Chapter 1

The Standard model of Particle Physics

The Standard Model (SM) of particle physics is the most successful quantum theory of our Universe describing electromagnetic, weak and strong interactions. In this chapter, we introduce the fundamental theoretical structure of the SM and its dynamics, which is based on a gauge symmetry.

Understanding the basic constituents of our Universe — the fundamental particles that make everything around us— has always been an integral part of human curiosity. This quest about the nature of existence can be traced back to the fifth century BCE, when the Greek philosopher Empedocles proposed that all structures in the world are made up of four fundamental elements : fire, water, air, and earth. He also introduced the concept of two opposing forces, Love and Strife, which govern how these fundamental elements combine and interact. In modern terms, Empedocles‘ idea can be interpreted as an early attempt to describe matter being made up of smaller substances that influence each other through attraction or repulsion.

Around the same time, Democritus, a contemporary of Empedocles, introduced a groundbreaking idea all matter consists of tiny, indestructible particles, which he called “atoms", derived from the Greek word *“atomos"*, meaning “indivisible".

Over the centuries, through the evolution of science, marked by various theories and experimental discoveries, the quest to uncover the fundamental constituents of matter has evolved, moving from the classical notion of earth, air, fire, and water to the discovery of a fundamental particle spectrum, which are millions of times smaller than the objects visible to the naked eye. We now believe we may have finally uncovered the ultimate *“atomos"*. These fundamental building blocks, known as quarks and leptons, are governed by four fundamental forces, forming the foundation of our current understanding of the universe.

The interactions between the fundamental matter particles are governed by the four fundamental forces : gravity, electromagnetism, the strong force, and the weak force. These forces vary in strength and range. The strengths of these fundamental forces are determined by how strongly their mediator particle interacts, which is described by the corresponding coupling constant.

The strongest among them is the aptly named, strong force, which is effective over a short range. Carried by gluons, the strong force binds together quarks to form hadrons, such as mesons, which consist of quark-antiquark pair ($q\bar{q}$), and baryons, which composed of three quarks (qqq). Quarks and gluon cannot be isolated from their parent hadron, without spontaneously forming new hadrons, i.e. an isolated quark or gluon cannot be observed as free particles. This is due to a inherent phenomenon of the strong interaction called color confinement.

The electromagnetic force, carried by photons, is 1/60 the strength of the strong force but has an infinite range. It acts on all charged particles, including quarks and charged leptons (electron, muon, and tau). This force is responsible for keeping electrons in orbits bound to atomic nuclei. Next is the weak nuclear force, which is about 10^{-4} times the strength of the electromagnetic force. Mediated by the $W^{\pm}$ and Z bosons, the weak force is responsible for processes like radioactive decay, where for example, it permits a proton to turn into a neutron and vice-versa. Like the strong force, it only acts over very short distances, about 10^{-18} m. The weak interaction is the only fundamental force that can change a quark's 'flavour,' an inherent property associated with each fermion that we will discuss in detail shortly.

1.1 The particle spectrum of the SM

All particles, whether elementary or composite, possess an inherent angular momentum known as quantum mechanical spin. Beyond being a measure of intrinsic angular momentum, spin determines the statistical behavior of particles. Particles with half-integer spins, $(1/2, 3/2, 5/2, \ldots)$, measured in units of Planck's constant ($\hbar$), are classified as fermions, while those with integer spins, $(0, 1, 2, \ldots)$ are classified as bosons. Fermions follow Fermi-Dirac statistics and are subject to the Pauli exclusion principle, which prohibits two identical fermions from occupying the same quantum state simultaneously. In contrast, bosons follow Bose-Einstein statistics, which allows multiple identical bosons to occupy the same quantum state.

The elementary fermions in the SM are divided into two main categories: quarks and leptons. Each category consists of six particles (excluding their anti-matter counterparts). Ordinary matter is composed of elementary spin 1/2 particles like quarks and leptons, making them fermions. In the SM, fermions are grouped based on their transformation properties under gauge group $SU(2)_L$. The left-handed fermions are organized as doublets under $SU(2)_L$, while the right handed fermions are treated as $SU(2)_L$ singlets. Each $SU(2)$ doublets consists of two fermion fields, referred to as family or generation of fermions.

Both the quark and leptonic sector consists of six flavours, categorized into three generations or families. 'Flavour' refers to the type of each fundamental particles having distinct properties, which is associated with specific flavour quantum numbers, such as isospin, strangeness, charmness, Baryon number, topness. In the quark sector, the up (u) and down (d) quarks are grouped in the first family, the charm (c) and strange (s) quarks in the second, while the third family consists of the top (t) and bottom (b) quarks. These families are arranged in the order of increasing mass.

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix} \tag{1.1}$$

Similarly, the lepton sector is also organized into three families: the electron (e) and its neutrino (ν_e) in the first family, the muon (μ) and muon-neutrino (ν_μ) in the second family, and the tau (τ) and tau-neutrino (ν_τ) forms the third family. In the SM, neutrinos are originally considered massless, although now we have strong experimental evidence that they possess a small mass.

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \tag{1.2}$$

The particles responsible for mediating the fundamental forces of the SM that we discussed – such as photons, $W^\pm$ and Z bosons, and gluons, possess a spin of 1, and are thus bosons. Furthermore, the Higgs boson, which we will discuss in details in the following section, has spin 0.

Now, to summarize the discussions thus far, it is helpful to illustrate various categories of particles in the SM, and the interactions governing them.

Type	Particle	Mass (approx.)	Interactions
Quarks	u	2.2 MeV/c²	Strong, Electromagnetic, Weak
	d	4.7 MeV/c²	Strong, Electromagnetic, Weak
	c	1.27 GeV/c²	Strong, Electromagnetic, Weak
	s	96 MeV/c²	Strong, Electromagnetic, Weak
	t	172.76 GeV/c²	Strong, Electromagnetic, Weak
	b	4.18 GeV/c²	Strong, Electromagnetic, Weak
Leptons	e	0.511 MeV/c²	Electromagnetic, Weak
	ν_e	$<$ 0.8 eV/c²	Weak
	μ	105.66 MeV/c²	Electromagnetic, Weak
	ν_μ	$<$ 0.8 eV/c²	Weak
	τ	1.776 GeV/c²	Electromagnetic, Weak
	ν_τ	$<$ 0.8 eV/c²	Weak
Gauge Bosons	γ	0	Electromagnetic
	W^+, W^-	80.38 GeV/c²	Weak
	Z^0	91.19 GeV/c²	Weak
	g	0	Strong
	H^0	125.1 GeV/c²	Electroweak

Table 1.1 Elementary particles of the Standard Model, their masses, and the interactions governing them. The values of the masses are taken from the reference [1].

1.2 Theoretical structure of the standard model for one generation

The mathematical structure of the Standard model is governed by fundamental principles, such as Lorentz invariance, local gauge invariance, and renormalizability. The SM is built upon the gauge symmetry formed by the direct product of three Lie groups, $SU(3)_C \times SU(2)_L \times U(1)_Y$, where $SU(3)_c$ corresponds to strong interactions, while $SU(2)_L \times U(1)_Y$ represents the electroweak interactions, which is the unified description of electromagnetism and the weak interactions. The subscripts refers to the quantum numbers, for instance, C in $SU(3)_C$ denotes the color charge, while the L in $SU(2)_L$ indicates that this group acts only on the left-handed particles. The Y in $U(1)_Y$ represents hypercharge, which distinguishes it from the $U(1)$ symmetry associated with electromagnetism.

We first consider one-family structure of the electroweak theory by introducing a doublet and a singlet of leptons transforming under the SU(2)$_L$×U(1)$_Y$ as ,

$$\psi^{\ell}_{L_1} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \sim (2,-1), \quad \psi^{\ell}_{R_1} = e_R \sim (1,-2). \tag{1.3}$$

We can write the free Lagrangian, invariant under the global SU(2)$_L$×U(1)$_Y$ symmetry as,

$$\mathscr{L} = \bar{\psi}^{\ell}_{L_1} i\gamma^{\mu}\partial_{\mu}\psi^{\ell}_{L_1} + \bar{\psi}^{\ell}_{R_1} i\gamma^{\mu}\partial_{\mu}\psi^{\ell}_{R_1}. \tag{1.4}$$

1.2.1 The gauge sector

To ensure gauge invariance under local gauge transformations, where the fields under $SU(2)_L$ transform as,

$$\psi^{\ell}_{L_1}(x) \to U_L(x)\psi^{\ell}_{L_1}(x), \quad U_L(x) = e^{-i\tau_a\alpha^a(x)}, \quad \tau_a = \frac{\sigma_a}{2}, \tag{1.5}$$

and the transformations under $U(1)_Y$ are,

$$\psi^{\ell}_{L_1} \to e^{-iY\beta(x)}\psi^{\ell}_{L_1}, \quad \psi^{\ell}_{R_1} \to e^{-iY\beta(x)}\psi^{\ell}_{R_1}, \tag{1.6}$$

the derivatives (∂_μ) in the kinetic terms of the SM Lagrangian are generalized into covariant derivatives (D_μ). Here, the hypercharge Y for left-handed and right-handed fields are different, as given in 1.3. The covariant derivative takes the form,

$$D_\mu = \partial_\mu - igW^a_\mu \tau^a - ig' B_\mu Y - ig_s G^b_\mu T^b, \tag{1.7}$$

where g, g', and g_s denote the coupling constants associated with $SU(2)_L$, $U(1)_Y$, and $SU(3)_c$, respectively, and μ is the spacetime index. Here, τ^a and T^b represent the generators of the Lie algebras for $SU(2)_L$ and $SU(3)_c$, respectively, where $a = 1, 2, 3$, and $b = 1, \ldots, 8$, refer to new spin-1 gauge bosonic fields, the three vector fields W^a_μ for $SU(2)_L$, and the eight vector fields G^b_μ for $SU(3)_c$, respectively. There is also a vector field B_μ for $U(1)_Y$. The σ_a ($a = 1, 2, 3$) in equation 1.5 represent the three Pauli matrices.

The dynamics of these vector fields is governed by writing the following kinetic terms,

$$L = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} G^b_{\mu\nu} G^{b\mu\nu} \tag{1.8}$$

where $B_{\mu\nu}$, $W^a_{\mu\nu}$, and $G^a_{\mu\nu}$ represent the corresponding field strengths that are introduced to ensure gauge-invariance in the kinetic term. They have the following explicit form,

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu, \end{aligned} \tag{1.9}$$

with $a, b, c = 1, 2, 3$, and

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \tag{1.10}$$

where $a, b, c = 1, \ldots, 8$.

The structure constants f^{abc}, that appear in equations 1.9 and 1.10, originate from the Lie algebras of $SU(2)_L$ and $SU(3)_c$, respectively, and are distinct for each.

They naturally obey the following relation for a specific group structure,

$$[\tau^a, \tau^b] = i f^{abc} \tau^c \tag{1.11}$$

This governs the non-abelian nature of the corresponding gauge-interactions.

The Lagrangian in equation 1.8 describes the propagation and self-interactions of the gauge-bosons. For non-abelian groups such as $SU(2)_L$ and $SU(3)_c$, these self-interactions are evident through the emergence of three- and four-point vertex terms.

We now turn to the electroweak sector of the SM, which is governed by the gauge group $SU(2)_L \times U(1)_Y$. Therefore, in the following discussion of theoretical formalism of the electroweak sector, we will omit the strong interactions associated with $SU(3)_c$. The local gauge transformation of the gauge fields under $U(1)_Y$ and $SU(2)_L$ is given by ,

$$B_\mu \to B_\mu + \frac{1}{g}\partial_\mu \lambda_Y(x), \tag{1.12}$$

$$W^a_\mu \to W^a_\mu + \frac{1}{g'}\partial_\mu \lambda^a_L(x) + \varepsilon^{abc} W^b_\mu \lambda^c_L(x) \tag{1.13}$$

where $\lambda_Y(x)$, $\lambda^a_L(x)$ are the parameter of local transformations under $U(1)_Y$ and $SU(2)_L$, respectively.

To account for the observed masses of the $W^\pm$ and Z bosons, a naive approach would be adding a term like $\frac{1}{2}m_W^2 W_\mu W^\mu$ to the Lagrangian. However, directly introducing such mass terms would lead to the appearance of terms like $\partial_\mu \lambda \partial^\mu \lambda$, which would violate the gauge symmetry under the transformations described in equations 1.12 and 1.13. Thus, the mass terms for gauge bosons are forbidden by the requirement of local gauge invariance.

We employ the Higgs mechanism, as discussed in the following section, to construct a renormalizable gauge theory that describes a short-range weak interaction while also including the long-range $U(1)_{\mathrm{EM}}$ symmetry of electromagnetism. This provides masses to three of the four vector bosons associated with the four generators of the gauge group $SU(2)_L \times U(1)_Y$, while preserving one massless boson corresponding to electromagnetism.

1.2.2 The Higgs Mechanism

The development of a renormalizable, short-range force for the weak interaction in the Standard Model begins with the introduction of the Higgs field, which is a complex scalar field φ, which is a doublet in the $\frac{1}{2}$ representation of $SU(2)$, given as,

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \tag{1.14}$$

The gauge-invariant Lagrangian corresponding to this field will include the kinetic term and a potential, as follows,

$$\mathscr{L}_\varphi = D_\mu \varphi_i^\dagger D^\mu \varphi_i - V(\varphi, \varphi^\dagger). \tag{1.15}$$

The potential $V(\phi)$ corresponds to the most general form of a gauge invariant potential,

$$V(\varphi, \varphi^\dagger) = -\mu^2 \varphi^\dagger \varphi + \lambda \left(\varphi^\dagger \varphi\right)^2. \tag{1.16}$$

where μ^2 and λ are real parameters. The term, $-\mu^2 \varphi^\dagger \varphi$, is responsible for destabilizing the potential at the origin when $\mu^2 > 0$. The second term, $\lambda(\varphi^\dagger \varphi)^2$, ensures that the potential remains bounded from below, provided $\lambda > 0$. The potential $V(\varphi, \varphi^\dagger)$ achieves its minimum energy configuration not at the value $|\varphi| = 0$ of the scalar field, but at

$$\langle \varphi^\dagger \varphi \rangle = \frac{\mu^2}{2\lambda} = \frac{v^2}{2}. \tag{1.17}$$

where v is the vacuum expectation value (VEV) of the field, responsible for breaking the symmetry. In the unitary gauge, the scalar field φ is expressed in a form that eliminates the un-physical Goldstone bosons, leaving only the physical degrees of freedom,

$$\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}, \tag{1.18}$$

where h represents the real scalar field, which is the physical Higgs boson.

The covariant derivative of the scalar fields takes the form,

$$D_\mu = \partial_\mu - igW_\mu^a T^a - ig' B_\mu Y, \tag{1.19}$$

where $Y = 1$ is the weak hypercharge of Higgs field.

To obtain the masses of the gauge bosons, we focus on the terms in the kinetic Lagrangian involving the gauge bosons. In the unitary gauge, the kinetic term in the scalar Lagrangian 1.15 takes the following form,

$$\begin{aligned} \left(D_\mu \varphi\right) \left(D^\mu \varphi\right)^\dagger = &\frac{1}{2} \left(\partial_\mu h\right) \left(\partial^\mu h\right) + \frac{1}{8} g^2 (v+h)^2 \left(W_\mu^1 - iW_\mu^2\right) \left(W^{1\mu} + iW^{2\mu}\right) \\ &+ \frac{1}{8}(v+h)^2 \left(-g' B_\mu + gW_\mu^3\right)^2. \end{aligned} \tag{1.20}$$

After substituting $h = 0$, we identify the terms quadratic in the gauge fields as,

$$\left(D_\mu \varphi\right) \left(D^\mu \varphi\right)^\dagger = \frac{1}{8} g^2 v^2 \left(W_\mu^1 - iW_\mu^2\right) \left(W^{1\mu} + iW^{2\mu}\right) + \frac{1}{8} v^2 \left(-g' B_\mu + gW_\mu^3\right)^2. \tag{1.21}$$

The first term in equation 1.21 involves the combination W^1_μ and W^2_μ. We can define,

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right). \tag{1.22}$$

Substituting this, the first term in equation 1.21 can rewritten in terms of the charged vector bosons ($W^{\pm}$) as,

$$\frac{1}{8}g^2v^2\left(W^1_\mu - iW^2_\mu\right)\left(W^{1\mu} + iW^{2\mu}\right) = \frac{1}{4}g^2v^2W^-_\mu W^{\mu +} \tag{1.23}$$

This provides the mass of the $W^{\pm}$ bosons as,

$$M_W = \frac{1}{2}gv. \tag{1.24}$$

On the other hand, the second term in equation 1.21, which is quadratic in the neutral fields, involves W^3_μ and B_μ, and can be written in the matrix representation as,

$$\frac{1}{8}v^2\begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix}. \tag{1.25}$$

We notice that the mass matrix $\frac{1}{8}v^2\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}$ is non-diagonal, indicating that the mass terms for gauge bosons are mixed. In order to obtain the mass terms for the physical gauge bosons, it needs to be diagonalized. To achieve this, we introduce the following orthogonal combinations of the fields,

$$A_\mu = \sin\theta_{\rm W} W_{3\mu} + \cos\theta_{\rm W} B_\mu, \tag{1.26}$$

$$Z_\mu = \cos\theta_{\rm W} W_{3\mu} - \sin\theta_{\rm W} B_\mu, \tag{1.27}$$

where θ_W is the Weinberg angle, which we will determine shortly. It refers to the mixing between $W_{3\mu}$ and B_μ fields in such a way that the resulting fields A_μ and Z_μ are mass eigenstates. The inverse relations can be written as,

$$W_{3\mu} = \sin\theta_W A_\mu + \cos\theta_W Z_\mu, \tag{1.28}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu. \tag{1.29}$$

Now, substituting these relations into the original quadratic term in neutral fields, which is the second term in equation 1.21, we obtain,

$$\begin{aligned}\frac{1}{8}v^2\left(-g'B_\mu + gW^3_\mu\right)^2 = \frac{1}{8}v^2\Big[&A^2_\mu\left(g\sin\theta_W - g'\cos\theta_W\right)^2 + Z^2_\mu\left(g\cos\theta_W + g'\sin\theta_W\right)^2 \\ &+2A_\mu Z_\mu\left(g\sin\theta_W - g'\cos\theta_W\right)\left(g\cos\theta_W + g'\sin\theta_W\right)\Big].\end{aligned} \tag{1.30}$$

In order to diagonalize the mass matrix in equation 1.25, the off-diagonal terms in equation 1.30 must vanish. This requirement leads to the following condition for the Weinberg angle,

$$g\sin\theta_W = g'\cos\theta_W \tag{1.31}$$

From this, we can solve for the sine and cosine of the Weinberg angle,

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{1.32}$$

Now, the quadratic form in equation 1.30 reduces to the expected form for a massive Z_μ field. Comparing it with the standard mass term for a gauge boson, we can infer the mass of the Z-boson as,

$$M_Z = \frac{v}{2}\left(g\cos\theta_W + g'\sin\theta_W\right) = \frac{v}{2}\sqrt{g^2+g'^2} = \frac{gv}{2\cos\theta_W}. \tag{1.33}$$

Also, the mass of the $W^{\pm}$ bosons can be related to the Z-boson mass through the following relation,

$$M_W = \cos\theta_{\mathrm{W}} M_Z. \tag{1.34}$$

Furthermore, through the equation 1.30 and 1.31, it is evident that the field A_μ lacks any mass term, which is associated with the $U(1)$ symmetry that remains unbroken. Thus, A_μ is identified as the only massless gauge field in the electroweak sector, which is the photon.

1.2.3 Generating the masses of the fermions

We now turn up to the mass generation in the fermionic sector. The masses of the fermions in the SM arise through the mechanism of Spontaneous Symmetry Breaking (SSB). A Lorentz invariant mass term for fermionic field (ψ) could be written as $m\overline{\psi}\psi$, where $\overline{\psi} = \psi^\dagger\gamma^0$. This term can be further decomposed in terms of the left and right-handed projections of the fermionic field as,

$$m\overline{\psi}\psi = m\overline{\psi}_R\psi_L + m\overline{\psi}_L\psi_R, \tag{1.35}$$

where $\psi_L = P_L\psi, \psi_R = P_R\psi$, while $P_L = \frac{1}{2}(1-\gamma^5)$ and $P_R = \frac{1}{2}(1+\gamma^5)$ are the left-handed and right-handed projection operators, respectively.

However, as we discussed earlier, since left-handed and right-handed fermion fields transform differently under $SU(2)_L$, it is not possible to express the fermion mass term in the form of the equation 1.35, as it is not gauge-invariant. It turns out that in order to retain gauge invariance and allow the generation of mass terms, it is essential to introduce the Higgs field (φ). The Lagrangian that governs fermion mass generation is referred to as the Yukawa interaction, which for a generic fermionic field ψ corresponding to any fermion f, assumes the general form,

$$\mathscr{L}_Y = -\sum_{i,j} y_{ij} \bar{\Psi}^{q,\ell}_{L_i} \varphi \psi^{q,\ell}_{R_j} + \text{H.c.}, \tag{1.36}$$

where y_{ij} are the Yukawa couplings, and indices i, j run over the fermionic generations. For instance, in the case of one-family model corresponding to leptons, where $\psi^\ell_{L_1} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, and $\psi^\ell_{R_1} = e_R$, and going to the unitary gauge, given in equation 1.18, just as before, we obtain

$$\bar{\Psi}^\ell_{L_1} \varphi \psi^\ell_{R_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \begin{pmatrix} 0 \\ v+h \end{pmatrix} e_R. \tag{1.37}$$

The corresponding interaction term in the Yukawa Lagrangian becomes,

$$\mathscr{L}^\ell_Y = -\frac{y^\ell_{11}}{\sqrt{2}} \bar{e}_L (v+h) e_R + \text{H.c.}, \tag{1.38}$$

From here, the quadratic term in the fermion fields corresponding to the electron can be identified as the Dirac mass term for the electron, which is given by

$$m_e = y^\ell_{11} \frac{v}{\sqrt{2}}. \tag{1.39}$$

Due to the absence of right handed neutrinos ν_{R_i} in the SM, neutrino masses cannot be generated through the Yukawa interactions, and as a consequence, the neutrinos remains massless in the SM.

The mass generation of quarks follows the similar mechanism to that of leptons, and the explicit form of the Yukawa Lagrangian for up-type and down-type quarks turns out to be,

$$-\mathscr{L}^q_Y = \sum_{i,j} [y^u_{ij} \bar{\Psi}^q_{L_i} \tilde{\varphi} \psi^u_{R_i} + y^d_{ij} \bar{\Psi}^q_{L_i} \varphi \psi^d_{R_i}] + \text{H.c.}, \tag{1.40}$$

where $\tilde{\varphi} = i\sigma_2\varphi^* = \begin{pmatrix} \varphi^{0*} \\ -\varphi^- \end{pmatrix}$, is the conjugate Higgs field, which transforms in the same manner under $SU(2)_L$ as the Higgs field, but has hypercharge $Y = -1$. This is required to retain gauge invariance of the Yukawa Lagrangian for the up-type quarks, as up-type and down-type quarks have different hypercharges under $U(1)_Y$ gauge symmetry.

Similar to the approach as before, we begin by considering single generation. The extension to other families is straightforward. The first family of quark sector includes a doublet $\bar{\Psi}^q_{L_1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, and two singlets u_R and d_R under $SU(2)_L$.

After spontaneous symmetry breaking, and turning to the unitary gauge of the Higgs field, as earlier, we have,

$$\mathscr{L}^q_Y = -\frac{(v+h)}{\sqrt{2}}\,(y^u_{11}\bar{u}_L u_R + y^d_{11}\bar{d}_L d_R) + \text{H.c.}, \tag{1.41}$$

from where the mass terms for up and down quarks can be read as,

$$m_u = y^u_{11}\frac{v}{\sqrt{2}}, \quad m_d = y^d_{11}\frac{v}{\sqrt{2}}. \tag{1.42}$$

In this way, through the spontaneous symmetry breaking, all elementary matter particles, including all the quarks and charged leptons, get their masses from their interaction with the Higgs field through Yukawa interactions. Neutrinos, on the other hand, may obtain their masses through a more indirect mechanism, such as, the see-saw mechanism, but the Higgs field remains essential for generating Dirac mass term for them as well.

1.2.4 Quark mixing

We now extend our discussion on the Yukawa interactions of the SM by explicitly considering the three families of quarks and leptons. The Yukawa Lagrangian involving the fermions and the Higgs field interactions, can be written as,

$$-\mathscr{L}_Y = \sum_{i,j=1}^{3} y^{\ell}_{ij}\bar{\Psi}^{\ell}_{L_i}\varphi\psi^{\ell}_{R_j} + \sum_{i,j=1}^{3}[y^{u}_{ij}\bar{\Psi}^{q}_{L_i}\tilde{\varphi}\psi^{u}_{R_i} + y^{d}_{ij}\bar{\Psi}^{q}_{L_i}\varphi\psi^{d}_{R_i}] + \text{H.c.}, \tag{1.43}$$

where the fermion fields $\psi^{\ell,q}_{L_i}$ and $\psi^{\ell,q}_{R_i}$ are fermion fields transforming under the gauge group $SU(2)_L$ and $U(1)_Y$, respectively, and y^{ℓ}_{ij}, y^{u}_{ij}, and y^{d}_{ij} are the corresponding Yukawa couplings.

Similar to the discussion before, once the Higgs field acquire its VEV through spontaneous symmetry breaking of the electroweak symmetry while retaining the residual $U(1)$ symmetry, the Yukawa interactions result in the generation of fermion mass terms as,

$$\begin{aligned}-\mathscr{L}_Y &= \sum_{i,j=1}^{3} y^{\ell}_{ij}\frac{v}{\sqrt{2}}\,\bar{\Psi}^{\ell}_{L_i}\ell_{R_j}\left(1+\frac{h}{v}\right) + \frac{v}{\sqrt{2}}\sum_{i,j=1}^{3}\left(y^{u}_{ij}\,\bar{\Psi}^{u}_{L_i}u_{R_j} + y^{d}_{ij}\,\bar{\Psi}^{d}_{L_i}d_{R_j}\right)\left(1+\frac{h}{v}\right) + \text{H.c.} \\ &= \left[\mathscr{M}^{\ell}_{ij}\bar{\Psi}^{\ell}_{L_i}\ell_{R_j} + \mathscr{M}^{u}_{ij}\bar{\Psi}^{u}_{L_i}u_{R_j} + \mathscr{M}^{d}_{ij}\bar{\Psi}^{d}_{L_i}d_{R_j}\right]\left(1+\frac{h}{v}\right) + \text{H.c.},\end{aligned} \tag{1.44}$$

where $\mathscr{M}^{\ell}_{ij}$, $\mathscr{M}^{u}_{ij}$, and $\mathscr{M}^{d}_{ij}$, are the 3×3 mass matrices for charged leptons, and up-type and down-type quarks respectively. They are given by,

$$\begin{aligned}\mathscr{M}^{\ell}_{ij} &= y^{\ell}_{ij}\frac{v}{\sqrt{2}}, \\ \mathscr{M}^{u}_{ij} &= y^{u}_{ij}\frac{v}{\sqrt{2}}, \\ \mathscr{M}^{d}_{ij} &= y^{d}_{ij}\frac{v}{\sqrt{2}},\end{aligned} \tag{1.45}$$

where v is the VEV of the Higgs field, while y^{ℓ}_{ij}, y^{u}_{ij}, and y^{d}_{ij} are the Yukawa couplings, which, in general, are complex numbers. In order to determine the physical mass eigenstates from the Lagrangian, the mass matrices must be diagonalized. The mass matrices are generally neither symmetric nor Hermitian. Their diagonalization is achieved by biunitary transformations, where one apply two distinct unitary matrices for each of the mass matrices.

For the diagonalization of the quarks mass matrices through biunitary transformations, we employ unitary matrices $V_L^{u,d}$, and $V_R^{u,d}$, which diagonalizes the mass matrices as,

$$V_L^u \mathscr{M}_u V_R^{u\dagger} = \mathrm{diag.}\,(m_u, m_c, m_t), \tag{1.46}$$

$$V_L^d \mathscr{M}_d V_R^{d\dagger} = \mathrm{diag.}\,(m_d, m_s, m_b), \tag{1.47}$$

where $V_L^{u^\dagger} V_L^u = V_R^{u^\dagger} V_R^u = 1$, and analogous unitarity condition holds for the matrices corresponding to the down-type quarks. These matrices, which are employed in biunitary diagonalization can also be used in expressing the physical mass eigenstates in terms of the gauge eigenstates through linear transformations, as follows,

$$\psi_L^u = V_L^{u^\dagger} \begin{pmatrix} u'_L \\ c'_L \\ t'_L \end{pmatrix}, \quad \psi_L^d = V_L^{d^\dagger} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix}, \tag{1.48}$$

and likewise for the right-handed fields. Here, the prime notation on the fields from this point onward in this section, indicates that they are gauge eigenstates, which interact diagonally with the gauge fields. This is to distinguish them from mass eigenstates (denoted by un-primed fields) that correspond to physical particles with definite masses after symmetry breaking and diagonalization of the mass matrices.

Thus, the masses of fermions, after diagonalization of the mass matrices, are expressed in terms of the Higgs VEV and Yukawa couplings as,

$$m_i^{u,d,\ell} = y_{ii}^{u,d,\ell} \frac{v}{\sqrt{2}}. \tag{1.49}$$

The Yukawa couplings y_{ii} represent the strength of the coupling between the Higgs and the corresponding fermion. These couplings are free parameters in the Standard Model, and their values are determined experimentally.

Now, we discuss the phenomenon of quark mixing, which is the consequence of the fact that mass eigenstate are different than gauge eigenstate, that we discussed just before. Mixing refers to the transition of one quark flavour into another, which is governed through the weak interactions. In the Standard Model (SM), there are two distinct types of interactions between fermions and the bosons mediating the weak force, the weak charged currents (CC) and the neutral currents (NC). These interactions arise from currents coupled to the charged $W^{\pm}$ bosons and the neutral Z boson, respectively.

For the leptonic sector, in terms of the gauge eigenstates, the weak charged current takes the form,

$$J_\ell^\mu = \sum_{i=1}^{3} \left(\bar{\nu}'_{iL} \gamma^\mu e'_{iL} + \bar{e}'_{iL} \gamma^\mu \nu'_{iL} \right). \tag{1.50}$$

Since neutrinos are assumed to be massless in the SM, this expression simplifies to,

$$J_\ell^\mu = \bar{\nu}'_L \gamma^\mu e'_L + \bar{e}'_L \gamma^\mu \nu'_L = \bar{\nu}_L V_L^{\ell\dagger} \gamma^\mu V_L^\ell e_L + \bar{e}_L V_L^{\ell\dagger} \gamma^\mu V_L^\ell \nu_L, \tag{1.51}$$

where V_L^ℓ is the matrix that diagonalizes the Yukawa mass matrix for the charged-leptons. Since neutrinos are massless in the SM, the same diagonalizing matrix can be used for them, as they remain massless under any unitary transformation.

From the unitarity of V_L^ℓ matrix, which is analogous to the quark case discussed earlier, we obtain

$$J_\ell^\mu = \sum_{i=1}^{3} \left(\bar{\nu}_{iL}\gamma^\mu e_{iL} + \bar{e}_{iL}\gamma^\mu \nu_{iL} \right), \tag{1.52}$$

where the un-primed fields represent the physical mass-eigenstates. This shows that no mixing of lepton flavours occurs at the level of the weak charged currents within the SM. This also explains why the three lepton numbers L_e, L_μ, L_τ are conserved in weak interactions.

However, for quarks, the situation is different because, unlike the massless neutrinos in the lepton sector, all the quarks have masses. The charged weak current in the quark sector is given by,

$$J_q^\mu = \sum_{i=1}^{3} \left(\bar{d}'_{iL}\gamma^\mu u'_{iL} + \bar{u}'_{iL}\gamma^\mu d'_{iL} \right) \tag{1.53}$$

This can be rewritten in terms of the physical mass eigenstates using the transformations 1.48 for the left-handed quarks fields as,

$$J_q^\mu = \bar{d}'_L\gamma^\mu u'_L + \bar{u}'_L\gamma^\mu d'_L = \bar{d}_L V_L^{d\dagger}\gamma^\mu V_L^u u_L + \bar{u}_L V_L^{u\dagger}\gamma^\mu V_L^d d_L, \tag{1.54}$$

where we have two different matrices V_L^u, V_L^d that are used to diagonalize the mass matrices of up-type and down-type quarks, respectively. Now, we define the Cabibbo-Kobayashi-Masakawa (CKM) matrix as,

$$V_{CKM} = V_L^{u\dagger} V_L^d. \tag{1.55}$$

The weak charged current expression for quarks in equation 1.54 now takes the following form,

$$J_q^{\mu} = \bar{d}_L V_{CKM}^{\dagger} \gamma^{\mu} u_L + \bar{u}_L V_{CKM} \gamma^{\mu} d_L. \tag{1.56}$$

This explicitly shows the mixing between quarks of different flavours, which is generated through the charged current of the weak interaction, and is quantified in terms of the elements of the CKM matrix.

The CKM matrix is a 3×3 unitary complex matrix, represented symbolically as,

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \tag{1.57}$$

The CKM matrix defined in equation 1.55 specifies the strength of the quark mixing, which allows quarks to alter their flavour during weak interactions mediated by $W^{\pm}$ bosons. For instance, a down-type quark can transform into an up-type quark through the exchange of a charged W boson, and the strength of such a transition is encoded into the element $|V_{ud}|$ of the CKM matrix. Such transitions, which involve a change in the quark's electric charge by one unit, referred to as flavour-changing charged currents (FCCC). These transitions are not confined to quarks within the same family and can occur across families. In contrast, flavour-changing neutral currents (FCNC) involve quark flavour transitions without altering electric charge. However, due to the flavour-diagonal coupling of the neutral Z boson, FCNCs are absent at tree level in the SM. Even at the loop level, they are highly suppressed by the Glashow-Iliopoulos-Maiani (GIM) mechanism.

In general, CKM matrix is parameterized in terms of three rotation angles $(\theta_{12}, \theta_{13}, \theta_{23})$ and one CP-violating phase (δ). In standard parametrization, it is given as,

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{1.58}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. When $\theta_{13} = \theta_{23} = 0$, the mixing involves only the first two generations, with θ_{12} corresponding to the Cabibbo angle (θ_C). The three mixing angles can be chosen such that they lie within the first quadrant, ensuring that $s_{ij} > 0$ and $c_{ij} > 0$. All the nine elements have to be determined experimentally, and there appears a hierarchical pattern of the quark mixing, as, $s_{13} \ll s_{23} \ll s_{12} \ll 1$.

1.3 Beyond the Standard Model

Since its inception, the SM, through its remarkably accurate predictions of numerous phenomena of particle physics and their subsequent experimental verification, has evolved into a robust and astonishing theoretical framework. However, the theoretical framework of the SM omits gravity, and consequently breaks down at energy scales near the Planck scale $M_{\rm Pl} \sim 10^{19}$ GeV, where quantum gravitational effects become significant. As such, in modern theory, the SM is viewed as an effective field theory, which is valid up to a cutoff scale much below $M_{\rm Pl}$. Thus, despite its immense success in providing precise experimental predictions, the SM still holds many key unresolved questions and limitations, which manifest themselves in various ways. We will briefly discuss some of these imperfections in the subsequent discussion in this section.

One of the most subtle yet captivating challenges within the SM is understanding the origin of the enigmatic pattern of masses and mixing of the three families of quarks and leptons—collectively known as the flavour problem or flavour puzzle of the SM.

For instance, it could be seen through the table 1.1, that among the three families of fermions, the masses of the third-family fermions significantly exceed those of the second family, which in turn are much larger than those in the first generation. Secondly, within each family, there is striking uneven pattern in the sense that in the second and the third family of quarks, the up-type quarks are much heavier than the corresponding down-type quarks. However, in the first family the mass of the down quark is almost of the same order that of the up-quark. The fermionic mass hierarchy is further complicated by the bizarre pattern of the observed mixing angles among families. For instance, the quark mixing angles exhibit the pattern, $\sin\theta^q_{12} \gg \sin\theta^q_{23} \gg \sin\theta^q_{13}$.

On the leptonic side, since SM lacks any mass term for neutrinos, only the charged leptons (e, μ, τ) within each family have masses in the SM. The discovery of neutrino oscillations, confirmed by the Super-Kamiokande and SNO experiments around the early 2000s, established that at least two neutrinos in the SM must have mass. However, the SM does not naturally include a mechanism to account for these masses.

Based on the subsections, 1.2.3 and 1.2.4, discussed earlier, we know the theoretical origin of masses of the quarks and leptons in the SM arise through the Higgs mechanism, where in order to account for the observed values of masses and mixing angles, the dimensionless Yukawa couplings ($y^{q,\ell}_{ii}$) scale themselves with respect to the reference value of the Higgs VEV $v \approx 246$ GeV. Therefore, while the top quark's Yukawa coupling is approximately of order 1, the coupling for the lightest charged lepton, the electron, is remarkably small, around $O(10^{-6})$. The theoretical rationale for this vast difference in the Yukawa couplings, spanning over several order of magnitude, remains an unresolved question in the SM, as these couplings were inserted in the SM only as a free parameter. In addition, the discovery of neutrino masses and mixing makes the flavour problem increasingly inescapable, as the fermion mass hierarchy now spans at least 12 orders of magnitude, ranging from the extremely light neutrinos to the massive top quark.

The SM faces another significant challenge from cosmological observations. These observations reveal that ordinary baryonic matter, which constitutes all the visible matter we observe—accounts for merely 5% of the Universe. In contrast, approximately 26% is attributed to dark matter, and 68% to dark energy. Dark matter, in particular, does not interact with ordinary matter or electromagnetic radiation, making it invisible. Its existence can be inferred solely through the gravitational effects it has on the visible matter. The SM, however, offers no explanation for dark matter. The current leading hypotheses in numerous extension to the SM propose potential fundamental candidates for dark matter, which are electrically neutral, such as weakly interacting massive particles (WIMPs) or axions for cold (non-relativistic) dark matter.

There is an observed imbalance between matter and antimatter in the universe, with matter vastly being abundant in comparison to antimatter. This is also referred to as baryon asymmetry. This discrepancy is also not explained by the SM, which predicts equal amounts of both.

Another limitation of the SM arises from the strong interactions, where there is strong CP problem. This refers to the apparent absence of the Charge-Parity (CP) violation in Quantum Chromodynamics (QCD). In principle, QCD allows for CP violation, which is characterized by a parameter θ, but experimental observations, such as the electric dipole moment of the neutron, indicate that θ must be less or around $O(10^{-10})$, which is extremely small and unnatural. A possible leading solution is the Peccei-Quinn mechanism, which introduces a new global symmetry that dynamically drives θ to zero. This theory predicts the existence of a new pseudoscalar particle, axion, which is also a potential dark matter candidate.

Among all the unresolved problems and limitations of the SM, some of which are discussed above, this thesis focuses specifically on the flavour problem and its solutions. This involves providing a natural explanation for the theoretical origin of the entire flavour structure of the SM, which includes the masses and mixing patterns of quarks and leptons, including neutrinos.