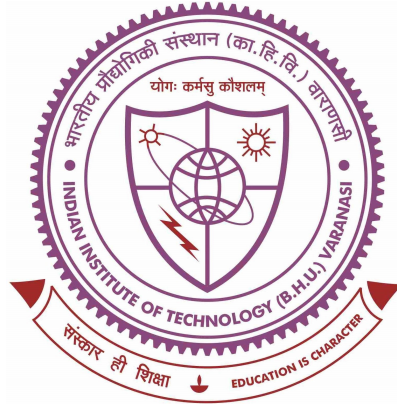


**A STUDY OF ONE-DIMENSIONAL MOVING  
BOUNDARY PROBLEMS INVOLVING PHASE  
CHANGE AND DIFFUSION LOGISTIC POPULATION  
MODELS**



*The thesis submitted in partial fulfillment  
for the award of degree  
DOCTOR OF PHILOSOPHY*

*by*

**ABHISHEK KUMAR**

**DEPARTMENT OF MATHEMATICAL SCIENCES  
INDIAN INSTITUTE OF TECHNOLOGY  
(BANARAS HINDU UNIVERSITY)  
VARANASI-221005**

**Roll No.: 17121024**

**Year of submission : 2023**



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23/05/23

**Dr. Rajeev**  
**Supervisor**  
**Associate Professor**  
**Department of Mathematical Sciences**  
**Indian Institute of Technology**  
**(Banaras Hindu University)**  
**Varanasi-221005**



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*Abhishek*  
23.05.2023

**Abhishek Kumar**

**(Roll No. 17121024)**

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*Dr. Rajeev*  
23-05-2023

**Dr. Rajeev**

**Supervisor**

**Associate Professor**

**Department of Mathematical Sciences**

**Indian Institute of Technology**

**(Banaras Hindu University)**

**Varanasi-221005**

*Subank*  
23.05.2023

**Signature of Head of the Department**

विभाग प्रमुख  
गणतीय विज्ञान विभाग  
Department of Mathematical Sciences  
भारतीय प्रौद्योगिकी संस्थान  
Indian Institute of Technology  
(काशी हिन्दू विश्वविद्यालय)  
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## ACKNOWLEDGEMENTS

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My Ph.D. journey was full of many unforgettable ups and downs accompanied with hardship, encouragement and trust. There are so many people who helped me a lot along this way. It is not possible to say thanks in few words. Still I would like to take this opportunity to express my deepest gratitude for their support, love and affection.

I owe a great debt of thanks to my respected supervisor, ***Dr. Rajeev, Associate Professor, Department of Mathematical Sciences, IIT(BHU), Varanasi***. It has been a great pleasure to work under his mentorship. He listened my queries, with utmost patience and curiosity, and then offered his perceptive that gave a new direction to my research. I am very grateful to him for introducing me a fascinating branch of mathematics. I owe the completion of this thesis to his great wisdom, careful guidance, continuous encouragement, and unending patience over the past years.

Besides my supervisor, I would like to express my deep sense of gratitude to the rest of my RPEC (Research Progress Evaluation Committee) members ***Prof. Ram Sharan Singh*** and ***Dr. Vineet Kumar Singh***, for their precious time, valuable suggestions and encouragement. I owe my special thanks to ***Prof. S. K. Pandey, Head of Department, Prof. T. Som, Prof. L.P. Singh, Prof. S. Das, and Dr. Ashok Gupta, Convener, DPGC of Department*** for their support throughout my research work. I also express my deep sense of gratitude to all faculty members of the department for their assistance and support. I express my thanks to all non teaching staff members of the department for their cooperation and help throughout my research tenure.

I gratefully acknowledge **University Grants Commission, India** for providing the fellowship in form of Junior Research Fellowship and Senior Research fellowship.

I am also grateful to my Institute, IIT(BHU), for providing necessary resources throughout my Ph.D. program.

This acknowledgment would be incomplete if the name of great visionary ***Pt. Madan Mohan Malaviya*** is not mentioned, who made this divine centre of knowledge. Deepest regards to him.

Above all, praises and thanks to the God, the almighty, for his showers of blessings throughout my research work, who has made everything possible.

**Varanasi-221005**

**Abhishek Kumar**

*Dedicated*  
*to*  
*My Parents*



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# Abstract

This thesis is concerned with the construction and solution of various models of moving boundary problems (MBPs) by different approaches. The main feature of these problems is that they consist variable thermal coefficients or diffusion coefficients (Space-dependent, temperature-dependent) and different types of boundary conditions, due to which analytical solutions of these problems have not been derived yet. Therefore, different approximate and numerical techniques are applied on these problems to derive the almost accurate solutions.

This thesis contains five chapters. In chapter 1, introduction of MBPs, literature survey of MBPs, some definitions and basic concept of some proposed techniques are included.

In chapter 2, a moving boundary problem with moving phase change material and time dependent thermal conductivity is considered in a moving domain. The boundary fixation of this moving boundary has been done by using Landau type transformation. After that, the finite difference discretizations have been done for space and time derivatives, and then explicit scheme has been used to derive the solution. The error analysis, stability analysis and convergence of the have also been established for the considered method. The dependence of temperature distribution in the domain on various parameters is also analyzed.

Next, a space fractional population logistic diffusion model with density dependent dispersal rate has been considered in chapter 3, and explicit scheme is applied on it to get the solution of the problem. The proposed scheme for the above mentioned problem is consistent. It is proved that the scheme is conditionally stable and convergence for the proposed method. The analysis of various parameters of the model on the population growth is also presented through the different figures.

In chapter 4, a time-fractional Stefan problem in a semi-infinite domain with a Robin boundary condition at the first fixed boundary and variable thermal conductivity is presented. The classical approach of the heat balance integral method with quadratic and exponential temperature profiles is applied to the problem to find an approximate solution. To test the validation of the proposed approach, our solution is compared with the analytical solutions for integer-order derivative and fractional order cases when thermal conductivity is a constant. In this study, the effects of variable thermal conductivity and Biot number ( $Bi$ ) on the temperature distribution and moving interface are also analyzed for the fractional-order system.

In chapter 5, an Operational Matrix Method based on Genocchi polynomials is used to determine the solution of a non-classical Stefan problem with space dependent thermal conductivity, variable latent heat and Robin boundary condition. The proposed method is easy to apply and the results obtained from this method are sufficiently accurate.

In chapter 6, the conclusion of the overall work and the scope for the future work related to this thesis is discussed.