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# List of Symbols

Symbol	Description
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+^m$	Nonnegative orthant in $\mathbb{R}^m$ (components $\geq 0$ )
$\mathbb{R}_{++}^m$	Strictly positive orthant in $\mathbb{R}^m$ (components $> 0$ )
$\mathbb{R}^n$	Euclidean space of dimension $n$
$\mathcal{B}(x)$	The Hessian approximation matrix at $x$
$I_{n \times n}$	Identity matrix of dimension $n$
$y_1 \succeq y_2$ or $y_2 \preceq y_1$	$y_1 - y_2$ consist of nonnegative components i.e., $y_1 - y_2 \in \mathbb{R}_+^m$
$y_1 \succ y_2$ or $y_2 \prec y_1$	$y_1 - y_2$ consist of positive components, i.e., $y_1 - y_2 \in \mathbb{R}_{++}^m$
$JF(x)$	Jacobian of $F$ at $x$
Image $JF(x)$	Image set of Jacobian of $F$ at $x$
$A \subset_{\infty} B$	$A$ is an infinite subset of $B$
$\ \cdot\ _2$	2-norm (Euclidean norm)

# Abbreviations

<b>Abbreviation</b>	<b>Description</b>
MOP	Multiobjective Optimization Problem
BFGS	Broyden-Fletcher-Goldfarb-Shanno
NMPRP	Nonmonotone Polak-Ribière-Polyak
NMCG	Nonmonotone Conditional Gradient Method
HSDY	Hestenes-Stiefel Dai-Yuan
HSPRP	Hestenes-Stiefel Polak-Ribière-Polyak
DYPRP	Dai-Yuan Polak-Ribière-Polyak
ALM	Augmented Lagrangian Method