$ for $\alpha = 0.9$, with different N, and exact solution $y_1^*(x)$ for $\alpha = 1$

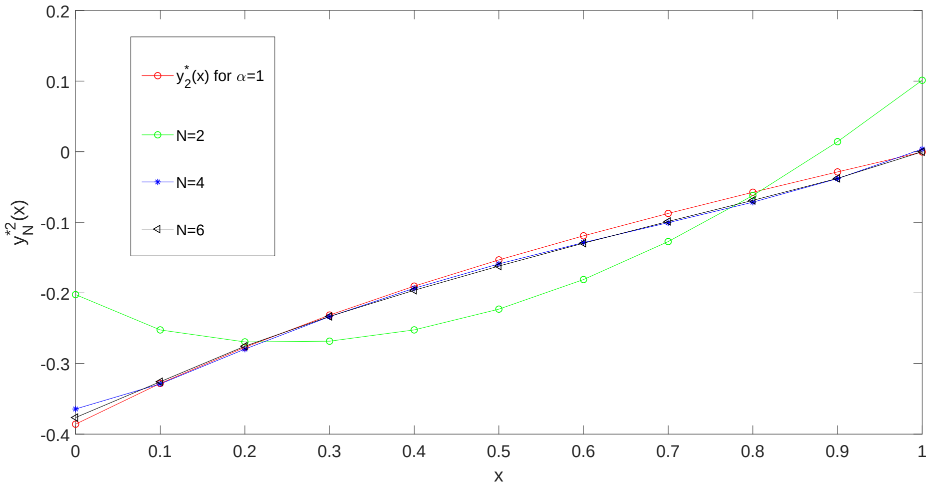

FIGURE 2.11: Comparison of the approximate solution $y_N^{*2}(x)$ for $\alpha = 0.9$, with different N, and exact solution $y_2^*(x)$ for $\alpha = 1$

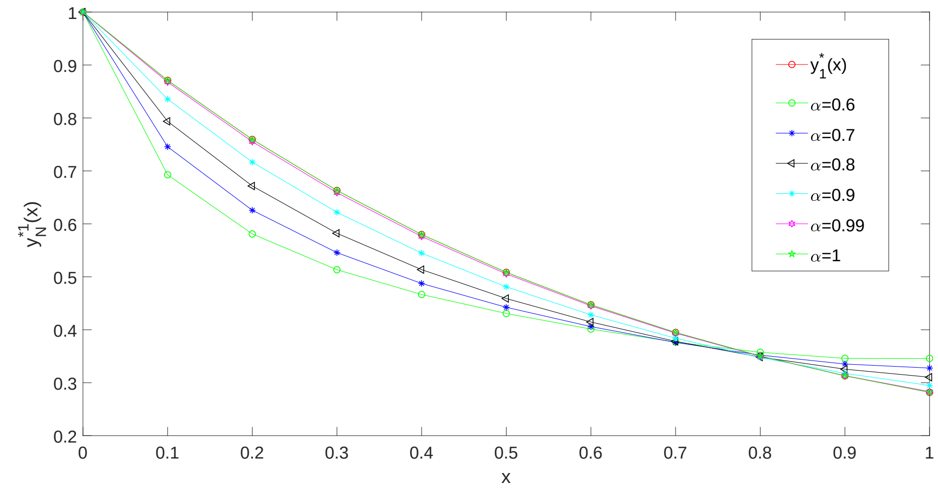

FIGURE 2.12: Approximate solution $y_N^{*1}(x)$ for $N = 4$ with different values of α, and exact solution $y_1^*(x)$ for $\alpha = 1$

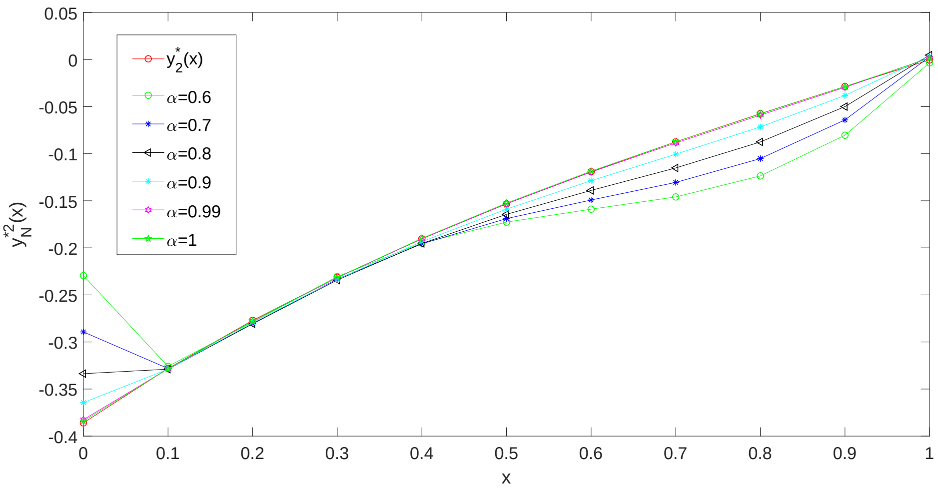

FIGURE 2.13: Approximate solution $y_N^{*2}(x)$ for $N = 4$ with various values of α, and exact solution $y_2^*(x)$ for $\alpha = 1$

TABLE 2.1: Absolute error calculation for fixed $\alpha = 0.9$ from the numerical solutions only.

t	$\lvert y_{16}^1(t) - y_8^1(t)\rvert$	$\lvert y_{16}^2(t) - y_8^2(t)\rvert$
0.2	1.697×10^{-5}	2.7×10^{-4}
0.4	9.667×10^{-6}	1.409×10^{-5}
0.6	7.01×10^{-6}	1.065×10^{-5}
0.8	3.4×10^{-7}	1.579×10^{-5}
1	2.3×10^{-7}	1.770×10^{-6}

2.7 Conclusions

For understanding a class of FVPs with several variables, an effective and precise numerical technique is presented in this chapter. Using Jacobi poly-fractonomials as a basis function and Rayleigh-Ritz method, we obtained the approximate solutions of the problem by solving a system of an algebraic equations. The convergence of the method has been broadly discussed, and illustrative test examples are considered to exhibit legitimacy and relevance of the method.

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