Extended Abstract

Nonsmooth Analysis of Interval-valued Functions and Optimization Methods for Set-valued Maps

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Abstract

Optimization aims at determining the maximum and minimum of a function within the observance of certain parameters and has its roots in differential calculus and mathematical analysis. In 1638, Fermat introduced the initial concept of differential calculus and provided the principle for calculating maximum and minimum values. Its applications span various fields like operations research, economics, and engineering, offering insights into optimal solutions under predefined conditions. Optimization deals with computational phenomena to obtain the best solution in all circumstances.

The optimization problems have special structures such as convex, nonconvex, linear, nonlinear, quadratic, semidefinite, dynamic, integer, stochastic programming, etc, and there are vast mathematical techniques and optimizing tools to handle them. A vast majority of machine learning algorithms train their models by solving optimization problems in which the designed objective is nonconvex. Due to randomness and imprecision in the real-world phenomenon, these problems have uncertainty in the given data in the form of sets. These problems are often modeled by optimization problems whose objective function is interval-valued or set-valued. These are represented as interval optimization problems (IOPs) or set-valued optimization problems (SOPs).

The first monograph on interval analysis was introduced by Moore in 1966. Since its inception, this field has seen rapid growth, driving research and precision in numerical computations performed by machines. When dealing with a problem where the given data and coefficient are in the form of intervals, interval-valued computational methods can yield the complete range of possible values for the solution.

Set-valued optimization deals with optimization problems where the objective map and/or the constraint maps are set-valued maps acting between abstract spaces. This framework is important to study as, in various scenarios, the decision maker's preference is based on comparing the elements in the image set and provides an important generalization and unification of scalar as well as vector optimization problems. Set optimization pertains to a branch of mathematical optimization that minimizes the set-valued mappings operating among two abstract spaces, subjected to the image space being partially ordered by a specific convex, closed, and pointed

cone. In order to define the set of minimal solutions using the set approach, a preorder relation is used on the power set of the image space. There are two principal approaches for the solution concepts of set-related problems: the set approach and the vector approach. Since the vector approach is unsuitable for comparing elements having images as sets, therefore we incorporate the set approach for our methodology to find efficient solutions to problems involving set-valued mappings with (or without) uncertainty.

Chapter 1 begins with an introduction to some properties of interval analysis, smooth and nonsmooth analysis of interval-valued functions (IVFs), and optimality conditions for interval optimization problems (IOPs). Interval arithmetic and some important properties of intervals are also explained. The definitions of continuity and convexity for IVFs and their basic results are explained. Next, an introduction to basic definitions and fundamental results of set-valued optimization is discussed.

In Chapter 2, an analysis of gH-subdifferential calculus for IVFs is given. The concept of weak efficient solution of IOPs is defined. With the help of this concept, a Fermat-type, a Fritz-John-type, and a KKT-type optimality condition for weak efficient solutions of nonsmooth IOPs that do not involve convexity assumption are given. A relation is proposed to estimate the weak efficient solution of a nonconvex composite problem to an IOP with the help of gH-subdifferential calculus of IVFs. The entire study is supported by suitable illustrative examples.

Chapter 3 discusses the effect of perturbations on an interval-valued objective. The objective function due to perturbations is named as interval-valued value function. Towards this, the concept of the Lagrangian of IVFs is given, followed by a weak duality theorem for IVFs. After that, the notion of interval-valued value function and the characterization of its gH-subdifferential set for IOPs is discussed. Further, the characterization of the stability of a solution to an IOP with gH-subdifferential set of an interval-valued value function is given. Also, an example demonstrating the results of the interval-valued value function is discussed.

In Chapter 4, the concepts of gH-Dini Hadamard ϵ-subdifferentiability for IVFs and $\mathbf{H}_\epsilon$-subgradient is given. These proposed concepts are observed to be more general than all the existing subdifferentials on IOPs in the literature and contain the set of gH-Dini Hadamard ϵ-subdifferential. A few relations between gH-Fréchet differentiability and gH-Dini Hadamard ϵ-subdifferentiability are given. Next, an important concept of $\mathbf{H}_\epsilon$-subgradient is given, which is based on the criterion of sponge of a set.

Further, a variational interpretation of gH-Dini Hadamard ϵ-subdifferential based on the sponge of a set is discussed. Furthermore, the concept of ϵ-efficient solution followed by necessary and sufficient efficient conditions for finding an ϵ-efficient solution to an IOP with the gH-Dini Hadamard ϵ-subgradient of its objective function is given. An example of applying proposed results in a sparsity regularizer for IOPs is given.

In Chapter 5 and 6, we have extended our study to the class of set-valued optimization problems. Bouza, in 2021, proposed a study on the steepest descent for set optimization problems having finite cardinality of set-valued maps. Motivated by this, we have proposed a Newton method and quasi-Newton method for set optimization problems with a strong convexity assumption. The proposed Newton method in this work exhibits a quadratic convergence near the optimal solution and works well for highly nonlinear objective functions. Next, the convergence of the proposed quasi-Newton method has been analysed. Further, we show the numerical implementation of our methods with the help of suitable examples. Finally, we compare the results of the proposed algorithms with the results of the steepest descent method presented in Bouza's work.

Finally, in Chapter 7, we conclude the thesis with some suggestions for future work.