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Tuning the controller gains using sensitivity analysis

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Abstract

This paper tries to solve the general problem of controller gain tuning. To that end, sensitivity analysis has been utilized. The study provides vital results to extend the usage of sensitivity analysis for the class of finite-time stable dynamics. The controller gain tuning procedure is elaborated by considering the examples of state feedback control and predefined time control. An insight into the selection of the gains or system parameters for these class of systems is presented with the help of sensitivity analysis. The sensitivity of the solution for these class of systems with the variation in the gains or system parameters is studied. Having know-how about the adjustment of these system gains or parameters will be beneficial from the point of view of practical applications. Thus, the proposed approach finds utility in controller gain tuning, which is an important aspect of control design.

KEYWORDS

fixed time convergence, nonlinear control, predefined time control, sensitivity analysis

1 | INTRODUCTION

Finite-time convergence and stabilization problems have been an important area of research [1–3]. The requirement of finite-time convergence emerges in many applications. One such application is observer design, where the objective is that the observed states track the real ones in finite time [4]. A revamped interest has developed in fractional calculus since it has successfully characterized the dynamics of some physical systems [3]. It also finds applications in signal processing, electrochemistry, fluid mechanics, mathematical biology, and many others. Finite-time stabilization of a fractional-order chain of integrators is shown in [3]. A finite-time cooperative control for multiagent systems is designed in [5]. Finite-time stabilization of nonlinear polytopic systems is studied in [6]. A comprehensive study on finite time stabilization can be found in [7, 8]. Sliding mode control algorithms exhibit finite time convergence [9, 10]. Sliding mode control is one of the preferred technique due to its inherent robustness to disturbances [11, 12]. A higher-order sliding mode has been utilized for achieving constrained stabilization in finite time in [13]. Gradually, it was realized that the problem with finite-time convergence was that the convergence time would often grow unbounded. This led to the development of a new class of systems termed as Fixed-time systems [14]. Fixed-time convergence can be viewed as a subclass of finite-time convergence, where the settling time (convergence time) is upper bounded, irrespective of the initial conditions [14]. It can be observed that sliding mode control has also been effectively utilized to attain fixed-time convergence [15, 16].

In recent years, the notion of prescribed-time convergence has gained popularity. The prescribed-time control approach in the context of stochastic nonlinear systems can be found in [17–20]. In [17], the reference trajectories can be tracked from below for stochastic strict-feedback nonlinear systems tracking problems. In [18], the prescribed-time mean-square stabilization and inverse optimality control problems are solved for stochastic strict-feedback nonlinear systems by developing a novel nonscaling backstepping design scheme. Whereas in [19], the prescribed-time output-feedback control of stochastic nonlinear systems is proposed. Additionally, in [20], a novel prescribed-time mean-nonovershooting design is proposed for stochastic nonlinear system with finite-time vanishing noise. Different from the existing results, this approach can tackle multiplicative and additive noise simultaneously.

Alternatively, the notion of predefined convergence time [21–27] has also emerged, which address the ease in stating the settling time-bound a priori. Another dimension of the fixed time convergence is captured by the predefined time convergence. Such a controller design for stabilization in predefined time has been proposed in [28]. Applications of predefined time control are explored in [29–35]. It has been shown that the special feature of this control strategy [28] is that the stabilization can be attained within any prespecified arbitrary time without worrying too much about the system parameters and initial conditions. It is, however, to be noted that the control involves a system parameter η, which can be tuned to alter the rate of convergence of the solutions. For practical applications, the study of sensitivity with respect to this system parameter is crucial since it would lead to its prudent selection. The sensitivity analysis [36] is well-developed for nonlinear smooth dynamics. However, the study of sensitivity analysis for systems showing finite-time convergence is still lacking. In fact, the idea of making sensitivity analysis as a tool to tune the again of the controller is very fresh and the same is pursued here. Sensitivity analysis for hybrid systems with application to trajectory tracking is studied in [37]. The authors proposed a technique to create the approximation of nearby perturbed state trajectories for a given system trajectory with state jump. It should be noted that sensitivity analysis has been instrumental in achieving the proposed results. Sensitivity analysis of uncertain dynamic systems using set-valued integration is discussed in [38]. The proposed method here relies on the sensitivity analysis capabilities of the state-of-the-art solvers. Sensitivity analysis of nonlinear fractional-order control systems with state delay is elaborated in [39]. In this study, the authors have shown the sensitivity of the state and control with regard to the parameters of the system. The rigorous study involving the usage of sensitivity analysis in control problems for stochastically forced systems can be found in [40–42]. In [40], a feedback regulator is constructed to stabilize the equilibrium and synthesize the dispersion of the random states around this equilibrium for the considered nonlinear dynamical stochastic system. Considering the stabilization problem in a nonlinear discrete-time system with incomplete information, the method of stochastic sensitivity synthesis is presented in [41]. Further stabilization problem of stochastically forced equilibria in nonlinear dynamic systems with incomplete information is addressed in [42].

Thus, we note that although some nice use cases of sensitivity-based analysis exist in the literature, there is hardly any study that utilizes sensitivity to tune the controller gains. Further, to the author's best knowledge, sensitivity-based analysis has not been exploited so far to tune the parameters in the case of finite/fixed-time convergent systems. To address these issues, we propose a sensitivity-based analysis for tuning the controller gains for state feedback control as well as predefined-time control. Moreover, this idea can be utilized for other systems as well. The key concept is to carry out the sensitivity analysis with respect to controller gains of a general control system. This would reveal some helpful information regarding the tuning of gains.

It is worth noting that the idea of parameterizing the gain has been utilized in this study. Some experiments may demand gains to be set at the start of the experiment. Some other experiments or real-time applications may allow the gain to be adjusted throughout the experiment. Thus, this justifies the idea of parameterizing the gain. Once gains are parameterized, it is useful to invoke sensitivity analysis to study the effect of variations in the gains. The sensitivity study is recommended to be carried out during simulation, that is, before the real-time experiment. Once know-how about the effect of gains has been established using sensitivity analysis, then one can choose some suitable gains. These gains can then be applied to the real-time experiment.

1.1 | Main contributions

The main contribution of the paper is that it finds a way of knowing some idea about tuning the parameters (or gains) based on sensitivity analysis. Such an analysis is presented here for the state feedback controller and predefined time controller. This strategy could be imitated for some other class as well. Thus, the generic problem of controller gain tuning, which is quite ubiquitous, can be relaxed to an extent based on this study. Another important feature of the given analysis is that it also considers the case of a class of finite-time stable dynamics, which differentiates it from the one given in [36]. Thus, in this context, this paper extends the purview of sensitivity analysis. The main contributions can be summarized as follows:

1. A novel controller gain tuning formalism based on sensitivity analysis is proposed.
2. Controller gain tuning is proposed for the state feedback control system and predefined-time control system (a subclass of finite-time control).
3. The methodology is generic and can be easily extended to any other class of systems as well.

1.2 | Structure

The rest of the paper is structured as follows: Section 2 presents definitions and preliminaries, Section 3 presents main results in the form of two key theorems, followed by the sensitivity function and sensitivity equation for the concerned class of system, Section 4 presents sensitivity analysis for the state feedback control and predefined time control, Section 5 presents the procedure to tune by considering the examples of a simple pendulum and a single inverted pendulum (SIP), and finally, Section 8 concludes the paper.

2 | PRELIMINARIES AND DEFINITIONS

Let us consider nonautonomous nonlinear system given by

$$\dot{x} = f(t, x, \lambda),\ x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ are the system states, $f : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a nonlinear function such that $f(t, 0, \lambda) = 0$, that is, origin $x = 0$ is an equilibrium point of (1), $\lambda \in \mathbb{R}^p$ are the tunable parameters of the system, t_0 is the initial time, and x_0 are the initial states.

Definition 1 (Global finite-time stable [14]). The origin of system (1) is globally finite-time stable if in addition to global asymptotic stability, any solution $x(t, t_0, x_0)$ of (1) converges to the origin at some finite time, that is, $x(t, t_0, x_0) = 0 \forall t \geq t_0 + \tau_s(t_0, x_0)$, where $\tau_s : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the settling time function.

Definition 2 (Fixed-time stable [14]). The origin of system (1) is fixed-time stable if it is globally finite-time stable and its settling time function is upper bounded; that is, there exists some time moment $\tau_{\max} > 0$: such that $\tau_s(t_0, x_0) \leq \tau_{\max} \forall x_0 \in \mathbb{R}^n$ and $\forall t_0 \in \mathbb{R}_+ \cup \{0\}$.

Definition 3 (Predefined time stable [28]). The origin of system (1) is called predefined time stable if

1. it is fixed-time stable,
2. $\exists \tau_a > 0$, which depends on some tunable system parameters λ, and can be calculated in advance for some given values λ,
3. it is possible to assign different values to τ_a by making possible and allowed changes in system parameters λ as a part of design process.
4. for some chosen λ, either of the following is true
   (a) $\tau_a \geq \tau_{tf}$ (weak predefined time stable)
   (b) $\tau_a = \tau_{tf}$ (strong predefined time stable)

   where τ_{tf} is the true fixed-time.

For a fixed-time stable system, if it is possible to predefine $\tau_{\max}$ arbitrarily for some values of the system parameters, then the origin of system (1) is termed as predefined time stable and the convergence to the origin for such a case is termed as predefined time convergence [28]. Such notions resulted in designing of controllers with arbitrary convergence time [28].

Remark 1. Please note that strong predefined-time convergence (condition 4(b) of Definition 3) holds similarity to the prescribed-time convergence in the sense that the chosen convergence time is equal to the true fixed-time of convergence. However, it should be noted that the prescribed-time convergence [43] was originally defined for systems that vanish at chosen time t_f, whereas strong predefined-time convergence (condition 4(b) of Definition 3) holds for the entire time horizon.

The key theorem which characterizes Lyapunov stability in [28] is reiterated in brief here for the purpose of clarity in understanding.

Theorem 1 ([28]). *For the given system (1), let us assume that set $\mathcal{D} \subset \mathbb{R}^n$ contains the equilibrium point, that is, origin. Consider continuous positive definite functions $\xi_1(x)$ and $\xi_2(x)$ on $\mathcal{D}$. Suppose there exists a real valued continuously differentiable function $V : \tau \times \mathcal{D} \to \mathbb{R}_+ \cup \{0\}$, where τ is some finite time interval. Assume that the following conditions hold:*

$$(i)\ \xi_1(x) \le V \le \xi_2(x), \forall t \in \tau \text{ and } \forall x \in D \backslash \{0\} \tag{2a}$$

$$(ii)\ V(t, 0) = 0 \forall t \in \tau \tag{2b}$$

$$(iii)\ \dot{V} \le \frac{-\eta(1 - e^{-V})}{(t_f - t)};\ \forall V \ne 0 \tag{2c}$$

then for all $\eta > 1$, the origin is predefined time stable with settling time bound given by t_f.

Proof. For the concerned system, it is not difficult to realize the uniform stability of the origin and the boundedness of the solutions [28]. Let us establish the predefined time stability. Let us assume that the given $V(t, x)$ satisfies the relation $\dot{V} \le \frac{-\eta(1-e^{-V})}{(t_f - t)}$; $V(t_0) = V(0)$. Let ρ be the solution of the differential equation

$$\dot{\rho} = \frac{-\eta(1 - e^{-\rho})}{(t_f - t)};\ \rho \in \mathbb{R} \tag{3}$$

where the real constant, $\eta > 1$. The solution to (3) is

$$\rho = \ln(k(t_f - t)^\eta + 1) \tag{4}$$

where integration constant $k = \frac{e^{\rho(t_0)} - 1}{(t_f - t_0)^\eta}$. From (4), we have

$$\dot{\rho} = \frac{-\eta k(t_f - t)^{\eta-1}}{k(t_f - t)^\eta + 1} \tag{5}$$

Now from (5), when $t = t_f$, then $\dot{\rho} = 0$, and from (4), when $t = t_f$, then $\rho = 0$. Therefore, $\forall t \ge t_f$, $\rho = 0$ is maintained. Similar kind of analysis can be done for $V(t)$ by applying comparison principle [36]. Here, it can be seen that t_f can be predefined arbitrarily for some $\eta > 1$. Thus, it can be inferred that system (1) is predefined time stable with prespecified settling time bound given by t_f. This completes the proof. □

Remark 2. Although τ_a is not an explicit function of system parameters, still it is very much evident that the arbitrary selection of τ_a is not allowed for all values of system parameters (controller gains). The restriction is imposed due to the stability conditions (see Theorem 1 above). For the n^{th}-order predefined time stable system, the condition on the selection of controller gains η_is is $\eta_i > n$, $i = \{1, 2, \cdots, n\}$ (see [29]).

3 | MAIN RESULTS

Two theorems, which are actually relaxed versions of (Theorem 3.4, 3.5, pp. 96-97 [36]), are introduced here followed by sensitivity function and sensitivity equation in context to these theorems.

Theorem 2. *Let $f(t, x)$ be piecewise continuous in t and x on $[t_0, t_1] \times W$, where $W \subset \mathbb{R}^n$ is an open connected set. Let $x(t)$ and $y(t)$ be solutions of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ and $\dot{y} = f(t, y) + g(t, y)$, $y(t_0) = y_0$ respectively, such that $x(t), y(t) \in W\ \forall t \in [t_0, t_1]$. Suppose that for all $(t, x) \in [t_0, t_1] \times W$, the following conditions hold:*

$$(i)\ \| f(t, x(t)) - f(t, y(t))\| \le \Gamma$$
$$(ii)\ \|g(t, x)\| \le \varsigma$$

where Γ and ς are some positive constants. Then

$$\|y(t) - x(t)\| \le \|y_0 - x_0\| + (\Gamma + \varsigma)(t - t_0)\ \forall t \in [t_0, t_1].$$

Proof. The solutions of $x(t)$ and $y(t)$ are given by

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

$$y(t) = y_0 + \int_{t_0}^{t} [f(s, y(s)) + g(s, y(s))]ds$$

subtracting $x(t)$ from $y(t)$ and then taking norm results in

$$\|y(t) - x(t)\| \le \|y_0 - x_0\| + \int_{t_0}^{t} \|g(s, y(s))\|ds + \int_{t_0}^{t} \|f(s, y(s)) - f(s, x(s))\|ds \le \|y_0 - x_0\| + (\varsigma + \Gamma)(t - t_0)$$

which completes the proof of the theorem. □

This theorem speaks about the closeness of the solutions. The next theorem examines the confinement in a tube, for the two solutions which are close to each other.

Theorem 3. *Let $f(t, x, \varphi)$ be continuous in t, x, φ on $[t_0, t_1] \times D \times \{\|\varphi - \varphi_0\| \leq c\}$, where $D \subset \mathbb{R}^n$ is an open connected set. Let $x(t, \varphi_0)$ be a solution of $\dot{x} = f(t, x, \varphi_0)$, with $x(t_0, \varphi_0) = x_0 \in D$. Suppose $x(t, \varphi_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$. Suppose $y(t, \varphi)$ is a unique solution of $\dot{x} = f(t, x, \varphi)$ defined on $[t_0, t_1]$, with $y(t_0, \varphi) = y_0$. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|y_0 - x_0\| < \delta$ and $\|\varphi - \varphi_0\| < \delta$ then $\|y(t, \varphi) - x(t, \varphi_0)\| < \epsilon$, $\forall t \in [t_0, t_1]$.*

Proof. Since $x(t, \varphi_0)$ is continuous in t, on a compact set $[t_0, t_1]$ therefore, $x(t, \varphi_0)$ is bounded on $[t_0, t_1]$. Let us define a tube U around the solution $x(t, \varphi_0)$ by

$$U = \{(t, x) \in [t_0, t_1] \times \mathbb{R}^n \mid \|x - x(t, \varphi_0)\| \leq \epsilon\}$$

Assume ϵ small enough to ensure that $U \subset [t_0, t_1] \times D$. The set U is compact and f is continuous in x on U; therefore, f remains bounded on U by a bound, say, Γ i.e. $\|f(t, x, \varphi) - f(t, y, \varphi)\| \leq \Gamma$. Since f is continuous in φ, therefore, $\forall \alpha > 0, \exists \beta > 0$ (with $\beta < c$) such that $\|f(t, x, \varphi) - f(t, x, \varphi_0)\| < \alpha$, for all $(t, x) \in U$ and for all $\|\varphi - \varphi_0\| < \beta$. Let us choose $\alpha < \epsilon$ and $\|y_0 - x_0\| < \alpha$. It is known that $y(t, \varphi)$ is a unique solution $\dot{x} = f(t, x, \varphi)$ defined on $[t_0, t_1]$ with $y(t_0, \varphi) = y_0$. It is to be established that $y(t, \varphi)$ remains confined to the tube U for all $t \in [t_0, t_1]$. Let us assume that t_{lu} is the time instant at which the solution leaves tube U for the first time. We will investigate the behavior of the solution for all $t \in [t_0, t_{lu}]$. Now we only need to show that the solution would leave the tube U only when $t_{lu} > t_1$. It can be observed that with $\varsigma = \alpha$, the conditions of Theorem 2 are satisfied for all $t \in [t_0, t_{lu}]$. Hence, we have $\|y(t, \varphi) - x(t, \varphi_0)\| \leq \|y_0 - x_0\| + (\alpha + \Gamma)(t - t_0)$. Now since we have assumed that $\|y_0 - x_0\| < \alpha$, therefore, we get $\|y(t, \varphi) - x(t, \varphi_0)\| < \alpha + (\alpha + \Gamma)(t - t_0)$. By selecting $\alpha \leq \left[\frac{\epsilon}{(1+t_1-t_0)} - \Gamma\right]$, we have

$$\begin{aligned} \|y(t, \varphi) - x(t, \varphi_0)\| &< \left[\frac{\epsilon}{(1 + t_1 - t_0)} - \Gamma\right] \\ &\quad + \left[\frac{\epsilon(t - t_0)}{(1 + t_1 - t_0)}\right] \\ &< \left[\frac{\epsilon(1 + t - t_0)}{(1 + t_1 - t_0)}\right] \end{aligned}$$

Now, since $\frac{(1+t-t_0)}{(1+t_1-t_0)} < 1$ for any $t < t_1$, therefore, $\|y(t, \varphi) - x(t, \varphi_0)\| < \epsilon$ for any $t < t_1$. Thus, if t_{lu} is the time instant at which the solution $y(t, \varphi)$ leaves the tube U for the first time, then it must be necessarily greater than t_1. Since, it is already assumed that $\|\varphi - \varphi_0\| < \beta$ and $\|y_0 - x_0\| < \alpha$; therefore, take $\delta = min\{\alpha, \beta\}$. This completes the proof of the theorem. □

Remark 3. For system (1) to represent some useful physical system, it turns out that the solution should continuously depend on the initial state (x_0), initial time (t_0) and parameters (λ) of the system. Continuous dependence on initial time t_0 is straightforward and can be seen by observing the expression of the solution $x(t)$. Continuous dependence on initial state (x_0) can be judged by observing the solutions starting close to the given initial state x_0. If these solutions remain close to the original solution obtained by considering the initial state x_0, then it is evident that the solution continuously depends on the initial state. Observe that Theorem 2 speaks about continuous dependence on the initial state. On the other hand, Theorem 3 formulates the conditions for continuous dependence on the parameters.

3.1 | Sensitivity function and Sensitivity equation

Consider the system $\dot{x} = f(t, x, \varphi)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a nonlinear vector field continuous in t, x and φ on $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p$. Suppose that the first partial derivatives of f with respect to x and φ are continuous on $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p$. Let φ_0 be the nominal value of system parameter φ. Let $x(t, \varphi_0)$ be the unique solution of the nominal state equation $\dot{x} = f(t, x, \varphi_0)$, $x(t_0, \varphi_o) = x_0$. Suppose the state equation $\dot{x} = f(t, x, \varphi)$, $x(t_0, \varphi) = x_0$; has a unique solution $x(t, \varphi)$ for $\|x(t_0, \varphi) - x(t_0, \varphi_o)\|$ and $\|\varphi - \varphi_0\|$ sufficiently small. Then by Theorem 3, the solutions $x(t, \varphi_0)$ and $x(t, \varphi)$ remain close to each other and also remain bounded in a tube U for all $t \in [t_0, t_1]$. The solution of the state equation $\dot{x} = f(t, x, \varphi)$ is given by

$$x(t, \varphi) = x_0 + \int_{t_0}^{t} f(s, x, \varphi) ds$$

Taking partial derivatives with respect to φ yields

$$x_\varphi(t, \varphi) = \int_{t_0}^{t} \left[\frac{\partial f(s, x(s, \varphi), \varphi)}{\partial x} x_\varphi(s, \varphi) + \frac{\partial f(s, x(s, \varphi)}{\partial \varphi}\right] ds \tag{6}$$

where $x_\varphi(t, \varphi) = \frac{\partial x(t,\varphi)}{\partial \varphi}$ and $\frac{\partial x_0}{\partial \varphi} = 0$, since x_0 is independent of φ. It is to be noted that $x_\varphi(t, \varphi)$ exists only if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \varphi}$ are continuous in x and φ, respectively. Taking time derivative of (6) gives

$$\frac{\partial x_\varphi(t,\varphi)}{\partial t} = A(t,\varphi)x_\varphi(t,\varphi) + B(t,\varphi),\ x_\varphi(t_0,\varphi) = 0$$

where $A(t,\varphi) = \frac{\partial f(t,x(t,\varphi),\varphi)}{\partial x}$ and $B(t,\varphi) = \frac{\partial f(t,x(t,\varphi),\varphi)}{\partial \varphi}$. For $\|\varphi - \varphi_0\|$ sufficiently small, the matrices A and B are defined on $[t_0, t_1]$. Therefore, $x_\varphi(t,\varphi)$ is also defined on $[t_0, t_1]$. Let $S(t) = x_\varphi(t,\varphi_0)$, then $S(t)$ is the unique solution of

$$\dot{S}(t) = A(t,\varphi_0)S(t) + B(t,\varphi_0),\ S(t_0) = 0 \tag{7}$$

The function $S(t)$ is called sensitivity function, and Equation (7) is called sensitivity equation. Sensitivity functions allow us to know the effect of parameter variations on solutions. $S(t)$ can be calculated by solving the nominal state equation and sensitivity equation simultaneously [36]. This can be done by appending the variational Equation (7) with the original state equation, then setting $\varphi = \varphi_0$ to obtain the $(n + np)$ augmented equation

$$\dot{x} = f(t,x,\varphi_0),\ x(t_0,\varphi_0) = x_0$$
$$\dot{S} = \left.\frac{\partial f(t,x,\varphi)}{\partial x}\right|_{\varphi=\varphi_0} S + \left.\frac{\partial f(t,x,\varphi)}{\partial \varphi}\right|_{\varphi=\varphi_0},\ S(t_0) = 0$$

which is solved numerically. In the later portions of the paper, we will study about the first and second order predefined time stable systems (which belong to finite-time convergent systems) for sensitivity analysis of the solutions with respect to the system parameter η. Note that for this class of systems, Theorems 2 and 3 are applicable, and hence, sensitivity analysis is possible for sufficiently small $\|\varphi - \varphi_0\|$.

Remark 4. Note that sensitivity functions are crucial to study the effect of the parameter variations on the solutions. In fact, they can also be used to approximate the solution under the condition that the parameters (λ) are close to their nominal values (λ_0).

4 | SENSITIVITY ANALYSIS AND SIMULATION RESULTS

In this section, sensitivity analysis is performed for both the class of functions: the system with asymptotic dynamics and the system with finite-time dynamics. The system with asymptotic dynamics adheres to the requirements of the theorems given in (Theorem 3.4,3.5,pp.96-97 [36]). The case of state feedback control is taken as an example of this class of system. For the other class of system with finite-time dynamics, the example of predefined time convergent dynamics is considered. The simulation results are also presented.

4.1 | State feedback control

Let us consider the second order system in the normal form:

$$\dot{x}_1 = x_2;\ \dot{x}_2 = u \tag{8}$$

where x_1 and x_2 are states and $u \in \mathbb{R}$ is the control. Let us assume a state feedback control law $u = -k_1x_1 - k_2x_2$, where k_1 and k_2 are constants. The idea here is to parameterize these constants to study the sensitivity of states with respect to these constants. Let us assume nominal values $k_{10} = k_{20} = 1$. Then the Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial k]$ can be evaluated as follows: $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$ and $\frac{\partial f}{\partial k} = \begin{bmatrix} 0 & 0 \\ -x_1 & -x_2 \end{bmatrix}$. At the nominal values, we have $\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $\left.\frac{\partial f}{\partial k}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 0 \\ -x_1 & -x_2 \end{bmatrix}$. Now the sensitivity function is given by

$$S = \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial k_1} & \frac{\partial x_1}{\partial k_2} \\ \frac{\partial x_2}{\partial k_1} & \frac{\partial x_2}{\partial k_2} \end{bmatrix}\right|_{\text{nominal}} \tag{9}$$

where x_3, x_4, x_5, and x_6 are the new introduced states that represent the sensitivity of the original systems states x_1 and x_2 with respect to the gains k_1 and k_2, at the nominal values of the gains. The sensitivity equation is $\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial k}\right|_{\text{nominal}}$ with initial condition $S(t_0) = 0$. To see that $S(t_0) = 0$, let us consider $x_3 = \left.\frac{\partial x_1}{\partial k_1}\right|_{\text{nominal}}$, now $x_3(0) = \left.\frac{\partial x_1(0)}{\partial k_1}\right|_{\text{nominal}} = 0$ (since $x_1(0)$ is constant). With the same reasoning, we obtain $x_4(0) = x_5(0) = x_6(0) = 0$. Moreover, observe that $\dot{x}_3 = \left.\frac{\partial \dot{x}_1}{\partial k_1}\right|_{\text{nominal}} = \left.\frac{\partial x_2}{\partial k_1}\right|_{\text{nominal}} = x_4$. Similarly, one can obtain the expressions for $\dot{x}_4$, $\dot{x}_5$, and $\dot{x}_6$. Appending these variational equations to the original state Equation (8), one obtains the following:

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(0) &= x_{10} \\
\dot{x}_2 &= -k_1x_1 - k_2x_2, & x_2(0) &= x_{20} \\
\dot{x}_3 &= x_4, & x_3(0) &= 0 \\
\dot{x}_4 &= -x_1 - x_3 - x_4, & x_4(0) &= 0 \\
\dot{x}_5 &= x_6, & x_5(0) &= 0 \\
\dot{x}_6 &= -x_2 - x_5 - x_6, & x_4(0) &= 0
\end{aligned} \tag{10}$$

Note that x_3 and x_5 are sensitivities of state x_1 with respect to k_1 and k_2, respectively. Similarly, x_4 and x_6 are sensitivities of state x_2 with respect to k_1 and k_2, respectively. This Equation 10 needs to be solved simultaneously to study the effect of parameterized gains on the states. For the simulation, we have assumed initial states as $x_{10} = 3$ and $x_{20} = 3$. Other required values are as discussed above. From the simulation results in Figure 1, it is clearly evident that both the states x_1 and x_2 are more sensitive to the variations in k_1 as compared to k_2. These observations will be utilized in tuning the gains for a mechanical system to be discussed later.

4.2 | Predefined time control

Now let us consider the example of finite-time dynamics. To that end, we consider the predefined time stable systems [28]. Note that the predefined time dynamics allows the user to prespecify the time of convergence of the states to the equilibrium. The examples of first-order and second-order predefined time stable systems are considered to elaborate the related sensitivity analysis.

4.2.1 | First-order system

Let us consider the scalar function

$$\dot{x} = u,\ x(t_0) = x_0 \tag{11}$$

For the predefined time convergence [28], the control u is defined as

$$u = \begin{cases} \frac{-\eta(1-e^{-x})}{(t_f-t)}, & \text{if } \ t_0 \le t < t_f \\ 0, & \text{otherwise} \end{cases}$$

Observe that the control u is zero for $t \ge 0$, also $x = 0$ for $t \ge 0$, then sensitivity function is zero for $t \ge 0$ (as is evident from the definition of sensitivity function (10)). Now, for the time interval $t_0 \le t < t_f$, the sensitivity function is given by

$$S = \left.\frac{\partial x}{\partial \eta}\right|_{\eta=\eta_0} = x_1 \text{ (say)}$$

where η_0 is the nominal value of the system parameter η. Sensitivity equation is given by

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\eta=\eta_0} S + \left.\frac{\partial f}{\partial \eta}\right|_{\eta=\eta_0} \tag{12}$$

For the given system, we have $\left.\frac{\partial f}{\partial x}\right|_{\eta=\eta_0} = \frac{-\eta_0 e^{-x}}{t_f-t}$ and $\left.\frac{\partial f}{\partial \eta}\right|_{\eta=\eta_0} = \frac{-(1-e^{-x})}{(t_f-t)}$. Therefore, using (11) and (12), one obtains the following set of equations

$$\dot{x} = \frac{-\eta_0(1-e^{-x})}{t_f - t};\ \dot{x}_1 = \frac{-\eta_0 e^{-x}}{(t_f-t)} x_1 - \frac{(1-e^{-x})}{(t_f-t)}$$

which needs to be solved simultaneously to perform the sensitivity analysis. For simulations, let us assume that $x(t_0) = 5, x_1(t_0) = 0, t_f = 6$s, and $\eta_0 = 2$. Figure 2 shows the simulation results. From the simulation results, it can be seen that the state x is less sensitive to parameter (gain η) at the beginning, and it becomes very sensitive to the parameter (gain η) as it approaches the convergence time, and then for time $t \ge 0$ as expected, it becomes insensitive to the parameter variations.

4.2.2 | Second-order system

Consider system (8) again, but this time with u being the predefined time control. The predefined time convergent system [28] can be designed with the control $u = -x_1 - \dot{\psi}_1 - \psi_2$, where $\eta_1 > 2$, $\eta_2 > 2$, $\psi_1 = \frac{\eta_1(1-e^{-x_1})}{(t_f-t)}$, $\psi_2 = \frac{\eta_2(1-e^{-z_2})}{(t_f-t)}$ with $z_2 = x_2 + \psi_1$ and $\dot{\psi}_1 = \frac{\eta_1[e^{x_1}-1+(t_f-t)x_2]}{e^{x_1}(t_f-t)^2}$.

Note that the following analysis is being done for $t_0 \le t < t_f$. For any time $t \ge t_f$, the control as well as sensitivity function disappears. Let the nominal values of the system parameters be η_{10} and η_{20}. Comparing with the standard form $\dot{x} = f(t, x, \varphi)$, we have $f_1 = x_2$ and $f_2 = u$. While computing $\frac{\partial f}{\partial x}$, one obtains $\frac{\partial f_1}{\partial x_1} = 0$, $\frac{\partial f_1}{\partial x_2} = 1$, $\frac{\partial f_2}{\partial x_1} = \phi_{f_2 x_1}$ and $\frac{\partial f_2}{\partial x_2} = \phi_{f_2 x_2}$, where

$$\phi_{f_2x_1} = -1 - \frac{\eta_{10}e^{-x_1}}{(t_f-t)^2}[1 - x_2(t_f - t)] - \frac{\eta_{10}\eta_{20}}{(t_f-t)^2} e^{-x_1} e^{-z_2}$$
$$\phi_{f_2x_2} = -\frac{\eta_{10}}{(t_f-t)} e^{-x_1} - \frac{\eta_{20}}{(t_f-t)} e^{-z_2}$$

Note that the above computations are done with nominal values of system parameter η. Therefore,

$$\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} = \begin{bmatrix} 0 & 1 \\ \phi_{f_2x_1} & \phi_{f_2x_2} \end{bmatrix}$$

To obtain the sensitivity equation, apart from $\left.\frac{\partial f}{\partial x}\right|_{\text{nominal}}$, one also needs to evaluate $\left.\frac{\partial f}{\partial \eta}\right|_{\text{nominal}}$, which is given by

$$\left.\frac{\partial f}{\partial \eta}\right|_{\text{nominal}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial \eta_1} & \frac{\partial f_1}{\partial \eta_2} \\ \frac{\partial f_2}{\partial \eta_1} & \frac{\partial f_2}{\partial \eta_2} \end{bmatrix}\right|_{\eta=\eta_0} = \begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix}$$

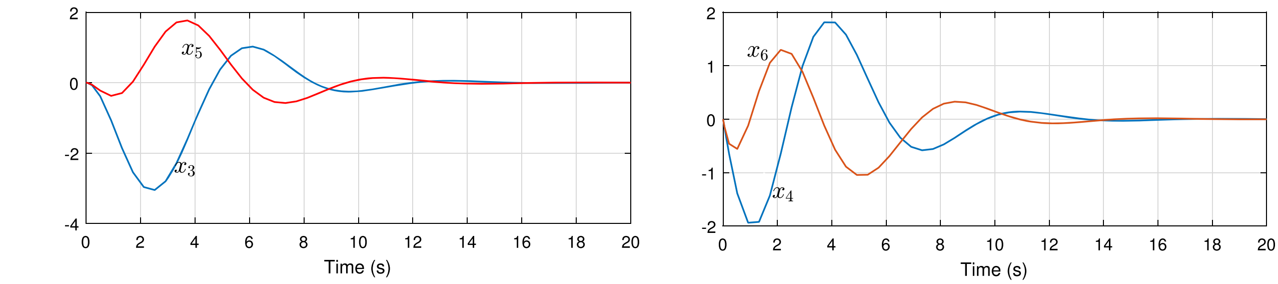
where $r_1 = -\frac{[1-e^{-x_1}+(t_f-t)x_2e^{-x_1}]}{((t_f-t)^2)}$ and $r_2 = -\frac{(1-e^{-z_2})}{(t_f-t)}$. Now the sensitivity function is given by

$$S = \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial \eta_1} & \frac{\partial x_1}{\partial \eta_2} \\ \frac{\partial x_2}{\partial \eta_1} & \frac{\partial x_2}{\partial \eta_2} \end{bmatrix}\right|_{\text{nominal}}$$

Now the sensitivity equation $\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \eta}\right|_{\text{nominal}}$ leads to

$$\begin{aligned} \begin{bmatrix} \dot{x}_3 & \dot{x}_5 \\ \dot{x}_4 & \dot{x}_6 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \phi_{f_2x_1} & \phi_{f_2x_2} \end{bmatrix} \begin{bmatrix} x_3 & x_5 \\ x_4 & x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r_1 & r_2 \end{bmatrix} \\ &= \begin{bmatrix} x_4 & x_6 \\ r_1 + x_3\phi_{f_2x_1} + x_4\phi_{f_2x_2} & r_2 + x_5\phi_{f_2x_1} + x_6\phi_{f_2x_2} \end{bmatrix} \end{aligned} \tag{13}$$

The augmented set of equations can be obtained by combining (8) and (13) with initial conditions $x_1(t_0) = x_{10}, x_2(t_0) = x_{20}, x_3(t_0) = x_4(t_0) = x_5(t_0) = x_6(t_0) = 0$ and needs to be solved simultaneously. For simulations, the assumed data are $x_1(t_0) = 4, x_2(t_0) = 2, t_f = 5$s, $\eta_{10} = 3$ and $\eta_{20} = 3$. The simulation results are shown in



(a) (b)

FIGURE 1 (a, b) The plot of sensitivity function S with time for second order system with state feedback control law.

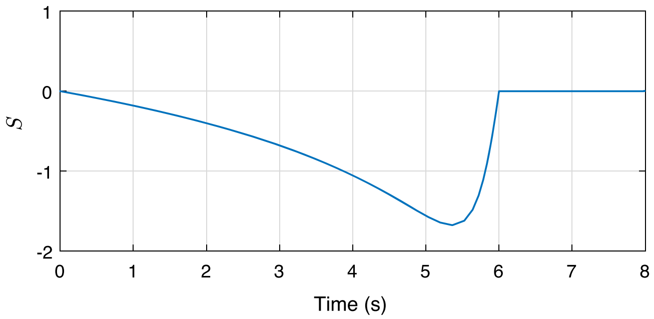


FIGURE 2 The plot of sensitivity function S with time for first order system.

Figure 3. x_3 and x_5 are sensitivities of x_1 with respect to η_1 and η_2, respectively. x_4 and x_6 are sensitivities of x_2 with respect to η_1 and η_2, respectively. It is evident that both the states x_1 and x_2 are more sensitive to variations in η_2 as compared to η_1.

5 | GUIDE TO TUNE: EXAMPLES

Here, we elaborate on the tuning procedure of the gains based on the proposed developments. For this, we consider two mechanical systems as examples. The first system is the simple pendulum being controlled by state feedback control law. This system exhibits Lipschitzian dynamics all through the range of operation. The other example is that of a SIP being controlled by a predefined time controller.

5.1 | Simple pendulum: Stabilization problem

Let us consider the simple pendulum [36] with the dynamical equation

$$\ddot{\phi} = -a\sin\phi - b\dot{\phi} + c\mathcal{T} \tag{14}$$

where ϕ denotes the angle subtended by the rod with the vertical axis, $a = g/l > 0, b = k/m \geq 0, c = 1/ml^2 > 0$, g is acceleration due to gravity, l is the length of the rod, m is the mass of bob, k is the coefficient of friction, and $\mathcal{T}$ is the control torque applied to the pendulum. It is assumed that the rod is rigid with zero mass. The objective here is to stabilize the pendulum at an angle $\phi = \delta$. To maintain equilibrium at $\phi = \delta$, the torque must have steady-state component given by $\mathcal{T}_{ss} = \frac{a}{c}\sin\delta$.

Let us assume state variables as $x_1 = \phi - \delta, x_2 = \dot{\phi}$ and the control variable as $u = \mathcal{T} - \mathcal{T}_{ss}$. Then the state equations are given by

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu \tag{15}$$

These state equations are in the standard form $\dot{x} = f(x,u)$, where $f(0,0) = 0$ and $f(x,u)$ is continuously differentiable in the domain $\mathcal{D}_x \times \mathcal{D}_u \subset \mathbb{R}^n \times \mathbb{R}^p$ that contains the origin $(x = 0, u = 0)$. The next step in the design is to linearize this system at the origin. To that end, we obtain a linear system $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix}; B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

It can be verified that the pair (A, B) is controllable. Taking $u = -Kx$, where $K = [k_1\ k_2]$, one obtains $A - BK = \begin{bmatrix} 0 & 1 \\ -a\cos\delta - ck_1 & -b - ck_2 \end{bmatrix}$. It is evident that the condition for the roots of the closed loop system to lie on the open left half complex plane can be obtained by applying the Routh table criterion to the polynomial on the left-hand side of the characteristic equation given by $|sI - (A - BK)| = 0$. As a result, we obtain

$$k_1 > -\frac{a}{c}\cos\delta \ \text{ and } \ k_2 > -\frac{b}{c} \tag{16}$$

The torque is given by $\mathcal{T} = \frac{a\sin\delta}{c} - k_1x_1 - k_2x_2$. For simulation, we assume $a = c = 10$, $\delta = \pi/4, b = 0$. The nominal values of k_1 and k_2 are the same as taken in Section 4.1, that is, $k_{10} = k_{20} = 1$. From the study in Section 4.1, it was seen that the system states (8) are more sensitive to variations in k_1 in comparison to k_2. Similar behavior is preserved

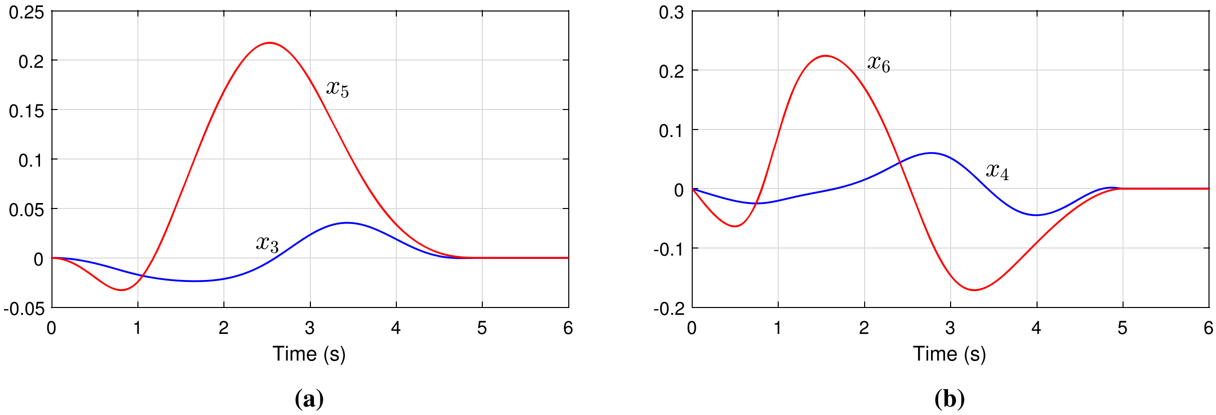


FIGURE 3 (a, b) The plot of sensitivity function S with time for second order system.

for the example of the simple pendulum. The objective of tuning is assumed to be faster convergence of the states. We will start with the nominal values of the gain and then progressively increase k_1 in steps of 0.1. Initial states must be chosen to lie in the region of attraction throughout the tuning process. We demonstrate the process of finding the region of attraction at the nominal values of gain. The control law $u = -Kx$ has been designed in view of the linearized model; however, it is actually applied to the original nonlinear system (15). Thus, the viability of the control u lies in the region of attraction. Nonlinear terms in (15) are assumed as perturbation terms. Thus, we have

$$\dot{x} = (A - BK)x + B_1 g(x_1)$$

where

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; g(x_1) = -a[\sin(x_1 + \delta) - \sin\delta] + a\cos(\delta)x_1$$

For the assumed constants, we have

$$g(x_1) = -10\sin(\pi/4 + x_1) + \frac{10}{\sqrt{2}} + \frac{10}{\sqrt{2}}x_1$$

It can be seen that $|g(x_1)| \leq 5x_1^2$. Assume a positive definite quadratic Lyapunov function $V = x^T Px$. Then the derivative of V along system trajectories is given by

$$\dot{V} = x^T[(A - BK)^T P + P(A - BK)]x + 2x^T PB_1 g(x_1)$$

Assume that $(A - BK)^T P + P(A - BK) = -Q$, where Q is an identity matrix. Then the solution of this equation yields positive definite symmetric $P = \begin{bmatrix} 1.1964 & 0.0293 \\ 0.0293 & 0.0529 \end{bmatrix}$. Then one obtains $PB_1 = \begin{bmatrix} 0.0293 \\ 0.0293 \end{bmatrix}$, which yields $\|PB_1\|_2 =$

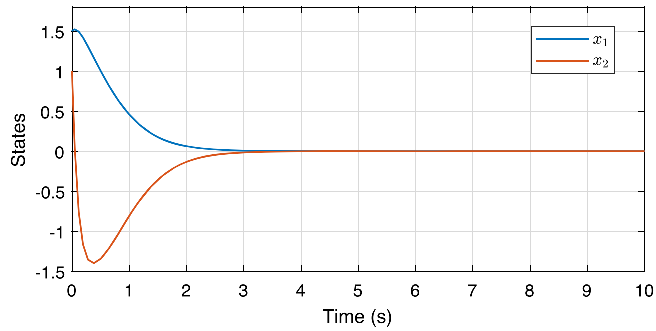


FIGURE 4 Evolution of states at the nominal values of gains.

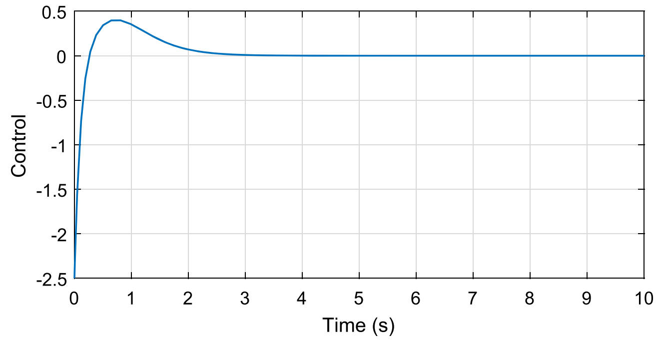


FIGURE 5 Evolution of control at the nominal values of gains.

0.0605. Then we obtain

$$\begin{aligned} \dot{V} &\leq -\|x\|_2^2 + 2\|PB_1\|_2\|x\|_2|g(x_1)| \\ &\leq -\|x\|_2^2 + 10\|PB_1\|_2\|x\|_2|x_1^2 \\ &\leq -(1 - 0.6050|x_1|)\|x\|_2^2 \end{aligned}$$

So, $\dot{V} < 0\,\forall |x_1| < 1.6529$. Thus, region of attraction is estimated by the set $\Omega_d = \{V(x) \leq d\} \subset \{|x_1| < 1.6529\}$, where d is some positive real number. It can be verified that

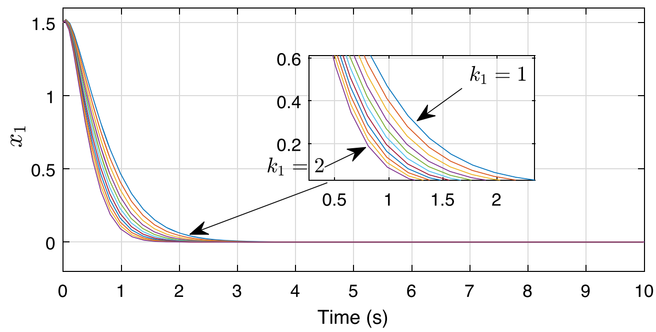


FIGURE 6 The swing angle for variations in gain k_1.

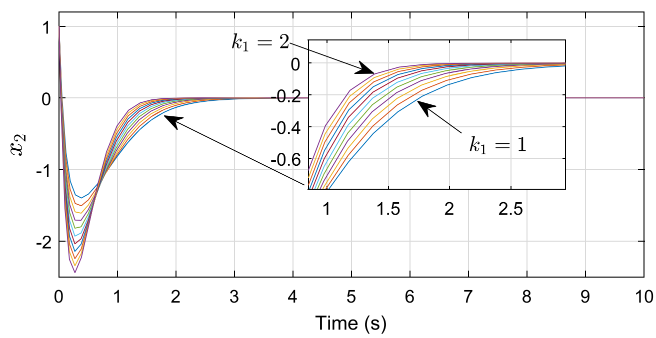


FIGURE 7 The swing velocity for variations in gain k_1.

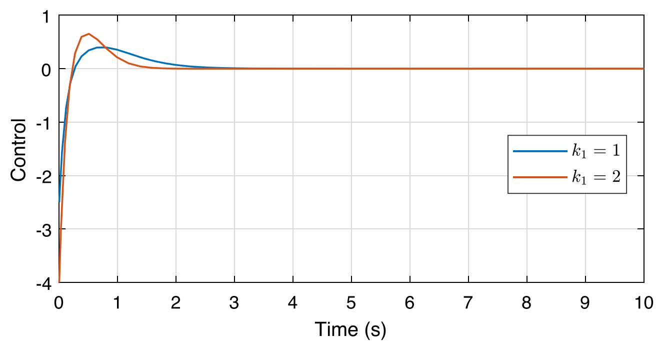


FIGURE 8 The control for $k_1 = 1$ and $k_1 = 2$ with fixed k_2.

with $L^T = [1\ 0]$, one obtains

$$d < \min_{|x_1|=1.6529} \{x^T P x\} = \frac{(1.6529)^2}{L^T P^{-1} L} = 3.2244$$

Similar analysis can be carried on to show that for gains $k_1 = 2, k_2 = 1$, required condition which satisfies region of attraction is $|x_1| < 1.8181$. For any value of gain $1 < k_1 < 2, k_2 = 1$, it is obvious that the required condition for satisfying region of attraction is $|x_1| < \epsilon$, where ϵ is some real number such that $1.6529 < \epsilon < 1.8181$. To comply with these requirements, we assume initial states as $x_1(t_0) = 1.5$, $x_2(t_0) = 1.0$, over the whole tuning process.

The states and control for the nominal values of gains, that is, $k_1 = 1$ and $k_2 = 1$ are shown in Figures 4 and 5, respectively. Now, as we know from the previous study that the states are more sensitivity to variations in k_1 as compared to k_2, therefore, variations in k_1 are done keeping k_2 fixed at the nominal value. As a result,it can be observed from Figures 6 and 7 that faster convergence is achieved. Finally, the control for $k_1 = 1$ and $k_2 = 1$, with k_2 being fixed at the nominal value of 1, is shown in Figure 8. It is evident that the control peaks to a higher value for $k_1 = 2$, thus justifying the reasons for faster convergence at this gain value.

Remark 5. In comparison to the classical pole placement technique that facilitates the placement of the closed-loop poles to the desired location with the help of gain matrix K, the current study allows studying the variation of the states with regard to the small changes in the gains. Thus, the practitioner would be well aware of what changes in states could be expected for slight changes in the gains.

5.2 | Simple pendulum: Tracking problem

Let us now consider the tracking problem for the simple pendulum system with dynamical state equations:

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -a \sin x_1 - b x_2 + cu;\ y = x_1 \tag{17}$$

where x_1 is swing angle, x_2 is swing velocity, y is output, u is control, and a, b, c are constants. For simulation we assume $a = c = 1$ and $b = 0.02$. The initial values of the system are selected as $x(0) = [1.5 1]^{\mathsf{T}}$. The nominal values of the gains are taken as $k_1 = k_2 = 12$. We want the output y to track a reference signal $r(t) = \sin(t/3)$. Let us assume the error signals as follows:

$$e_1 = x_1 - r = x_1 - r_1, e_2 = x_2 - \dot{r} = x_2 - r_2,$$

then the error dynamics results in the following:

$$\dot{e}_1 = e_2;\ \dot{e}_2 = -a \sin x_1 - b x_2 + cu - \ddot{r} \tag{18}$$

Observe that the original tracking problem for the states x_1 and x_2 (17) is now formulated as a stabilization problem for the error variables e_1 and e_2 (18). To solve this stabilization problem, we use feedback linearization. To that end, we assume the control $u = \kappa(u_{\text{nom}} + u_{\text{sfc}})$, where u_{nom} is the nominal control, u_{sfc} is the state feedback control, and $\kappa = \frac{1}{c}$.

The nominal control is chosen as $u_{\text{nom}} = a \sin x_1 + b x_2 + \ddot{r}$. The state feedback control u_{sfc} is selected as $u_{\text{sfc}} = -k_1 e_1 - k_2 e_2$. Although the objective is that x_1 tracks the reference $r(= r_1)$, for the chosen design, we also obtain tracking of $\dot{r}(= r_2)$ by state x_2. At the nominal values

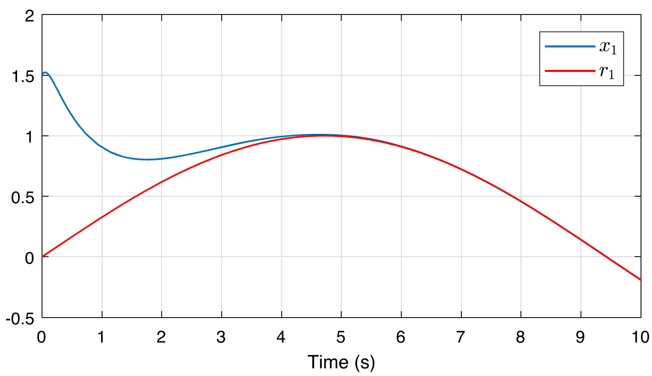


FIGURE 9 The plot of state x_1 and track signal r_1.

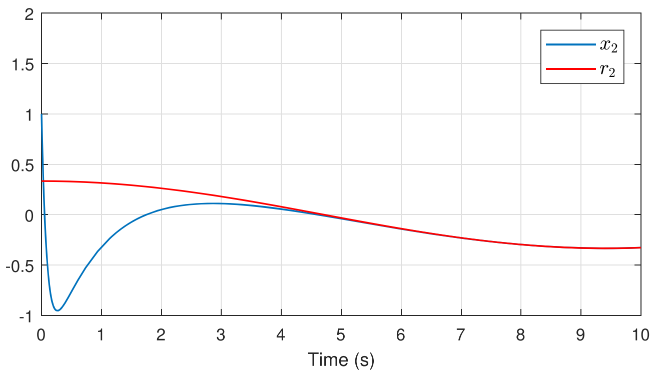


FIGURE 10 The plot of state x_2 and track signal r_2.

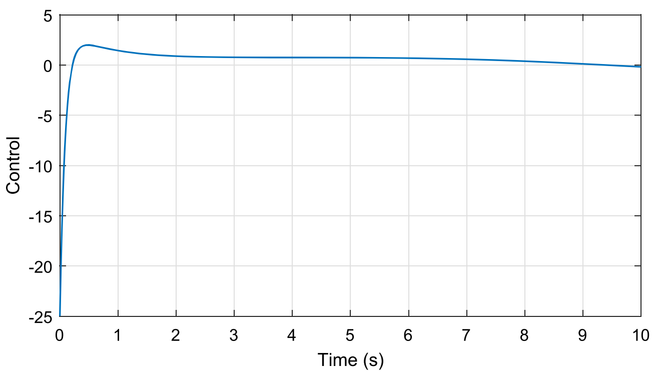


FIGURE 11 The control for tracking at the nominal gains $k_1 = k_2 = 12$.

of the gains, the simulation results for the tracking are shown in Figures 9 and 10, whereas the control is shown in Figure 11. Now, starting from the nominal values of the gains k_1 and k_2, we vary k_1 in steps of 0.4 and keep k_2 fixed at the nominal value. The objective assumed here is better tracking performance. The simulation results for this case are shown in Figures 12 and 13. It is clearly visible from the zoomed-in section of Figures 12 and 13 that improvement in tracking is obtained on increasing the gain k_1. The control for extreme values of gain k_1 is shown in Figure 14.

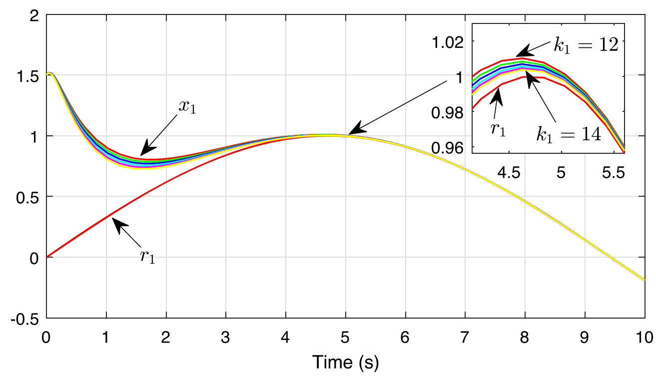


FIGURE 12 The state x_1 and the track signal r_1 for variations in gain k_1 with fixed gain $k_2 = 12$.

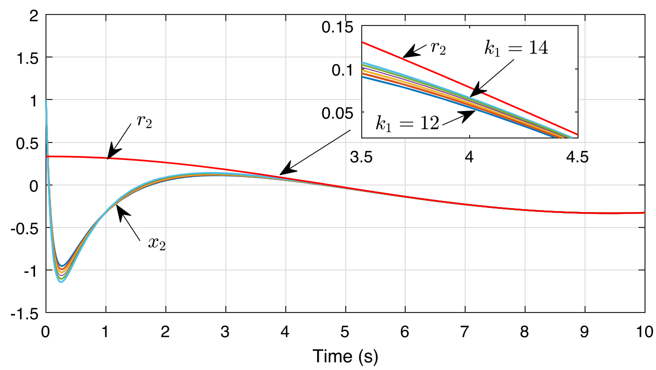


FIGURE 13 The track signal r_2 and the swing state x_2 for variations in gain k_1.

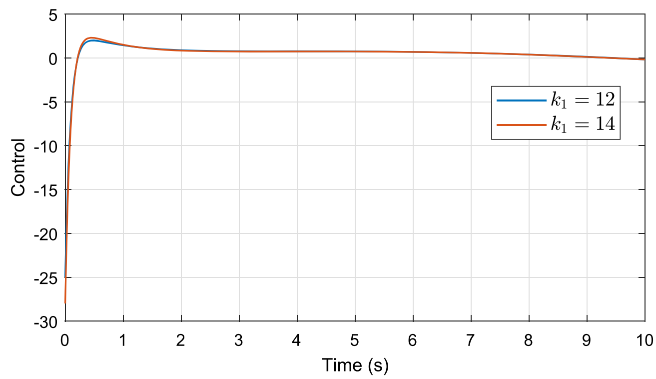


FIGURE 14 The control for tracking at gain values $k_1 = 12$ and $k_1 = 14$ with fixed $k_2 = 12$.

5.3 | SIP

Let us now see how the parameter η can be tuned to obtain the desired response. The procedure is investigated for the case of SIP system [44]. The parameters of the SIP are given in Table 1. Although the example considered here is simple, still the procedure elaborated is generic in nature and can be suitably applied to many other cases as well.

TABLE 1 Parameters of SIP.

Symbols	Description	Values
M_c	Cart's mass	1 kg
m_p	Pendulum's mass	0.1 kg
L	Length to pendulum center of mass	0.5 m
g	Acceleration due to gravity	9.8 m/s^2

Abbreviation: SIP, single inverted pendulum.

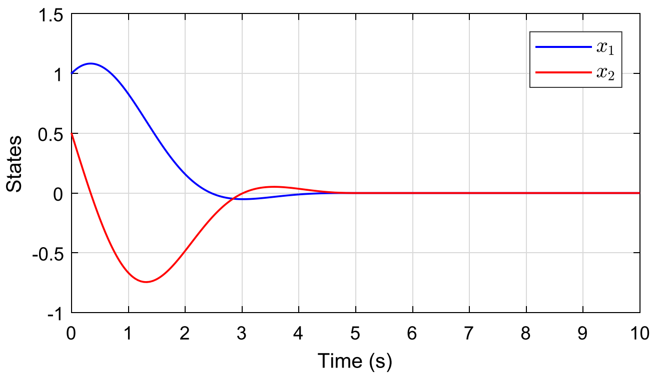


FIGURE 15 The states at the nominal values of parameter η.

For SIP system, the cart's acceleration is assumed to be the input, and the states x_1 and x_2 represent the swing angle and swing velocity, respectively. The swing angle is measured from the vertical top position where $x_1 = 0$. Then the dynamic equations are given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin x_1 - m_p L x_2^2 \cos x_1 \sin x_1/(M_c + m_p)}{L\left(\frac{4}{3} - m_p \cos^2 x_1/(M_c + m_p)\right)} + \frac{\cos x_1/(M_c + m)}{L\left(\frac{4}{3} - m_p \cos^2 x_1/(M_c + m_p)\right)} u \tag{19}$$

The control u is assumed to be consisting of two parts given by $u = \kappa(u_{\text{nominal}} + u_{ptc})$, where u_{nominal} is the nominal control, u_{ptc} is the predefined time convergent control, and

$$\kappa = \frac{L\left(\frac{4}{3} - m_p \cos^2 x_1/(M_c + m_p)\right)}{\cos x_1/(M_c + m)}$$

It is to be noted that for κ to remain finite, it is necessary that $|x_1| \neq \frac{\pi}{2}$. It is necessary to refrain κ from becoming unbounded because such a thing will also be reflected in the required control. To avoid such a situation, it is necessary to assume that $|x_1(0)| \leq \frac{\pi}{2} - \Delta$, where Δ is a small positive constant. The nominal control is chosen as

$$u_{\text{nominal}} = -\frac{g\sin x_1 - m_p L x_2^2 \cos x_1 \sin x_1/(M_c + m_p)}{L\left(\frac{4}{3} - m_p \cos^2 x_1/(M_c + m_p)\right)} \tag{20}$$

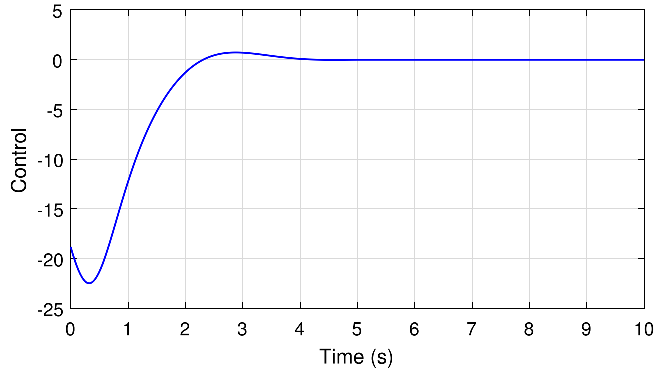


FIGURE 16 The control at the nominal values of parameter η.

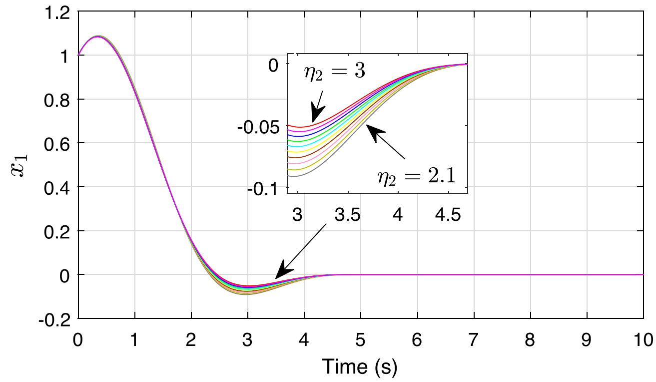


FIGURE 17 The swing angle for variations in parameter η.

For such choices of u and $u_{nominal}$, the given system is reduced to the chain of integrators form given $\dot{x}_1 = x_2$, $\dot{x}_2 = u_{ptc}$, then u_{ptc} can be chosen in a similar manner as given in Section 4.2.2. Here, it is assumed that the nominal values of η are known. To be consistent with the results obtained in Section 4.2.2, the same nominal values are assumed, that is, $\eta_{10} = 3$ and $\eta_{20} = 3$, then it can be easily realized that sensitivity with respect to η_2 is higher as compared to η_1.

One objective while obtaining predefined time convergence is that the states should die out completely in a close vicinity of the desired convergence time t_f. The designed predefined time convergent controllers assure that the states completely die out after t_f, but before t_f, the states can be made to die out close to t_f by tuning the parameter η.

For simulation analysis of SIP system (19), desired convergence time has been considered as $t_f = 5s$ and initial states are considered as $x_1(0) = 1$ and $x_2(0) = 0.5$, then the states and control for nominal values of η are given Figures 15 and 16, respectively. As can be observed in Figure 15, the states are dying out completely before t_f; our objective is to make the convergence to zero happen in close vicinity of t_f, that is, the states should have

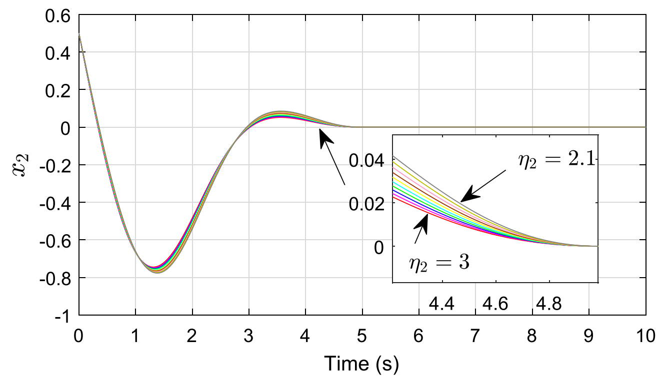


FIGURE 18 The swing velocity for variations in parameter η.

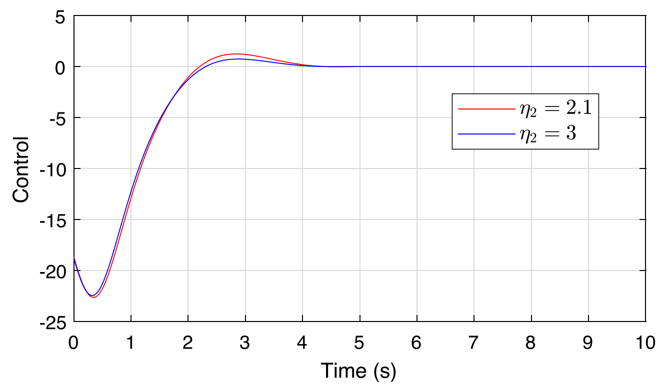


FIGURE 19 The control for $\eta_2 = 2.1$ and $\eta_2 = 3$.

significant values before t_f. This can be done by tuning the parameter η. Since sensitivity is higher with respect to η_2 as compared to η_1, therefore, one approach would be to vary parameter η_2, while keeping η_1 fixed. Here, the η_1 has been kept fixed at the nominal value and η_2 is reduced from 3 to 2.1 in steps of 0.1. The states x_1 and x_2 are plotted for these variations in η in Figures 17 and 18, respectively. It can be observed that with reduction in η_2, the states become more significant. The control for $\eta_2 = 2.1$ and $\eta_2 = 3$ has been given in Figure 19. Note that η_1 has been kept fixed at nominal value of 3. It can be observed that the required control is higher at $\eta_2 = 2.1$ as compared to $\eta_2 = 3$. For tuning, some other objectives can also be considered; for example, one might also consider the minimization of control effort.

Remark 6. To our best knowledge, there is no strategy available to tune the gain of the considered class of controllers. In this regard, the current study not only provides such a strategy but also gives knowledge about the variations in states with fine changes in the gains.

Remark 7. It is recommended to use the proposed approach for creating a database of effective gain conditions with regard to a given system. This database can be created in an off-line manner and can be utilized later. For real-time applications, one can start with the nominal gain values and proceed with the preliminary knowledge about the effect of gains on the states for the chain of integrators in the normal form (see Section (4)). In this case, the choice of step size is important, and the user can start with small step size and could eventually increase it if required. Thus, the choice of the control strategy and the results for the case of a chain of integrators are required in order to utilize the proposed approach in real time.

Remark 8. In practice, the parameters of the simple pendulum models are determined using a combination of theoretical analysis, experimental analysis, or optimization techniques. The theoretical analysis relies on fundamental laws of physics, such as the relationship between the period (T) and the length (l) of the pendulum, as described by $T = 2\pi\sqrt{\frac{l}{g}}$, where g is the acceleration due to gravity. By rearranging this equation, one can determine the length of the pendulum given the period and g. Experimental analysis involves direct measurements, while optimization techniques aim to minimize the error between predicted and actual values by adjusting parameters iteratively. These methods collectively contribute to the accurate determination of parameters for simple pendulum models.

6 | PARAMETRIC UNCERTAINTY CONSIDERATIONS

Parametric uncertainty is very often encountered in real-world systems. Therefore, it is important to see the feasibility of the current formulation in such a scenario. To elaborate, let us consider the example of a simple pendulum with state equations given in (15). Assume that the parameters a, b, and c exhibit variations as follows: $a_{min} \leq a \leq a_{max}$, $b_{min} \leq b \leq b_{max}$, and $c_{min} \leq c \leq c_{max}$. Observe that the proposed approach still works with some modifications in (16). The gains k_1 and k_2 should now be chosen as follows:

$$k_1 > -\frac{a_{min}}{c_{max}} \cos\delta \text{ and } k_2 > -\frac{b_{min}}{c_{max}} \tag{21}$$

Note that a, b, c being nonnegative, this choice of gains remains valid for the entire range of values that these parameters may assume.

On the other hand, for the example of SIP in Section (5.3), it should be noted that the implied predefined time control design relies on feedback linearization [28], which assumes complete knowledge of the parameters. So, if there are known parameter variations in the SIP, then predefined time control can be applied, and the proposed approach would still work. Thus, it is concluded that the choice of the control is important for the feasibility of the proposed approach in the event of parametric uncertainty.

7 | LIMITATIONS

Few points need to be taken care of while using the proposed approach. Firstly, the information about the nominal gains is required just at the start. Secondly, the analysis is restricted to the vicinity of nominal values. This second point is an attribute of sensitivity analysis. However, this limitation could be laid off if it is possible to assume different nominal values. Then it would be possible to broaden the domain of analysis by considering various sets of nominal values.

8 | CONCLUSIONS

The study elaborated the procedure for controller gain tuning based on sensitivity analysis. The proposed technique has been satisfactorily applied for the case of state feedback controller and predefined time controller. The sensitivity function and sensitivity equation have been presented for these class of systems. For the case of the state feedback controller, the study reveals that the sensitivity of states is higher with respect k_1 as compared to k_2. The state feedback law applied to the simple pendulum system is tuned with the proposed procedure. For the predefined time stable system, this analysis reveals that sensitivity of state with respect to system parameter η is higher just before the ultimate convergence time t_f for the first- order system. It is also shown that for the second-order system, the sensitivity of states is higher with respect to η_2 as compared to η_1. These findings are consistent for other initial states as well. The tuning process has been elaborated by taking the examples of simple pendulum and SIP. This analysis provides some clues to the designer for proper tuning of the controller gain.

The future work would focus on developing similar strategies for higher-order systems as well as perturbed systems. Controller gain tuning considering some other pertinent objectives (e.g., control effort minimization) can also be considered as parts of future investigations.

AUTHOR CONTRIBUTIONS

Anil Kumar Pal: Conceptualization; investigation; writing—original draft. **Sunil Kumar:** Data curation; formal analysis. **Shyam Kamal:** Project administration; Methodology; supervision; validation; visualization. **Henry Leung:** Methodology; resources; software; supervision. **Xiaogang Xiong:** Resources; project administration; validation.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflicts of interest.

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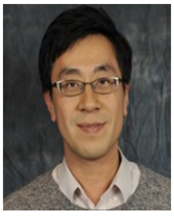

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