Chapter 4

Multistability Analysis of Octonion-Valued Neural Networks with Time-Varying Delays

4.1 Introduction

In this chapter, the multistability analysis is studied for n-dimensional octonion valued neural networks (OVNNs) with time-varying delays for a general class of activation functions. In OVNNs, state variables, synaptic connection strengths, activation functions, and external inputs are all octonions. The octonions were discovered in 1843 by John T. Graves, inspired by his friend William Rowan Hamilton's discovery of quaternions. Octonion numbers are neither commutative nor associative. One neuron of OVNNs can store eight times the information stored by one neuron of RVNNs. Due to octonion valued neurons, OVNNs have large storage capacity, and as a result, those can be used in many practical applications such as associative memories, signal processing, image processing, etc. Considering the potential applications of octonions in NNs, Popa in [48] has introduced the feedforward OVNNs, with the states, synaptic connections strength, activation functions and external inputs are as octonions. OVNNs are frequently used in the processing of color images, time series prediction, signal transmission etc [49, 50, 51]. Systems designed for applications such as associative memories, pattern recognition, and signal final processing must have high storage capacity, and stable equilibrium points play a significantly larger role in these applications. Therefore, performing multistability analysis of OVNNs is an interesting task.

4.2 Model Description and Preliminaries

In this chapter, the following Hopfield-type OVNNs [52, 7] with time-varying delays is considered as

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_j(t))) + k_i, \tag{4.1}$$

where $i, j = 1, 2, ..., n$ and n is the number of neurons in the network; $x_i(t) \in \mathbb{O}$ represents the state of i-th neuron at time t; $c_i \in \mathbb{R}$ with $c_i > 0$ is state decaying constant; $a_{ij}, b_{ij} \in \mathbb{O}$ represent the instantaneous and delayed synaptic connection strengths from j-th to i-th neuron of the network; $\tau_j(t) \geq 0$ be time-varying delays which are bounded by τ_M, $\forall\, j = 1, 2, ..., n$ and continuous on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$; $k_i \in \mathbb{O}$ is an external input to each neuron of the network; the mapping $g_j(.) : \mathbb{O} \to \mathbb{O}$ denotes octonion-valued activation functions; $x_i(s) = \phi_i(s),\ \forall s \in [t_0 - \tau_M, t_0]$ is the initial condition of the equation (4.1), where $\phi_i(\cdot) : [t_0 - \tau_M, t_0] \to \mathbb{O}$ are continuous functions for all i. Equation (4.1) can be rewritten in a vector form as

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + K, \tag{4.2}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{O}^n$, $C = \text{diag}(c_1, c_2, ..., c_n) \in \mathbb{R}^{n\times n}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n} \in \mathbb{O}^{n\times n}$, $K = (k_1, k_2, ..., k_n)^T \in \mathbb{O}^n$, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), ..., g_n(x_n(t)))^T \in \mathbb{O}^n$. In an octonion algebra, the activation functions $g_j(x_j(t)) \in \mathbb{O}$ for $j = 1, 2, ..., n$, is choosen such that

$$g_j(x_j(t)) = \sum_{b=0}^{7} g_j^{(b)}(x_j^{(b)}(t))e_b, \tag{4.3}$$

where, e_b is octonion unit and $g_j^{(b)}(\cdot) : \mathbb{R} \to \mathbb{R}$ be the real valued continuous function defined as

$$g_j^{(b)}(z) = \begin{cases} r_j^{(b)}, & \text{if } z \in (-\infty, p_j^{(b)}) \\ \bar{g}_j^{(b)}(z), & \text{if } z \in [p_j^{(b)}, q_j^{(b)}] \\ s_j^{(b)}, & \text{if } z \in (q_j^{(b)}, +\infty) \end{cases}, \tag{4.4}$$

where $b = 0, 1, ..., 7$, $\bar{g}_j^{(b)}(z)$ are continuously differentiable and monotonically non-decreasing functions on $[p_j^{(b)}, q_j^{(b)}]$.

Assumption 4.1. For $j = 1, 2, ..., n$,

$$0 \leq \alpha_j^{(b)} \leq \frac{\bar{g}_j^{(b)}(z_1) - \bar{g}_j^{(b)}(z_2)}{z_1 - z_2}, \; b = 0, 1, ...7,$$

where $z_1, z_2 \in [p_j^{(b)}, q_j^{(b)}]$ and $\alpha_j^{(b)}$ is a real constant.

4.3 Main Results

In this section, some important sufficient conditions on existence and exponential stability of equilibria for OVNNs (4.1) have been derived.

Using the multiplication operation performed in the 1.16, the octonion differential equation (4.1) can be separted into eight real-valued differential equations as

$$\begin{aligned}
\dot{x_i}^{(0)}(t) =& -c_i x_i^{(0)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(0)}(x_j^{(0)}(t)) - a_{ij}^{(1)} g_j^{(1)}(x_j^{(1)}(t)) - a_{ij}^{(2)} g_j^{(2)}(x_j^{(2)}(t)) - a_{ij}^{(3)} \\
& \times g_j^{(3)}(x_j^{(3)}(t)) - a_{ij}^{(4)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(5)} g_j^{(5)}(x_j^{(5)}(t)) - a_{ij}^{(6)} g_j^{(6)}(x_j^{(6)}(t)) - a_{ij}^{(7)} g_j^{(7)}(x_j^{(7)}(t))\Big] \\
& + \sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) - \\
& \quad b_{ij}^{(3)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) - b_{ij}^{(4)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) - b_{ij}^{(5)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) - \\
& \quad b_{ij}^{(6)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(7)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t)))\Big] + k_i^{(0)},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\dot{x_i}^{(1)}(t) =& -c_i x_i^{(1)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(1)}(x_j^{(1)}(t)) + a_{ij}^{(1)} g_j^{(0)}(x_j^{(0)}(t)) + a_{ij}^{(2)} g_j^{(3)}(x_j^{(3)}(t)) - a_{ij}^{(3)} \\
& \times g_j^{(2)}(x_j^{(2)}(t)) + a_{ij}^{(4)} g_j^{(5)}(x_j^{(5)}(t)) - a_{ij}^{(5)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(6)} g_j^{(7)}(x_j^{(7)}(t)) + a_{ij}^{(7)} g_j^{(6)}(x_j^{(6)}(t))\Big] \\
& + \sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(1)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(2)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) - \\
& \quad b_{ij}^{(3)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(4)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) - b_{ij}^{(5)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) - \\
& \quad - b_{ij}^{(6)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t)))\Big] + k_i^{(1)},
\end{aligned} \tag{4.6}$$

$$\dot{x_i}^{(2)}(t) = -c_i x_i^{(2)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(2)}(x_j^{(2)}(t)) + a_{ij}^{(2)} g_j^{(0)}(x_j^{(0)}(t)) - a_{ij}^{(1)} g_j^{(3)}(x_j^{(3)}(t)) + a_{ij}^{(3)} g_j^{(1)}(x_j^{(1)}(t))$$
$$+ a_{ij}^{(4)} g_j^{(6)}(x_j^{(6)}(t)) - a_{ij}^{(6)} g_j^{(4)}(x_j^{(4)}(t) + a_{ij}^{(5)} g_j^{(7)}(x_j^{(7)}(t)) - a_{ij}^{(7)} g_j^{(5)}(x_j^{(5)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(2)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) +$$
$$b_{ij}^{(3)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(4)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(6)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) +$$
$$b_{ij}^{(5)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t)))\Big] + k_i^{(2)}, \tag{4.7}$$

$$\dot{x_i}^{(3)}(t) = -c_i x_i^{(3)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(3)}(x_j^{(3)}(t)) + a_{ij}^{(3)} g_j^{(0)}(x_j^{(0)}(t)) + a_{ij}^{(1)} g_j^{(2)}(x_j^{(2)}(t)) - a_{ij}^{(2)} g_j^{(1)}(x_j^{(1)}(t))$$
$$+ a_{ij}^{(4)} g_j^{(7)}(x_j^{(7)}(t)) - a_{ij}^{(7)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(5)} g_j^{(6)}(x_j^{(6)}(t)) + a_{ij}^{(6)} g_j^{(5)}(x_j^{(5)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) + b_{ij}^{(3)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(1)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) -$$
$$b_{ij}^{(2)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(4)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(5)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t)))\Big] + k_i^{(3)}, \tag{4.8}$$

$$\dot{x_i}^{(4)}(t) = -c_i x_i^{(4)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)} g_j^{(4)}(x_j^{(4)}(t)) + a_{ij}^{(4)} g_j^{(0)}(x_j^{(0)}(t)) - a_{ij}^{(1)} g_j^{(5)}(x_j^{(5)}(t)) + a_{ij}^{(5)} g_j^{(1)}(x_j^{(1)}(t))$$
$$- a_{ij}^{(2)} g_j^{(6)}(x_j^{(6)}(t)) + a_{ij}^{(6)} g_j^{(2)}(x_j^{(2)}(t)) - a_{ij}^{(3)} g_j^{(7)}(x_j^{(7)}(t)) + a_{ij}^{(7)} g_j^{(3)}(x_j^{(3)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) + b_{ij}^{(4)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) +$$
$$b_{ij}^{(5)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) -$$
$$b_{ij}^{(3)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t)))\Big] + k_i^{(4)}, \tag{4.9}$$

$$\dot{x_i}^{(5)}(t) = -c_i x_i^{(5)}(t) + \sum_{j=1}^{n} \Big[a_{ij}^{(0)} g_j^{(5)}(x_j^{(5)}(t)) + a_{ij}^{(5)} g_j^{(0)}(x_j^{(0)}(t)) + a_{ij}^{(1)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(4)} g_j^{(1)}(x_j^{(1)}(t))$$
$$- a_{ij}^{(2)} g_j^{(7)}(x_j^{(7)}(t)) + a_{ij}^{(7)} g_j^{(2)}(x_j^{(2)}(t)) + a_{ij}^{(3)} g_j^{(6)}(x_j^{(6)}(t)) - a_{ij}^{(6)} g_j^{(3)}(x_j^{(3)}(t)) \Big] +$$
$$\sum_{j=1}^{n} \Big[b_{ij}^{(0)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) + b_{ij}^{(5)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(1)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(4)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) +$$
$$b_{ij}^{(3)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(6)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) \Big] + k_i^{(5)}, \tag{4.10}$$

$$\dot{x_i}^{(6)}(t) = -c_i x_i^{(6)}(t) + \sum_{j=1}^{n} \Big[a_{ij}^{(0)} g_j^{(6)}(x_j^{(6)}(t)) + a_{ij}^{(6)} g_j^{(0)}(x_j^{(0)}(t)) + a_{ij}^{(2)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(4)} g_j^{(2)}(x_j^{(2)}(t))$$
$$+ a_{ij}^{(1)} g_j^{(7)}(x_j^{(7)}(t)) - a_{ij}^{(7)} g_j^{(1)}(x_j^{(1)}(t)) - a_{ij}^{(3)} g_j^{(5)}(x_j^{(5)}(t)) + a_{ij}^{(5)} g_j^{(3)}(x_j^{(3)}(t)) \Big] +$$
$$\sum_{j=1}^{n} \Big[b_{ij}^{(0)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(2)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(4)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(1)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) -$$
$$b_{ij}^{(3)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) + b_{ij}^{(5)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) \Big] + k_i^{(6)}, \tag{4.11}$$

$$\dot{x_i}^{(7)}(t) = -c_i x_i^{(7)}(t) + \sum_{j=1}^{n} \Big[a_{ij}^{(0)} g_j^{(7)}(x_j^{(7)}(t)) + a_{ij}^{(7)} g_j^{(0)}(x_j^{(0)}(t)) - a_{ij}^{(1)} g_j^{(6)}(x_j^{(6)}(t)) + a_{ij}^{(6)} g_j^{(1)}(x_j^{(1)}(t))$$
$$+ a_{ij}^{(2)} g_j^{(5)}(x_j^{(5)}(t)) - a_{ij}^{(5)} g_j^{(2)}(x_j^{(2)}(t)) + a_{ij}^{(3)} g_j^{(4)}(x_j^{(4)}(t)) - a_{ij}^{(4)} g_j^{(3)}(x_j^{(3)}(t)) \Big] +$$
$$\sum_{j=1}^{n} \Big[b_{ij}^{(0)} g_j^{(7)}(x_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)} g_j^{(0)}(x_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)} g_j^{(6)}(x_j^{(6)}(t-\tau_j(t))) +$$
$$b_{ij}^{(6)} g_j^{(1)}(x_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(2)} g_j^{(5)}(x_j^{(5)}(t-\tau_j(t))) - b_{ij}^{(5)} g_j^{(2)}(x_j^{(2)}(t-\tau_j(t))) +$$
$$b_{ij}^{(3)} g_j^{(4)}(x_j^{(4)}(t-\tau_j(t))) - b_{ij}^{(4)} g_j^{(3)}(x_j^{(3)}(t-\tau_j(t))) \Big] + k_i^{(7)}. \tag{4.12}$$

Now, for $i = 1, 2, ..., n$ and $z \in \mathbb{R}$, let us define the following real-valued continuous functions as

$$\hat{G}_i^{(b)}(z) = -c_i z + (a_{ii}^{(0)} + b_{ii}^{(0)}) g_i^{(b)}(z) + \hat{\eta}_i^{(b)}, \tag{4.13}$$

$$\check{G}_i^{(b)}(z) = -c_i z + (a_{ii}^{(0)} + b_{ii}^{(0)}) g_i^{(b)}(z) + \check{\eta}_i^{(b)}, \tag{4.14}$$

$$\tilde{G_i}^{(b)}(z) = -c_i z + (a_{ii}^{(0)} + b_{ii)}^{(0)}) g_i^{(b)}(z), \tag{4.15}$$

where $b = 0, 1, ..., 7$, and the real constants $\hat{\eta}_i^{(b)}$ and $\check{\eta}^{(b)}$ are defined as

$$\begin{aligned}\hat{\eta}_i^{(0)} = &\sum_{j=1, j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)}) r_j^{(0)}, (a_{ij}^{(0)} + b_{ij}^{(0)}) s_j^{(0)}\} - \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(1)} + b_{ij}^{(1)}) r_j^{(1)}, (a_{ij}^{(1)} + b_{ij}^{(1)}) s_j^{(1)}\} \\ &+ \min\{(a_{ij}^{(2)} + b_{ij}^{(2)}) r_j^{(2)}, (a_{ij}^{(2)} + b_{ij}^{(2)}) s_j^{(2)}\} + \min\{(a_{ij}^{(3)} + b_{ij}^{(3)}) r_j^{(3)}, (a_{ij}^{(3)} + b_{ij}^{(3)}) s_j^{(3)}\} + \\ &\min\{(a_{ij}^{(4)} + b_{ij}^{(4)}) r_j^{(4)}, (a_{ij}^{(4)} + b_{ij}^{(4)}) s_j^{(4)}\} + \min\{(a_{ij}^{(5)} + b_{ij}^{(5)}) r_j^{(5)}, (a_{ij}^{(5)} + b_{ij}^{(5)}) s_j^{(5)}\} + \\ &\min\{(a_{ij}^{(6)} + b_{ij}^{(6)}) r_j^{(6)}, (a_{ij}^{(6)} + b_{ij}^{(6)}) s_j^{(6)}\} + \min\{(a_{ij}^{(7)} + b_{ij}^{(7)}) r_j^{(7)}, (a_{ij}^{(7)} + b_{ij}^{(7)}) s_j^{(7)}\}\Big] + k_i^{(0)},\end{aligned}$$

$$\begin{aligned}\check{\eta}_i^{(0)} = &\sum_{j=1, j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)}) r_j^{(0)}, (a_{ij}^{(0)} + b_{ij}^{(0)}) s_j^{(0)}\} - \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(1)} + b_{ij}^{(1)}) r_j^{(1)}, (a_{ij}^{(1)} + b_{ij}^{(1)}) s_j^{(1)}\} \\ &+ \max\{(a_{ij}^{(2)} + b_{ij}^{(2)}) r_j^{(2)}, (a_{ij}^{(2)} + b_{ij}^{(2)}) s_j^{(2)}\} + \max\{(a_{ij}^{(3)} + b_{ij}^{(3)}) r_j^{(3)}, (a_{ij}^{(3)} + b_{ij}^{(3)}) s_j^{(3)}\} + \\ &\max\{(a_{ij}^{(4)} + b_{ij}^{(4)}) r_j^{(4)}, (a_{ij}^{(4)} + b_{ij}^{(4)}) s_j^{(4)}\} + \max\{(a_{ij}^{(5)} + b_{ij}^{(5)}) r_j^{(5)}, (a_{ij}^{(5)} + b_{ij}^{(5)}) s_j^{(5)}\} + \\ &\max\{(a_{ij}^{(6)} + b_{ij}^{(6)}) r_j^{(6)}, (a_{ij}^{(6)} + b_{ij}^{(6)}) s_j^{(6)}\} + \max\{(a_{ij}^{(7)} + b_{ij}^{(7)}) r_j^{(7)}, (a_{ij}^{(7)} + b_{ij}^{(7)}) s_j^{(7)}\}\Big] + k_i^{(0)},\end{aligned}$$

$$\begin{aligned}\hat{\eta}_i^{(1)} = &\sum_{j=1, j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)}) r_j^{(1)}, (a_{ij}^{(0)} + b_{ij}^{(0)}) s_j^{(1)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(1)} + b_{ij}^{(1)}) r_j^{(0)}, (a_{ij}^{(1)} + b_{ij}^{(1)}) s_j^{(0)}\} \\ &+ \max\{(a_{ij}^{(2)} + b_{ij}^{(2)}) r_j^{(3)}, (a_{ij}^{(2)} + b_{ij}^{(2)}) s_j^{(3)}\} - \min\{(a_{ij}^{(3)} + b_{ij}^{(3)}) r_j^{(2)}, (a_{ij}^{(3)} + b_{ij}^{(3)}) s_j^{(2)}\} + \\ &\max\{(a_{ij}^{(4)} + b_{ij}^{(4)}) r_j^{(5)}, (a_{ij}^{(4)} + b_{ij}^{(4)}) s_j^{(5)}\} - \min\{(a_{ij}^{(5)} + b_{ij}^{(5)}) r_j^{(4)}, (a_{ij}^{(5)} + b_{ij}^{(5)}) s_j^{(4)}\} - \\ &\min\{(a_{ij}^{(6)} + b_{ij}^{(6)}) r_j^{(7)}, (a_{ij}^{(6)} + b_{ij}^{(6)}) s_j^{(7)}\} + \max\{(a_{ij}^{(7)} + b_{ij}^{(7)}) r_j^{(6)}, (a_{ij}^{(7)} + b_{ij}^{(7)}) s_j^{(6)}\}\Big] + k_i^{(1)},\end{aligned}$$

$$\check{\eta}_i^{(1)} = \sum_{j=1,j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(1)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(1)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(0)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(0)}\}$$
$$+ \min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(3)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(3)}\} - \max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(2)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(2)}\}+$$
$$\min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(5)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(5)}\} - \max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(4)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(4)}\}-$$
$$\max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(7)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(7)}\} + \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(6)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(6)}\}\Big] + k_i^{(1)},$$

$$\hat{\eta}_i^{(2)} = \sum_{j=1,j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(2)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(2)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(0)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(0)}\}$$
$$- \min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(3)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(3)}\} + \max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(1)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(1)}\}+$$
$$\max\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(6)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(6)}\} - \min\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(4)}, (a_{ij}^{(6)} + b_{i,j}^{(6)})s_j^{(4)}\}+$$
$$\max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(7)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(7)}\} - \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(5)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(5)}\}\Big] + k_i^{(2)},$$

$$\check{\eta}_i^{(2)} = \sum_{j=1,j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(2)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(2)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(0)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(0)}\}$$
$$- \max\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(3)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(3)}\} + \min\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(1)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(1)}\}+$$
$$\min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(6)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(6)}\} - \max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(4)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(4)}\}+$$
$$\min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(7)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(7)}\} - \max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(5)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(5)}\}\Big] + k_i^{(2)},$$

$$\hat{\eta}_i^{(3)} = \sum_{j=1,j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(3)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(3)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(0)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(0)}\}$$
$$+ \max\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(2)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(2)}\} - \min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(1)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(1)}\}+$$
$$\max\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(7)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(7)} - \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(4)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(4)}\}-$$
$$\min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(6)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(6)}\} + \max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(5)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(5)}\}\Big] + k_i^{(3)},$$

$$\check{\eta}_i^{(3)} = \sum_{j=1, j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(3)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(3)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(0)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(0)}\}$$
$$+ \min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(2)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(2)}\} - \max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(1)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(1)}\}+$$
$$\min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(7)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(7)}\} - \max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(4)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(4)}\}-$$
$$- \max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(6)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(6)}\} + \min\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(5)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(5)}\}\Big] + k_i^{(3)},$$

$$\hat{\eta}_i^{(4)} = \sum_{j=1, j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(4)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(4)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(0)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(0)}\}$$
$$- \min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(5)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(5)}\} + \max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(1)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(1)}\}-$$
$$\min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(6)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(6)}\} + \max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(2)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(2)}\}-$$
$$\min\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(7)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(7)}\} + \max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(3)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(3)}\}\Big] + k_i^{(4)},$$

$$\check{\eta}_i^{(4)} = \sum_{j=1, j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(4)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(4)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(0)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(0)}\}$$
$$- \max\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(5)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(5)}\} + \min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(1)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(1)}\}-$$
$$\max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(6)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(6)}\} + \min\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(2)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(2)}\}-$$
$$\max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(7)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(7)}\} + \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(3)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(3)}\}\Big] + k_i^{(4)},$$

$$\hat{\eta}_i^{(5)} = \sum_{j=1, j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(5)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(5)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(0)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(0)}\}$$
$$+ \max\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(4)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(4)}\} - \min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(1)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(1)}\}-$$
$$\min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(7)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(7)}\} + \max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(2)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(2)}\}+$$
$$+ \max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(6)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(6)}\} - \min\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(3)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(3)}\}\Big] + k_i^{(5)},$$

$$\check{\eta}_i^{(5)} = \sum_{j=1,j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(5)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(5)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(0)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(0)}\}$$
$$+ \min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(4)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(4)}\} - \max\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(1)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(1)}\} -$$
$$\max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(7)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(7)}\} + \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(2)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(2)}\} +$$
$$\min\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(6)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(6)}\} - \max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(3)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(3)}\}\Big] + k_i^{(5)},$$

$$\hat{\eta}_i^{(6)} = \sum_{j=1,j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(6)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(6)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(0)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(0)}\}$$
$$+ \max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(4)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(4)}\} - \min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(2)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(2)}\} +$$
$$\max\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(7)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(7)}\} - \min\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(1)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(1)}\} -$$
$$\min\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(5)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(5)}\} + \max\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(3)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(3)}\}\Big] + k_i^{(6)},$$

$$\check{\eta}_i^{(6)} = \sum_{j=1,j\neq i}^{n} \min\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(6)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(6)}\} + \sum_{j=1}^{n} \Big[\min\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(0)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(0)}\}$$
$$+ \min\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(4)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(4)}\} - \max\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(2)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(2)}\} +$$
$$\min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(7)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(7)}\} - \max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(1)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(1)}\} -$$
$$\max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(5)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(5)}\} + \min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(3)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(3)}\}\Big] + k_i^{(6)},$$

$$\hat{\eta}_i^{(7)} = \sum_{j=1,j\neq i}^{n} \max\{(a_{ij}^{(0)} + b_{ij}^{(0)})r_j^{(7)}, (a_{ij}^{(0)} + b_{ij}^{(0)})s_j^{(7)}\} + \sum_{j=1}^{n} \Big[\max\{(a_{ij}^{(7)} + b_{ij}^{(7)})r_j^{(0)}, (a_{ij}^{(7)} + b_{ij}^{(7)})s_j^{(0)}\}$$
$$- \min\{(a_{ij}^{(1)} + b_{ij}^{(1)})r_j^{(6)}, (a_{ij}^{(1)} + b_{ij}^{(1)})s_j^{(6)}\} + \max\{(a_{ij}^{(6)} + b_{ij}^{(6)})r_j^{(1)}, (a_{ij}^{(6)} + b_{ij}^{(6)})s_j^{(1)}\} +$$
$$\max\{(a_{ij}^{(2)} + b_{ij}^{(2)})r_j^{(5)}, (a_{ij}^{(2)} + b_{ij}^{(2)})s_j^{(5)}\} - \min\{(a_{ij}^{(5)} + b_{ij}^{(5)})r_j^{(2)}, (a_{ij}^{(5)} + b_{ij}^{(5)})s_j^{(2)}\} +$$
$$\max\{(a_{ij}^{(3)} + b_{ij}^{(3)})r_j^{(4)}, (a_{ij}^{(3)} + b_{ij}^{(3)})s_j^{(4)}\} - \min\{(a_{ij}^{(4)} + b_{ij}^{(4)})r_j^{(3)}, (a_{ij}^{(4)} + b_{ij}^{(4)})s_j^{(3)}\}\Big] + k_i^{(7)},$$

$$\check{\eta}_i^{(7)} = \sum_{j=1,j\neq i}^{n} \min\{(a_{ij}^{(0)}+b_{ij}^{(0)})r_j^{(7)},(a_{ij}^{(0)}+b_{ij}^{(0)})s_j^{(7)}\} + \sum_{j=1}^{n}\Big[\min\{(a_{ij}^{(7)}+b_{ij}^{(7)})r_j^{(0)},(a_{ij}^{(7)}+b_{ij}^{(7)})s_j^{(0)}\}$$
$$-\max\{(a_{ij}^{(1)}+b_{ij}^{(1)})r_j^{(6)},(a_{ij}^{(1)}+b_{ij}^{(1)})s_j^{(6)}\}+\min\{(a_{ij}^{(6)}+b_{ij}^{(6)})r_j^{(1)},(a_{ij}^{(6)}+b_{ij}^{(6)})s_j^{(1)}\}+$$
$$\min\{(a_{ij}^{(2)}+b_{ij}^{(2)})r_j^{(5)},(a_{ij}^{(2)}+b_{ij}^{(2)})s_j^{(5)}\}-\max\{(a_{ij}^{(5)}+b_{ij}^{(5)})r_j^{(2)},(a_{ij}^{(5)}+b_{ij}^{(5)})s_j^{(2)}\}+$$
$$\min\{(a_{ij}^{(3)}+b_{ij}^{(3)})r_j^{(4)},(a_{ij}^{(3)}+b_{ij}^{(3)})s_j^{(4)}\}-\max\{(a_{ij}^{(4)}+b_{ij}^{(4)})r_j^{(3)},(a_{ij}^{(4)}+b_{ij}^{(4)})s_j^{(3)}\}\Big]+k_i^{(7)}.$$

From the definitions of $\hat{\eta}_i^{(b)}$ and $\check{\eta}_i^{(b)}$, it can be concluded that $\hat{G}_i^{(b)}(z) \geq \check{G}_i^{(b)}(z)$ for all $b = 0, 1, ..., 7$, and $i = 1, 2, ..., n$. Moreover, the geometric configurations of the $\hat{G}_i^{(b)}(z)$ and $\check{G}_i^{(b)}(z)$ are given in Figure 4.1.

Lemma 4.1. For $b = 0, 1, ..., 7$,and $i = 1, 2, ..., n$, let us assume that the Assumption 4.1 imposed on activation functions holds with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$, then

1. $\tilde{G}_i^{(b)}(z)$ is strictly decreasing on $(-\infty, p_i^{(b)})$.
2. $\tilde{G}_i^{(b)}(z)$ is strictly increasing on $[p_i^{(b)}, q_i^{(b)}]$.
3. $\tilde{G}_i^{(b)}(z)$ is strictly decreasing on $(q_i^{(b)}, +\infty)$.

Proof. We have $\tilde{G}_i^{(b)}(z) = -c_i z + (a_{ii}^{(0)} + b_{ii)}^{(0)})g_i^{(b)}(z)$. Since $c_i > 0$, by using definition of $g_i^{(b)}(\cdot)$, it is easy to see that $\tilde{G}_i^{(b)}(z)$ is strictly decreasing on $(-\infty, p_i^{(b)})$ and $(q_i^{(b)}, +\infty)$. Meanwhile, it follows from $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ and Assumption 4.1 that $\tilde{G}_i^{(b)}(z)$ is strictly increasing on $[p_i^{(b)}, q_i^{(b)}]$. □

Remark 4.1. From the definition of $\hat{G}_i^{(b)}$, $\check{G}_i^{(b)}$ and $\tilde{G}_i^{(b)}$, it can be shown that $\hat{G}_i^{(b)}$, $\check{G}_i^{(b)}$ are vertical shifts of $\tilde{G}_i^{(b)}$. Hence $\hat{G}_i^{(b)}$, $\check{G}_i^{(b)}$ and $\tilde{G}_i^{(b)}$ have same monotonic intervals for all $b = 0, 1, ..., 7$.

Remark 4.2. From the definitions of $\hat{\eta}_i^{(b)}$,$\check{\eta}_i^{(b)}$, we conclude that $\hat{G}_i^{(b)} \geq \check{G}_i^{(b)}$, for $i = 1, 2, ...n$, and $b = 0, 1, ..., 7$. Since $c_i > 0$, we can see that

$$\lim_{z\to+\infty} \hat{G}_i^{(b)} = -\infty, \ \lim_{z\to-\infty} \check{G}_i^{(b)} = +\infty,$$

for $b = 0, 1, ..., 7$, thus, there exist $\check{p}_i^{(b)} < 0$ and $\hat{q}_i^{(b)} > 0$, such that $\check{p}_i^{(b)} < p_i^{(b)}$, $\hat{q}_i^{(b)} > q_i^{(b)}$ and

$$\hat{G}_i^{(b)}(\check{p}_i^{(b)}) \geq \check{G}_i^{(b)}(\check{p}_i^{(b)}) > 0,\ \check{G}_i^{(b)}(\hat{q}_i^{(b)}) \leq \hat{G}_i^{(b)}(\hat{q}_i^{(b)}) < 0. \tag{4.16}$$

Assumption 4.2. For $i = 1, 2, ..., n$,

$$\hat{G}_i^{(b)}(p_i^{(b)}) < 0, \quad \check{G}_i^{(b)}(q_i^{(b)}) > 0,\ b = 0, 1, ...7.$$

Lemma 4.2. If Assumption 4.1 with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ and Assumption 4.2 hold, then there exist six constants $\check{a}_i^{(b)}$, $\hat{a}_i^{(b)}$, $\check{b}_i^{(b)}$, $\hat{b}_i^{(b)}$, $\check{c}_i^{(b)}$, $\hat{c}_i^{(b)}$ with $\hat{a}_i^{(b)} < \hat{b}_i^{(b)} < \hat{c}_i^{(b)}$ and $\check{a}_i^{(b)} < \check{b}_i^{(b)} < \check{c}_i^{(b)}$ such that $\hat{G}_i^{(b)}(\hat{a}_i^{(b)}) = \hat{G}_i^{(b)}(\hat{b}_i^{(b)}) = \hat{G}_i^{(b)}(\hat{c}_i^{(b)}) = 0$ and $\check{G}_i^{(b)}(\check{a}_i^{(b)}) = \check{G}_i^{(b)}(\check{b}_i^{(b)}) = \check{G}_i^{(b)}(\check{c}_i^{(b)}) = 0$.

Proof. Since $g_i^{(b)}$'s are continuous, then $\hat{G}_i^{(b)}, \check{G}_i^{(b)}$ are also continuous. From Lemma 4.1 and Remark 4.1, it can be said that $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ are strictly decreasing on $(-\infty, p_i^{(b)})$ for having inequalities $\hat{G}_i^{(b)}(\check{p}_i^{(b)}) \geq \check{G}_i^{(b)}(\check{p}_i^{(b)}) > 0$ and $\check{G}_i^{(b)}(p_i^{(b)}) \leq \hat{G}_i^{(b)}(p_i^{(b)}) < 0$. Therefore by applying intermediate value property there exist $\hat{a}_i^{(b)}$ and $\check{a}_i^{(b)}$ in $(\check{p}_i^{(b)}, p_i^{(b)})$ such that $\hat{G}_i^{(b)}(\hat{a}_i^{(b)}) = 0$ and $\check{G}_i^{(b)}(\check{a}_i^{(b)}) = 0$. Similarly we can obtain the remaining constants. This completes the proof. □

Now we denote the intervals by $I_i^{(b)l} = [\check{a}_i^{(b)}, \hat{a}_i^{(b)}]$, $I_i^{(b)m} = [\hat{b}_i^{(b)}, \check{b}_i^{(b)}]$, and $I_i^{(b)r} = [\check{c}_i^{(b)}, \hat{c}_i^{(b)}]$, where $b = 0, 1, .., 7$. Herein "l," "m," and "r", respectively represent "left," "middle," and "right." Then, for each index $\beta = (\beta_1, \beta_2, ..., \beta_{8n})$ where β_i is "l," "m,"

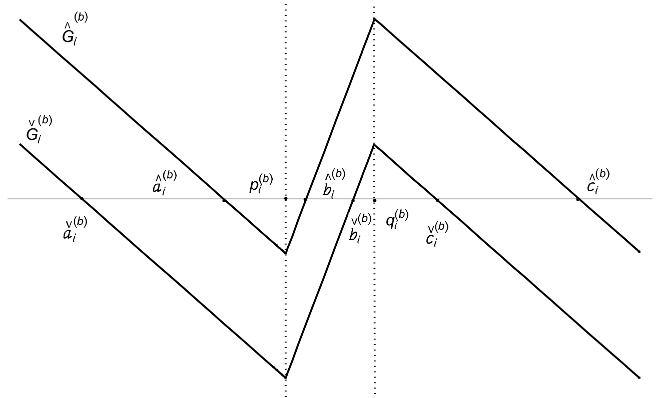

FIGURE 4.1: Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$

.

or "r," we denote

$$\begin{aligned}\Pi^\beta =&\big\{(x_1, x_2, ..., x_n)^T \in \mathbb{O}^n | x_i^{(0)} \in I_i^{(0)\beta_i}, x_i^{(1)} \in I_i^{(1)\beta_{n+i}}, x_i^{(2)} \in I_i^{(2)\beta_{2n+i}}, ..., x_i^{(7)} \in I_i^{(7)\beta_{7n+i}},\\ &\text{for } i = 1, 2, .., n\big\}, \text{ and}\\ \Pi =&\big\{\Pi^\beta | \beta = (\beta_1, \beta_2, ..., \beta_{8n}) \text{ with } \beta_i \text{ is "}l\text{", "}m\text{", or "}r\text{"}\big\}.\end{aligned} \tag{4.17}$$

From construction, it is seen that there are 3^{8n} elements of the type Π^β in Π which are disjoint regions in $\mathbb{O}^n$. Now some more real valued continuous functions are introduced for given $x = (x_1, x_2, ..., x_n)^T \in \mathbb{O}^n$. Let us define functions $G_i^{(b)} : \mathbb{R} \to \mathbb{R}$ such that

$$G_i^{(b)}(z) = -c_i z + (a_{ii}^{(0)} + b_{ii}^{(0)}) g_i^{(b)}(z) + \eta_i^{(b)}, \tag{4.18}$$

where $b = 0, 1, ...7,\ i = 1, 2, ...n$, and

$$\begin{aligned}\eta_i^{(0)} =& \sum_{j=1, j\neq i}^{n} (a_{ii}^{(0)} + b_{ii}^{(0)}) g_j^{(0)}(x_j^{(0)}) - \sum_{j=1}^{n} \Big[(a_{ij}^{(1)} + b_{ij}^{(1)}) g_j^{(1)}(x_j^{(1)}) + (a_{ij}^{(2)} + b_{ij}^{(2)}) g_j^{(1)}(x_j^{(2)}) + (a_{ij}^{(3)} +\\ & b_{ij}^{(3)}) g_j^{(3)}(x_j^{(3)}) + (a_{ij}^{(4)} + b_{ij}^{(4)}) g_j^{(4)}(x_j^{(4)}) + (a_{ij}^{(5)} + b_{ij}^{(5)}) g_j^{(5)}(x_j^{(5)}) + (a_{ij}^{(6)} + b_{ij}^{(6)}) g_j^{(6)}(x_j^{(6)}) +\\ & (a_{ij}^{(7)} + b_{ij}^{(7)}) g_j^{(7)}(x_j^{(7)})\Big] + k_i^{(0)},\end{aligned}$$

$$\eta_i^{(1)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(1)}(x_j^{(1)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(0)}(x_j^{(0)}) + (a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(3)}(x_j^{(3)}) - (a_{ij}^{(3)}+ b_{ij}^{(3)})g_j^{(2)}(x_j^{(2)}) + (a_{ij}^{(4)}+b_{ij}^{(4)})g_j^{(5)}(x_j^{(5)}) - (a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(4)}(x_j^{(4)}) - (a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(7)}(x_j^{(7)}) + (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(6)}(x_j^{(6)})\Big] + k_i^{(1)},$$

$$\eta_i^{(2)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(2)}(x_j^{(2)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(0)}(x_j^{(0)}) - (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(3)}(x_j^{(3)}) + (a_{ij}^{(3)}+ b_{ij}^{(3)})g_j^{(1)}(x_j^{(1)}) + (a_{ij}^{(4)}+b_{ij}^{(4)})g_j^{(6)}(x_j^{(6)}) - (a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(4)}(x_j^{(4)}) + (a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(7)}(x_j^{(7)}) - (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(5)}(x_j^{(5)})\Big] + k_i^{(2)},$$

$$\eta_i^{(3)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(3)}(x_j^{(3)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(3)}+b_{ij}^{(3)})g_j^{(0)}(x_j^{(0)}) + (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(2)}(x_j^{(2)}) - (a_{ij}^{(2)}+ b_{ij}^{(2)})g_j^{(1)}(q_j^{(1)}) + (a_{ij}^{(4)}+b_{ij}^{(4)})g_j^{(7)}(x_j^{(7)}) - (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(4)}(x_j^{(4)}) - (a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(6)}(x_j^{(6)}) + (a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(5)}(x_j^{(5)})\Big] + k_i^{(3)},$$

$$\eta_i^{(4)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(4)}(x_j^{(4)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(4)}+b_{ij}^{(4)})g_j^{(0)}(x_j^{(0)}) - (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(5)}(x_j^{(5)}) + (a_{ij}^{(5)}+ b_{ij}^{(5)})g_j^{(1)}(x_j^{(1)}) - (a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(6)}(x_j^{(6)}) + (a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(2)}(x_j^{(2)}) - (a_{ij}^{(3)}+b_{ij}^{(3)})g_j^{(7)}(x_j^{(7)}) + (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(3)}(x_j^{(3)})\Big] + k_i^{(4)},$$

$$\eta_i^{(5)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(5)}(x_j^{(5)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(0)}(x_j^{(0)}) + (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(4)}(x_j^{(4)}) - (a_{ij}^{(4)}+ b_{ij}^{(4)})g_j^{(1)}(x_j^{(1)}) - (a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(7)}(x_j^{(7)}) + (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(2)}(x_j^{(2)}) + (a_{ij}^{(3)}+b_{ij}^{(3)})g_j^{(6)}(x_j^{(6)}) - (a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(3)}(x_j^{(3)})\Big] + k_i^{(5)},$$

$$\eta_i^{(6)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(6)}(x_j^{(6)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(6)}+b_{ij}^{(6)})g_j^{(0)}(x_j^{(0)}) + (a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(4)}(x_j^{(4)}) - (a_{ij}^{(4)}+ b_{ij}^{(4)})g_j^{(2)}(x_j^{(2)}) + (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(7)}(x_j^{(7)}) - (a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(1)}(x_j^{(1)}) - (a_{ij}^{(3)}+b_{ij}^{(3)})g_j^{(5)}(x_j^{(5)}) + (a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(3)}(x_j^{(3)})\Big] + k_i^{(6)},$$

$$\eta_i^{(7)} = \sum_{j=1,j\neq i}^{n}(a_{ii}^{(0)}+b_{ii}^{(0)})g_j^{(7)}(x_j^{(7)}) + \sum_{j=1}^{n}\Big[(a_{ij}^{(7)}+b_{ij}^{(7)})g_j^{(0)}(x_j^{(0)}) - (a_{ij}^{(1)}+b_{ij}^{(1)})g_j^{(6)}(x_j^{(6)}) + (a_{ij}^{(6)}+ b_{ij}^{(6)})g_j^{(1)}(x_j^{(1)}) + (a_{ij}^{(2)}+b_{ij}^{(2)})g_j^{(5)}(x_j^{(5)}) - (a_{ij}^{(5)}+b_{ij}^{(5)})g_j^{(2)}(x_j^{(2)}) + (a_{ij}^{(3)}+b_{ij}^{(3)})g_j^{(4)}(x_j^{(4)}) - (a_{ij}^{(4)}+b_{ij}^{(4)})g_j^{(3)}(x_j^{(3)})\Big] + k_i^{(7)}.$$

From the definitions of $\check{G}_i^{(b)}(z)$, $\hat{G}_i^{(b)}(z)$ and $G_i^{(b)}(z)$, we can conclude that

$$\check{G}_i^{(b)}(z) \leq G_i^{(b)}(z) \leq \hat{G}_i^{(b)}(z), \quad \forall z \in \mathbb{R}. \tag{4.19}$$

Theorem 4.1. If Assumption 4.1 with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ and Assumption 4.2 hold, then there exist 3^{8n} equilibrium points for the system (4.1) or (4.2).

Proof. For an arbitrary $x = (x_1, x_2, ..., x_n) \in \mathbb{O}^n$, let us define a continuous real valued function $G_i^{(b)}$ in the form of equation (4.18). From inequality (4.19) and Assumption 4.2, we obtain

$$G_i^{(b)}(p_i^{(b)}) < 0, \; G_i^{(b)}(q_i^{(b)}) > 0. \tag{4.20}$$

Since the graph of $G_i^{(b)}(z)$ lies between $\check{G}_i^{(b)}(z)$ and $\hat{G}_i^{(b)}(z)$, clearly $\check{G}_i^{(b)}(z)$ and $\hat{G}_i^{(b)}(z)$ are vertical shifts of $G_i^{(b)}(z)$, so by using Lemma 4.1 and Remark 4.1, it is easy to say that $G_i^{(b)}$ is strictly decreasing on $(-\infty, p_i^{(b)})$ and $(q_i^{(b)}, +\infty)$ and strictly increasing on $[p_i^{(b)}, q_i^{(b)}]$. From Remark 4.2 ($\check{G}_i^{(b)}(\check{p}_i^{(b)}) > 0, \hat{G}_i^{(b)}(\hat{q}_i^{(b)}) < 0$) and inequality (4.19), we have $G_i^{(b)}(\check{p}_i^{(b)}) > 0$ and $G_i^{(b)}(\hat{q}_i^{(b)}) < 0$. Since $G_i^{(b)}$'s are continuous, by applying intermediate value property, there exist exactly three points $\underline{z}_i^{(b)l} \in I_i^{(b)l}$, $\underline{z}_i^{(b)m} \in I_i^{(b)m}$ and $\underline{z}_i^{(b)r} \in I_i^{(b)r}$, such that $G_i^{(b)}(\underline{z}_i^{(b)l}) = G_i^{(b)}(\underline{z}_i^{(b)m}) = G_i^{(b)}(\underline{z}_i^{(b)r}) = 0$.

Suppose index $\beta = (\beta_1, \beta_2, ..., \beta_{8n})$, where β_i is "I", "m", or "r". By choosing $x = (x_1, x_2, ..., x_n)^T \in \Pi^\beta$ arbitrary and putting it into (4.18), we get the corresponding functions $G_i^{(b)}(z)$, and exactly one $\underline{z} = (\underline{z}_1, \underline{z}_2, ..., \underline{z}_n)^T \in \Pi^\beta$, where $\underline{z}_i = \underline{z}_i^{(0)\beta_i} e_0 + \underline{z}_i^{(1)\beta_{n+i}} e_1 + \underline{z}_i^{(2)\beta_{2n+i}} e_2 + ... + \underline{z}_i^{(7)\beta_{7n+i}} e_7$, such that $G_i^{(b)}(\underline{z}_i^{(b)\beta_{bn+i}}) = 0$ with $b = 0, 1, ..., 7$ and $i = 1, 2, ..., n$. Then we can define a mapping $H : \Pi^\beta \to \Pi^\beta$ by $H(x) = \underline{z} = (\underline{z}_1, \underline{z}_2, ..., \underline{z}_n)$. It is easy to check that mapping H is continuous function. By Brouwer's fixed point theorem, H has at least one fixed point $x = \underline{z} \in \Pi^\beta$ which is also a equilibrium point of the system (4.1). Consequently, there exist 3^{8n} equilibrium points of the system (4.1) and each of those lies in one of the 3^{8n} regions Π^β. □

Next, for the index $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{8n})$ where γ_i is "l" or "r", let us denote

$$\Omega^\gamma = \Big\{(x_1, x_2, ..., x_n)^T \in \mathbb{O}^n | x_i^{(0)} \in \tilde{I}_i^{(0)\gamma_i}, x_i^{(1)} \in \tilde{I}_i^{(1)\gamma_{n+i}}, x_i^{(2)} \in \tilde{I}_i^{(2)\gamma_{2n+i}}, ..., x_i^{(7)} \in \tilde{I}_i^{(7)\gamma_{7n+i}}, \text{ for } i = 1, 2, .., n\Big\},$$

where $\tilde{I}_i^{(b)l} = (\check{p}_i^{(b)}, p_i^{(b)})$, $\tilde{I}_i^{(b)r} = (q_i^{(b)}, \hat{q}_i^{(b)})$.

Let

$$\Omega = \Big\{\Omega^\gamma | \gamma = (\gamma_1, \gamma_2, ..., \gamma_{8n}) \text{ with } \gamma_i \text{ is "}l\text{" or "}r\text{"}\Big\}. \tag{4.21}$$

From the above constructions, it can be said that, there are 2^{8n} elements Ω^γ in Ω, which are disjoint regions in $\mathbb{O}^n$.

Theorem 4.2. If Assumption 4.1 with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ and Assumption 4.2 hold, then each region $\Omega^\gamma \in \Omega$ is positively invariant.

Proof. Consider any one of the 2^{8n} sets Ω^{γ}. Without loss of generality, let us assume that the index $\gamma = (l, l, ..., l)$, and suppose that $x_i(s) = \phi_i(s),\ s \in [-\tau_M, 0]$ be the initial condition with $x_i^{(b)}(s) = \phi_i^{(b)}(s) \in (\check{p}_i^{b)}, p_i^{(b)})$, where $b = 0, 1..., 7$, and $i = 1, 2, ..., n$. It is claimed that for all $t \geq 0$, the solution $x(t)$ with the initial condition satisfies $\check{p}_i^{(b)} < x_i^{(b)}(t) < p_i^{(b)}$, where $b = 0, 1, 2, ..., 7, i = 1, 2, ..., n$. If possible, let us suppose that, it is not true, then there exists a time t_1 at which $x(t)$ first escapes from Ω^{γ}. Suppose that the real part of i_1-th component $x_{i_1}^{(0)}$ of $x(t)$ which is the first (or one of the first ones) that escapes from $(-\check{p}_i^{(0)}, p_i^{(0)})$ at time t_1. That is, for all $i = 1, 2, ..., n$ and $b = 0, 1, ..., 7$ $-\check{p}_i^{(b)} < x_i^{(b)}(t) < p_i^{(b)}, t \in [-\tau, t_1)$ and also any of the following two cases holds:

$$x_{i_1}^{(0)}(t_1) = p_{i_1}^{(0)} \text{ and } \dot{x}_{i_1}^{(0)}(t_1) > 0, \tag{4.22}$$

$$x_{i_1}^{(0)}(t_1) = \check{p}_{i_1}^{(0)} \text{ and } \dot{x}_{i_1}^{(0)}(t_1) < 0. \tag{4.23}$$

For the first case, it is observed that $g_{i_1}^{(0)}(x_{i_1}^{(0)}(t_1)) = g_{i_1}^{(0)}(p_{i_1}^{(0)}) = r_{i_1}^{(0)} = g_{i_1}^{(0)}(x_{i_1}^{(0)}(t_1 - \tau_{i_1}(t_1)))$. Then we have

$$
\begin{aligned}
\dot{x_{i_1}}^{(0)}(t_1) = & -c_{i_1}x_{i_1}^{(0)}(t_1) + \sum_{j=1}^{n}\Big[a_{i_1j}^{(0)}g_j^{(0)}(x_j^{(0)}(t_1)) - a_{i_1j}^{(1)}g_j^{(1)}(x_j^{(1)}(t_1)) - a_{i_1j}^{(2)}g_j^{(2)}(x_j^{(2)}(t_1)) - \\
& a_{i_1j}^{(3)}g_j^{(3)}(x_j^{(3)}(t_1)) - a_{i_1j}^{(4)}g_j^{(4)}(x_j^{(4)}(t_1)) - a_{i_1j}^{(5)}g_j^{(5)}(x_j^{(5)}(t_1)) - a_{i_1j}^{(6)}g_j^{(6)}(x_j^{(6)}(t_1)) - \\
& a_{i_1j}^{(7)}g_j^{(7)}(x_j^{(7)}(t_1))\Big] + \sum_{j=1}^{n}\Big[b_{i_1j}^{(0)}g_j^{(0)}(x_j^{(0)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(1)}g_j^{(1)}(x_j^{(1)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(2)} \\
& \times g_j^{(2)}(x_j^{(2)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(3)}g_j^{(3)}(x_j^{(3)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(4)}g_j^{(4)}(x_j^{(4)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(5)} \\
& \times g_j^{(5)}(x_j^{(5)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(6)}g_j^{(6)}(x_j^{(6)}(t_1 - \tau_j(t_1))) - b_{i_1j}^{(7)}g_j^{(7)}(x_j^{(7)}(t_1 - \tau_j(t_1)))\Big] + k_{i_1}^{(0)} \\
\leq & -c_{i_1}p_{i_1}^{(0)} + a_{i_1i_1}^{(0)}g_{i_1}^{(0)}(p_{i_1}^{(0)}) + b_{i_1i_1}^{(0)}g_{i_1}^{(0)}(p_{i_1}^{(0)}) + \sum_{j=1, j\neq i_1}^{n}\max\{(a_{i_1j}^{(0)} + b_{i_1j}^{(0)})r_j^{(0)}, (a_{i_1j}^{(0)} + b_{i_1j}^{(0)}) \\
& \times s_j^{(0)}\} - \sum_{j=1}^{n}\Big[\min\{(a_{i_1j}^{(1)} + b_{i_1j}^{(1)})r_j^{(1)}, (a_{i_1j}^{(1)} + b_{i_1j}^{(1)})s_j^{(1)}\} + \min\{(a_{i_1j}^{(2)} + b_{i_1j}^{(2)})r_j^{(2)}, (a_{i_1j}^{(2)} + \\
& b_{i_1j}^{(2)})s_j^{(2)}\} + \min\{(a_{i_1j}^{(3)} + b_{i_1j}^{(3)})r_j^{(3)}, (a_{i_1j}^{(3)} + b_{i_1j}^{(3)})s_j^{(3)}\} + \min\{(a_{i_1j}^{(4)} + b_{i_1j}^{(4)})r_j^{(4)}, (a_{i_1j}^{(4)} + b_{i_1j}^{(4)}) \\
& \times s_j^{(4)}\} + \min\{(a_{i_1j}^{(5)} + b_{i_1j}^{(5)})r_j^{(5)}, (a_{i_1j}^{(5)} + b_{i_1j}^{(5)})s_j^{(5)}\} + \min\{(a_{i_1j}^{(6)} + b_{i_1,j}^{(6)})r_j^{(6)}, (a_{i_1j}^{(6)} + b_{i_1j}^{(6)}) \\
& \times s_j^{(6)}\} + \min\{(a_{i_1j}^{(7)} + b_{i_1j}^{(7)})r_j^{(7)}, (a_{i_1j}^{(7)} + b_{i_1j}^{(7)})s_j^{(7)}\}\Big] + k_{i_1}^{(0)} \\
= & \hat{G}_{i_1}^{(0)}(p_{i_1}^{(0)}) < 0,
\end{aligned}
\tag{4.24}
$$

which is a contradiction. For the second case, we can observe that $g_{i_1}^{(0)}(x_{i_1}^{(0)}(t_1)) = g_{i_1}^{(0)}(\check{p}_{i_1}^{(0)}) = r_{i_1}^{(0)} = g_{i_1}^{(0)}(x_{i_1}^{(0)}(t_1 - \tau_{i_1}(t_1)))$.

Then,

$$
\begin{aligned}
\dot{x_{i_1}}^{(0)}(t_1) = & -c_{i_1}x_{i_1}^{(0)}(t_1) + \sum_{j=1}^{n}\Big[a_{i_1j}^{(0)}g_j^{(0)}(x_j^{(0)}(t_1)) - a_{i_1j}^{(1)}g_j^{(1)}(x_j^{(1)}(t_1)) - a_{i_1j}^{(2)}g_j^{(2)}(x_j^{(2)}(t_1)) - \\
& a_{i_1j}^{(3)}g_j^{(3)}(x_j^{(3)}(t_1)) - a_{i_1j}^{(4)}g_j^{(4)}(x_j^{(4)}(t_1)) - a_{i_1j}^{(5)}g_j^{(5)}(x_j^{(5)}(t_1)) - a_{i_1j}^{(6)}g_j^{(6)}(x_j^{(6)}(t_1)) - \\
& a_{i_1j}^{(7)}g_j^{(7)}(x_j^{(7)}(t_1))\Big] + \sum_{j=1}^{n}\Big[b_{i_1j}^{(0)}f_j^{(0)}(x_j^{(0)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(1)}g_j^{(1)}(x_j^{(1)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(2)} \\
& \times g_j^{(2)}(x_j^{(2)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(3)}g_j^{(3)}(x_j^{(3)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(4)}g_j^{(4)}(x_j^{(4)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(5)} \\
& \times g_j^{(5)}(x_j^{(5)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(6)}g_j^{(6)}(x_j^{(6)}(t_1-\tau_j(t_1))) - b_{i_1j}^{(7)}g_j^{(7)}(x_j^{(7)}(t_1-\tau_j(t_1)))\Big] + k_{i_1}^{(0)} \\
\geq & -c_{i_1}\check{p}_{i_1}^{(0)} + a_{i_1i_1}^{(0)}g_{i_1}^{(0)}(\check{p}_{i_1}^{(0)}) + b_{i_1i_1}^{(0)}g_{i_1}^{(0)}(\check{p}_{i_1}^{(0)}) + \sum_{j=1, j\neq i_1}^{n}\min\{(a_{i_1j}^{(0)}+b_{i_1j}^{(0)})u_j^{(0)}, (a_{i_1j}^{(0)}+b_{i_1j}^{(0)}) \\
& \times v_j^{(0)}\} - \sum_{j=1}^{n}\Big[\max\{(a_{i_1j}^{(1)}+b_{i_1j}^{(1)})u_j^{(1)}, (a_{i_1j}^{(1)}+b_{i_1j}^{(1)})v_j^{(1)}\} + \max\{(a_{i_1j}^{(2)}+b_{i_1j}^{(2)})u_j^{(2)}, (a_{i_1j}^{(2)}+ \\
& b_{i_1j}^{(2)})v_j^{(2)}\} + \max\{(a_{i_1j}^{(3)}+b_{i_1j}^{(3)})u_j^{(3)}, (a_{i_1j}^{(3)}+b_{i_1j}^{(3)})v_j^{(3)}\} + \max\{(a_{i_1j}^{(4)}+b_{i_1j}^{(4)})u_j^{(4)}, (a_{i_1j}^{(4)}+b_{i_1j}^{(4)}) \\
& \times v_j^{(4)}\} + \max\{(a_{i_1j}^{(5)}+b_{i_1j}^{(5)})u_j^{(5)}, (a_{i_1j}^{(5)}+b_{i_1j}^{(5)})v_j^{(5)}\} + \max\{(a_{i_1j}^{(6)}+b_{i_1,j}^{(6)})u_j^{(6)}, (a_{i_1j}^{(6)}+b_{i_1j}^{(6)}) \\
& \times v_j^{(6)}\} + \max\{(a_{i_1j}^{(7)}+b_{i_1j}^{(7)})u_j^{(7)}, (a_{i_1j}^{(7)}+b_{i_1j}^{(7)})v_j^{(7)}\}\Big] + k_{i_1}^{(0)} \\
= & \check{G}_{i_1}^{(0)}(\check{p}_{i_1}^{(0)}) > 0, \qquad (4.25)
\end{aligned}
$$

which is again a contradiction. So our assumption is wrong. Hence, each region $\Omega^\gamma \in \Omega$ is positively invariant. □

For the index $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, ..., \tilde{\omega}_{8n})$ where $\tilde{\omega}_i$ is "l" or "r" and for any $\hat{d}_i^{(b)}$ and $\check{d}_i^{(b)}$, where $\hat{d}_i^{(b)} \in (\hat{a}_i^{(b)}, \hat{b}_i^{(b)})$, $\check{d}_i^{(b)} \in (\check{b}_i^{(b)}, \check{c}_i^{(b)})$, let us denote

$$
\begin{aligned}
\tilde{\Delta}^{\tilde{\omega}} = \Big\{ & (x_1, x_2, ..., x_n)^T \in \mathbb{O}^n | x_i^{(0)} \in \overline{I}_i^{(0)\tilde{\omega}_i}, x_i^{(1)} \in \overline{I}_i^{(1)\tilde{\omega}_{n+i}}, x_i^{(2)} \in \overline{I}_i^{(2)\tilde{\omega}_{2n+i}}, ..., \\
& x_i^{(7)} \in \overline{I}_i^{(7)\tilde{\omega}_{7n+i}}, \text{for } i = 1, 2, .., n\Big\},
\end{aligned}
$$

where $\overline{I}_i^{(b)l} = (\check{p}_i^{(b)}, \hat{d}_i^{(b)})$, $\overline{I}_i^{(b)r} = (\check{d}_i^{(b)}, \hat{p}_i^{(b)})$.

Let

$$\tilde{\Delta} = \left\{ \tilde{\Delta}^{\tilde{\omega}} | \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, ..., \tilde{\omega}_{8n}) \text{ with } \tilde{\omega}_i \text{ is "}l,\text{" or "}r\text{"} \right\}. \tag{4.26}$$

Corollary 4.1. If Assumption 4.1 with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i, b_{ii}^{(0)} \geq 0$ and Assumption 4.2 hold, for $i = 1, 2, ..., n$. Then each region $\tilde{\Delta}^{\tilde{\omega}} \in \tilde{\Delta}$ is positively invariant.

Remark 4.3. From Theorem 4.2 and Corollary 4.1, we can say that under one more condition ($b_{ii} \geq 0$) the size of the positively invariant set can vary for any $\hat{d}_i^{(b)}$ and $\check{d}_i^{(b)}$, where $\hat{d}_i^{(b)} \in (\hat{a}_i^{(b)}, \hat{b}_i^{(b)})$ and $\check{d}_i^{(b)} \in (\check{b}_i^{(b)}, \check{c}_i^{(b)})$.

Theorem 4.3. If Assumption 4.1 with $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ and Assumption 4.2 hold, there exist 2^{8n} equilibria for the system (4.1) or (4.2), which are locally exponentially stable.

Proof. From Theorem 4.2, Ω^γ is a positive invariant set for the system (4.1), and from Theorem 4.1, system (4.1) or (4.2) must have an unique equilibrium point x^* lying in Ω^γ. Now using the translation $\overline{x}(t) = x(t) - x^*$, the system (4.1) can be transformed into the following equation as

$$\dot{\overline{x}}_i(t) = -c_i \overline{x}_i(t) + \sum_{j=1}^{n} a_{ij} \overline{g}_j(\overline{x}_j(t)) + \sum_{j=1}^{n} b_{ij} \overline{g}_j(\overline{x}_j(t - \tau_j(t))), \tag{4.27}$$

where $\overline{x}_i^{(b)} = x_i^{(b)}(t) - x_i^{*(b)}$ and $\overline{g}_j^{(b)}(\overline{x}_j^{(b)}(t)) = g_j^{(b)}(x_j^{(b)}(t)) - g_j^{(b)}(x_j^{*(b)})$. Equation (4.27) breaks into eight real states delay differential equations as

$$\begin{aligned}
\dot{\overline{x}}_i^{(0)}(t) = -c_i\overline{x}_i^{(0)}(t) + \sum_{j=1}^{n}\Big[& a_{ij}^{(0)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) - a_{ij}^{(1)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t)) - a_{ij}^{(2)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) - a_{ij}^{(3)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t)) \\
& - a_{ij}^{(4)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(5)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) - a_{ij}^{(6)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) - a_{ij}^{(7)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t))\Big] + \\
& \sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) - \\
& b_{ij}^{(3)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t))) - b_{ij}^{(4)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) - b_{ij}^{(5)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t))) - \\
& b_{ij}^{(6)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(7)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t)))\Big],
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\dot{\overline{x}}_i^{(1)}(t) = -c_i\overline{x}_i^{(1)}(t) + \sum_{j=1}^{n}\Big[& a_{ij}^{(0)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t)) + a_{ij}^{(1)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) + a_{ij}^{(2)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t)) - a_{ij}^{(3)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) \\
& + a_{ij}^{(4)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) - a_{ij}^{(5)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(6)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) + a_{ij}^{(7)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t))\Big] + \\
& \sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(1)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(2)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t))) - \\
& b_{ij}^{(3)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(4)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t))) - b_{ij}^{(5)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) - \\
& b_{ij}^{(6)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t)))\Big],
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\dot{\overline{x}}_i^{(2)}(t) = -c_i\overline{x}_i^{(2)}(t) + \sum_{j=1}^{n}\Big[& a_{ij}^{(0)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) + a_{ij}^{(2)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) - a_{ij}^{(1)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t)) + a_{ij}^{(3)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t)) \\
& + a_{ij}^{(4)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) - a_{ij}^{(6)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) + a_{ij}^{(5)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) - a_{ij}^{(7)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t))\Big] + \\
& \sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(2)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t))) + \\
& b_{ij}^{(3)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(4)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(6)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) + \\
& b_{ij}^{(5)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t)))\Big],
\end{aligned} \tag{4.30}$$

$$\dot{\overline{x}}_i^{(3)}(t) = -c_i\overline{x}_i^{(3)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t)) + a_{ij}^{(3)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) + a_{ij}^{(1)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) - a_{ij}^{(2)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t))$$
$$+ a_{ij}^{(4)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) - a_{ij}^{(7)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(5)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) + a_{ij}^{(6)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t))) + b_{ij}^{(3)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(1)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) -$$
$$b_{ij}^{(2)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(4)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(5)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t)))\Big], \tag{4.31}$$

$$\dot{\overline{x}}_i^{(4)}(t) = -c_i\overline{x}_i^{(4)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) + a_{ij}^{(4)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) - a_{ij}^{(1)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) + a_{ij}^{(5)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t))$$
$$-a_{ij}^{(2)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) + a_{ij}^{(6)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) - a_{ij}^{(3)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) + a_{ij}^{(7)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) + b_{ij}^{(4)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t)))$$
$$+ b_{ij}^{(5)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) -$$
$$b_{ij}^{(3)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t)))\Big], \tag{4.32}$$

$$\dot{\overline{x}}_i^{(5)}(t) = -c_i\overline{x}_i^{(5)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) + a_{ij}^{(5)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) + a_{ij}^{(1)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(4)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t))$$
$$- a_{ij}^{(2)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) + a_{ij}^{(7)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) + a_{ij}^{(3)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) - a_{ij}^{(6)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t))) + b_{ij}^{(5)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(1)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(4)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) - b_{ij}^{(2)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) +$$
$$b_{ij}^{(3)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) - b_{ij}^{(6)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t)))\Big], \tag{4.33}$$

$$\dot{\overline{x}}_i^{(6)}(t) = -c_i\overline{x}_i^{(6)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) + a_{ij}^{(6)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) + a_{ij}^{(2)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(4)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t))$$
$$+ a_{ij}^{(1)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) - a_{ij}^{(7)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t)) - a_{ij}^{(3)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) + a_{ij}^{(5)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) + b_{ij}^{(6)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) + b_{ij}^{(2)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) -$$
$$b_{ij}^{(4)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) + b_{ij}^{(1)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) - b_{ij}^{(7)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) -$$
$$b_{ij}^{(3)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t))) + b_{ij}^{(5)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t)))\Big], \tag{4.34}$$

$$\dot{\overline{x}}_i^{(7)}(t) = -c_i\overline{x}_i^{(7)}(t) + \sum_{j=1}^{n}\Big[a_{ij}^{(0)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t)) + a_{ij}^{(7)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t)) - a_{ij}^{(1)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t)) + a_{ij}^{(6)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t))$$
$$+ a_{ij}^{(2)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t)) - a_{ij}^{(5)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t)) + a_{ij}^{(3)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t)) - a_{ij}^{(4)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t))\Big] +$$
$$\sum_{j=1}^{n}\Big[b_{ij}^{(0)}\overline{g}_j^{(7)}(\overline{x}_j^{(7)}(t-\tau_j(t))) + b_{ij}^{(7)}\overline{g}_j^{(0)}(\overline{x}_j^{(0)}(t-\tau_j(t))) - b_{ij}^{(1)}\overline{g}_j^{(6)}(\overline{x}_j^{(6)}(t-\tau_j(t))) +$$
$$b_{ij}^{(6)}\overline{g}_j^{(1)}(\overline{x}_j^{(1)}(t-\tau_j(t))) + b_{ij}^{(2)}\overline{g}_j^{(5)}(\overline{x}_j^{(5)}(t-\tau_j(t))) - b_{ij}^{(5)}\overline{g}_j^{(2)}(\overline{x}_j^{(2)}(t-\tau_j(t))) +$$
$$b_{ij}^{(3)}\overline{g}_j^{(4)}(\overline{x}_j^{(4)}(t-\tau_j(t))) - b_{ij}^{(4)}\overline{g}_j^{(3)}(\overline{x}_j^{(3)}(t-\tau_j(t)))\Big]. \tag{4.35}$$

Let

$$\mathbf{X}_i(t) = e^{\epsilon t}\,|\overline{x}_i(t)| \text{ for } i = 1, 2, ..., n, \tag{4.36}$$

where ϵ is a constant with $0 < \epsilon < min\{c_1, c_2, ..., c_n\}$. Let $\delta > 1$ and $\mathbf{K} = \max_{1\leq i\leq n}\sup_{\theta\in[-\tau,0]}|x_i(\theta) - x_i^*|$. It follows that $\mathbf{X}_i < \mathbf{K}\delta$ for $t \in [-\tau, 0]$ and $i = 1, 2, ..., n$. We have to prove that $\mathbf{X}_i(t) < \mathbf{K}\delta$ for all $t > 0$, $i = 1, 2, ..., n$. If possible let us suppose that it does not hold, then there exists an index $k \in \{1, 2, ..., n\}$ and a time $t_1 > 0$ such that $\mathbf{X}_\mathbf{i}(t) \leq \mathbf{K}\delta$, $t \in [-\tau, t_1]$ and $\mathbf{X}_k(t_1) = \mathbf{K}\delta$ with $\dot{\mathbf{X}}_k(t_1) \geq 0$. Since $\mathbf{X}_k(t_1) = \mathbf{K}\delta > 0$, we have $\mathbf{X}_k(t_1) \neq 0$ which implies $|\overline{x}_k(t)|$ and $\mathbf{X}_k(t)$ are differentiable at $t = t_1$. Note that, by using positive invariant and the definition of

$g_j^{(b)}(\cdot)$, we get

$$\overline{g}_j^{(b)}(\overline{x}_j(t_1)) = \overline{g}_j^{(b)}(\overline{x}_j(t_1 - \tau_j(t_1))) = 0, \tag{4.37}$$

where $j = 1, 2, ..., n$. Now the derivative of $|\overline{g}_k(t)|$ can be computed at t_1 by using the equations (4.28)-(4.35) and (4.37).

$$\begin{aligned}
\frac{d}{dt}|\overline{x}_k(t)| &= \frac{d}{dt}[(\overline{x}_k^{(0)}(t))^2 + (\overline{x}_k^{(1)}(t))^2 + ... + (\overline{x}_k^{(7)}(t))^2]^{\frac{1}{2}}_{t=t_1} \\
&= \frac{1}{|\overline{x}_k(t_1)|}[\overline{x}_k^{(0)}(t_1)\dot{\overline{x}}_k^{(0)}(t_1) + \overline{x}_k^{(1)}(t_1)\dot{\overline{x}}_k^{(1)}(t_1) + x_k^{(2)}(t_1)\dot{\overline{x}}_k^{(2)}(t_1) + ... + \overline{x}_k^{(7)}(t_1)\dot{\overline{x}}_k^{(7)}(t_1)] \\
&= \frac{-c_k}{|\overline{x}_k(t_1)|}[(\overline{x}_k^{(0)}(t_1))^2 + (\overline{x}_k^{(1)}(t_1))^2 + (\overline{x}_k^{(2)}(t_1))^2 +, ... + (\overline{x}_k^{(0)}(t_1))^2] \\
&= -c_k|x_k(t_1)|.
\end{aligned} \tag{4.38}$$

Therefore, we have

$$\begin{aligned}
\frac{\mathbf{X}_k(t_1)}{dt} &= \epsilon e^{\epsilon t_1}|\overline{x}_k(t_1)| - e^{\epsilon t_1}c_k|\overline{x}_k(t_1)| \\
&= -(c_k - \epsilon)\mathbf{X}_k(t_1) \\
&= -(c_k - \epsilon)\mathbf{K}\delta < 0,
\end{aligned} \tag{4.39}$$

which contradicts $\dot{\mathbf{X}}_k(t_1) \geq 0$. So we must have $\mathbf{X}_i(t) < \mathbf{K}\delta$ for all $t > 0$ and $i = 1, 2, ..., n$. By taking $\delta \to 1^+$, we can write $\mathbf{X}_i(t) < \mathbf{K}$ for all $t > 0$, $i = 1, 2, ..., n$ and then

$$|x_i(t) - x_i^*| < e^{-\epsilon t} \max_{1 \leq i \leq n}(\sup_{\theta \in [-\tau, 0]} |x_i(\theta) - x_i^*|). \tag{4.40}$$

Therefore $x(t)$ is exponentially convergent to x^* with convergent rate $\epsilon > 0$. This completes the proof. □

4.4 Numerical Example

In this section two numerical examples have been considered to illustrate the effectiveness and accuracy of the derived results.

Example 4.1. *Assume the parameters of OVNNs* (4.1) *or* (4.2) *as*

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{4.41}$$

$$\begin{aligned} k_1 =& 0.1e_0 - 0.2e_1 - 0.2e_2 + 0.3e_3 + 0.1e_4 - 0.2e_5 - 0.2e_6 + 0.3e_7, \\ k_2 =& -0.2e_0 + 0.1e_1 + 0.2e_2 - 0.1e_3 - 0.2e_4 + 0.1e_5 + 0.2e_6 - 0.1e_7, \end{aligned} \tag{4.42}$$

$\tau_j(t) = 1 \ \forall \ j = 1, 2,$

$$g_1^{(b)}(z) = g_2^{(b)}(z) = \frac{|z+1| - |z-1|}{2}, \ \textit{clearly} \ g_1^{(b)}(z) = g_2^{(b)}(z) = \begin{cases} -1 & , \ (-\infty, -1) \\ z & , \ z \in [-1, 1] \\ 1 & , \ z \in (1, +\infty) \end{cases}, \tag{4.43}$$

where $b = 0, 1, 2, ..., 7,$ *and*

$$\begin{aligned}
a_{11} =& 8e_0 + 0.5e_1 - 0.5e_2 - e_3 + e_4 + 0.5e_5 - 0.5e_6 - e_7, \\
a_{12} =& e_0 - 0.8e_1 + 0.5e_2 + 0.2e_3 + e_4 - 0.8e_5 + 0.5e_6 + 0.2e_7, \\
a_{21} =& e_0 + 0.6e_1 + 0.6e_2 - 0.5e_3 + e_4 + 0.6e_5 + 0.6e_6 - 0.5e_7, \\
a_{22} =& 5e_0 - 0.8e_1 + 0.4e_2 + 0.5e_3 + e_4 - 0.8e_5 + 0.4e_6 + 0.5e_7, \\
b_{11} =& 7e_0 - e_1 - 0.3e_2 + 0.2e_3 + e_4 - e_5 - 0.3e_6 + 0.2e_7, \\
b_{12} =& 0.5e_0 + 0.6e_1 - 0.4e_2 + 0.8e_3 + 0.5e_4 + 0.6e_5 - 0.4e_6 + 0.8e_7, \\
b_{21} =& - 0.5e_0 - e_1 + 0.2e_2 + e_3 - 0.5e_4 - e_5 + 0.2e_6 + e_7, \\
b_{22} =& 8e_0 + 0.6e_1 + 0.4e_2 - 0.5e_3 + 1e_4 + 0.6e_5 + 0.4e_6 - 0.5e_7. \qquad (4.44)
\end{aligned}$$

Here $r_i^{(b)} = -1,\ s_i^{(b)} = 1,\ p_i^{(b)} = -1,\ q_i^{(b)} = 1,$ *and* $\alpha_i^{(b)} = 1,$ *where* $i = 1, 2, b = 0, 1, 2, ..., 7.$ *Clearly Assumption 4.1 and condition* $\alpha_i^{(b)}(a_{ii}^{(0)} + b_{ii}^{(0)}) > c_i$ *hold. Now we can compute* $\hat{G}_i^{(b)}(z)$, $\check{G}_i^{(b)}(z)$ *for* $i = 1, 2, b = 0, 1, ..., 7$ *as*

$$\begin{aligned}
\hat{G}_1^{(b)}(z) =& - c_1 z + (a_{11}^{(0)} + b_{11}^{(0)})g_1^{(b)}(z) + \hat{\eta}_1^{(b)} = -2z + 15g_1^{(b)}(z) + \hat{\eta}_1^{(b)}, \\
\check{G}_1^{(b)}(z) =& - c_1 z + (a_{11}^{(0)} + b_{11}^{(0)})g_1^{(b)}(z) + \check{\eta}_1^{(b)} = -2z + 15g_1^{(b)}(z) + \check{\eta}_1^{(b)}, \\
\hat{G}_2^{(b)}(z) =& - c_2 z + (a_{22}^{(0)} + b_{22}^{(0)})g_2^{(b)}(z) + \hat{\eta}_2^{(b)} = -3z + 13g_1^{(b)}(z) + \hat{\eta}_2^{(b)}, \\
\check{G}_2^{(b)}(z) =& - c_2 z + (a_{22}^{(0)} + b_{22}^{(0)})g_2^{(b)}(z) + \check{\eta}_2^{(b)} = -3z + 13g_2^{(b)}(z) + \check{\eta}_2^{(b)},
\end{aligned}$$

where $\hat{\eta}_1^{(0)} = 11.9 = \hat{\eta}_1^{(4)}, \hat{\eta}_1^{(1)} = 11.6 = \hat{\eta}_1^{(5)}, \hat{\eta}_1^{(2)} = 11.6 = \hat{\eta}_1^{(6)}, \hat{\eta}_1^{(3)} = 12.1 = \hat{\eta}_1^{(7)}, \check{\eta}_1^{(0)} = -11.7 = \check{\eta}_1^{(4)}, \check{\eta}_1^{(1)} = -12 = \check{\eta}_1^{(5)}, \check{\eta}_1^{(2)} = -12 = \check{\eta}_1^{(6)}, \check{\eta}_1^{(3)} = -11.5 = \check{\eta}_1^{(7)}, \hat{\eta}_2^{(0)} = 8.2 = \hat{\eta}_2^{(4)}, \hat{\eta}_2^{(1)} = 8.5 = \hat{\eta}_2^{(5)}, \hat{\eta}_2^{(2)} = 8.6 = \hat{\eta}_2^{(6)}, \hat{\eta}_2^{(3)} = 8.3 = \hat{\eta}_1^{(7)}, \check{\eta}_2^{(0)} = -8.6 = \check{\eta}_2^{(4)}, \check{\eta}_2^{(1)} = -8.3 = \check{\eta}_2^{(5)}, \check{\eta}_2^{(2)} = -8.2 = \check{\eta}_2^{(6)}, \check{\eta}_2^{(3)} = -8.5 = \check{\eta}_2^{(7)}$. *Putting these*

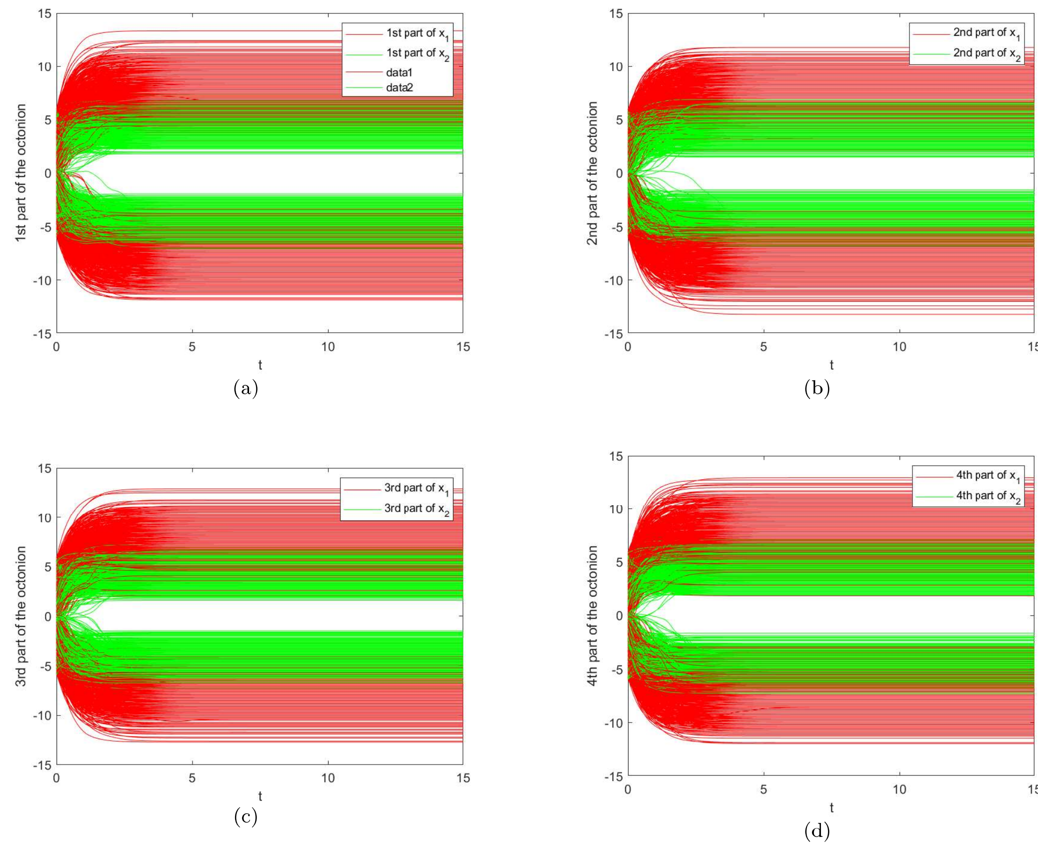

FIGURE 4.2: Figures (a)-(d) demonstrate state trajectories of the 1st part-4th part of the system (4.1), respectively.

values in $\hat{G}_1^{(b)}(z), \check{G}_1^{(b)}(z), \hat{G}_2^{(b)}(z), \check{G}_2^{(b)}(z)$, *we get*

$$\hat{G}_1^{(0)}(p_i^{(0)}) = -1.1 < 0, \hat{G}_1^{(1)}(p_i^{(1)}) = -1.4 < 0, \hat{G}_1^{(2)}(p_i^{2}) = -1.4 < 0, \hat{G}_1^{(3)}(p_i^{(3)}) = -0.9 < 0,$$

$$\hat{G}_1^{(4)}(p_i^{(4)}) = -1.1 < 0, \hat{G}_1^{(5)}(p_i^{(5)}) = -1.4 < 0, \hat{G}_1^{(6)}(p_i^{(6)}) = -1.4 < 0, \hat{G}_1^{(7)}(p_i^{(7)}) = -0.9 < 0,$$

$$\check{G}_1^{(0)}(q_i^{(0)}) = 1.3 > 0, \check{G}_1^{(1)}(q_i^{(1)}) = 1 > 0, \check{G}_1^{(2)}(q_i^{(2)}) = 1 > 0, \check{G}_1^{(3)}(q_i^{(3)}) = 1.5 > 0,$$

$$\check{G}_1^{(4)}(q_i^{(4)}) = 1.3 > 0, \check{G}_1^{(5)}(q_i^{(5)}) = 1 > 0, \check{G}_1^{(6)}(q_i^{(6)}) = 1 > 0, \check{G}_1^{(7)}(q_i^{(7)}) = 1.5 > 0,$$

$$\hat{G}_2^{(0)}(p_i^{(0)}) = -1.8 < 0, \hat{G}_2^{(1)}(p_i^{(1)}) = -1.5 < 0, \hat{G}_2^{(2)}(p_i^{(2)}) = -1.4 < 0, \hat{G}_2^{(3)}(p_i^{(3)}) = -1.7 < 0,$$

$$\hat{G}_2^{(4)}(p_i^{(4)}) = -1.8 < 0, \hat{G}_2^{(5)}(p_i^{5}) = -1.5 < 0, \hat{G}_2^{(6)}(p_i^{(6)}) = -1.4 < 0, \hat{G}_2^{(7)}(p_i^{(7)}) = -1.7 < 0,$$

$$\check{G}_2^{(0)}(q_i^{(0)}) = 1.4 > 0, \check{G}_2^{(1)}(q_i^{(1)}) = 1.7 > 0, \check{G}_2^{(2)}(q_i^{(2)}) = 1.8 > 0, \check{G}_2^{(3)}(q_i^{(3)}) = 1.5 > 0,$$

$$\check{G}_2^{(4)}(q_i^{(4)}) = 1.4 > 0, \check{G}_2^{(5)}(q_i^{(5)}) = 1.7 > 0, \check{G}_2^{(6)}(q_i^{(6)}) = 1.8 > 0, \check{G}_2^{(7)}(q_i^{(7)}) = 1.5 > 0.$$

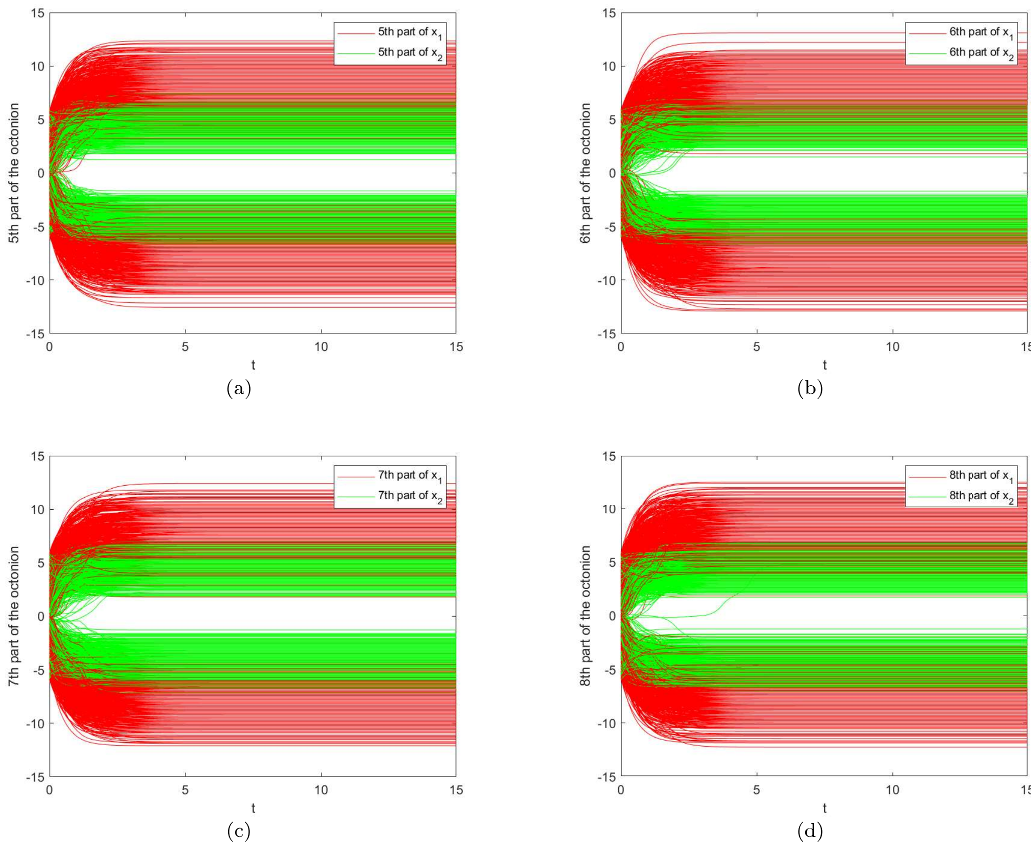

FIGURE 4.3: Figures (a)-(d) demonstrate state trajectories of the 5th part-8th part of the system (4.1), respectively.

Thus it can be concluded that the conditions from Assumption 4.2 viz., $\hat{G}_i^{(b)}(p_i^{(b)}) < 0$, $\check{G}_i^{(b)}(q_i^{(b)}) > 0$ *are satisfied. Hence from Theorem 4.1 and Theorem 4.3, the OVNNs* (4.1) *have* 3^{16} *equilibrium points, out of which* 2^{16} *are exponentially stable. We have extended the Quaternion Toolbox in MATLAB to implement the above OVNNs and leveraged to simulate the said example by using fourth order Runge Kutta method. By setting* 100000 *initial conditions randomly, one can obtain* 100000 *corresponding numerical solutions. These numerical solutions converge into* 65336 *stable state, three of which are listed as follows:*

$$\begin{pmatrix} 8.3500e_0 + 5.5000e_1 - 8.5000e_2 - 5.6000e_3 - 2.9500e_4 - 8.8000e_5 + 11.0000e_6 + 8.5000e_7 \\ 3.1333e_0 - 6.6667e_1 - 3.5333e_2 + 3.1667e_3 + 3.9333e_4 - 4.7667e_5 + 5.2667e_6 - 4.5667e_7 \end{pmatrix},$$

$$\begin{pmatrix} 8.3500e_0 + 7.6000e_1 + 4.5000e_2 + 7.9500e_3 + 5.4500e_4 - 10.1000e_5 - 10.4000e_6 - 7.7500e_7 \\ 6.2667e_0 + 5.2333e_1 - 4.6000e_2 - 5.1667e_3 - 3.4667e_4 + 3.7667e_5 - 4.0667e_6 + 4.7000e_7 \end{pmatrix},$$

$$\begin{pmatrix} 8.3500e_0 - 7.6000e_1 - 8.2000e_2 - 9.0500e_3 + 10.0500e_4 + 7.7000e_5 + 8.3000e_6 - 7.3500e_7 \\ 5.1333e_0 - 4.1000e_1 - 4.0667e_2 - 4.7000e_3 - 4.2000e_4 - 4.2000e_5 - 1.6667e_6 - 4.1667e_7 \end{pmatrix}.$$

Here, only three equilibrium points are listed due to space limitations. The dynamics of the considered system are demonstrated in Figs. 4.2(a)–4.3(d), where the initial conditions are chosen by 1000 *random constant octonion-valued vectors. From the Figs.4.2(a)–4.3(d), it can be concluded that each neuron state converges into the stable state.*

Next an application of associative memory of OVNNs is illustrated, which shows their efficiency over quaternion and complex valued NNs.

Example 4.2. *Consider the color image pattern "V" as shown in Fig 4.4 and design the OVNNs in the form of the considered model* (4.1) *for associatively memorizing this color image. As shown in Fig. 4.4, the size of the image "V" is 12 × 12 pixels. We have been able to design OVNNs* (4.1) *composed of 72 neurons that have a 72-D equilibrium point for storing the colors of the pattern. Assume that the parameters of OVNNs* (4.1) *are given as*

$$c_i = 1, \tag{4.45}$$

$$a_{ij} = \begin{cases} 4e_o + 0.4e_1 - 0.3e_2 + 0.5e_3 + 0.4e_4 - 0.5e_5 + 0.5e_6 - 0.3e_7, & i = j \\ 0.4e_o - 0.5e_1 + 0.5e_2 - 0.3e_3 + 0.4e_4 - 0.5e_5 + 0.5e_6 - 0.3e_7, & i \neq j \end{cases}, \tag{4.46}$$

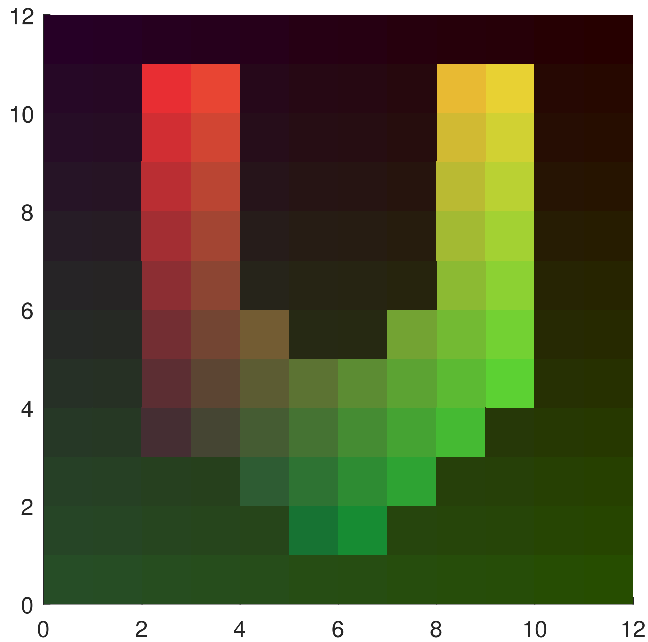

FIGURE 4.4: Plot of the original color image of pattern "V".

$$b_{ij} = \begin{cases} -0.2e_o + 0.2e_1 - 0.5e_2 + 0.4e_3 - 0.2e_4 + 0.2e_5 - 0.5e_6 + 0.4e_7, & i < j \\ 2e_o + 0.3e_1 - 0.2e_2 - 0.3e_3 - 0.2e_4 + 0.2e_5 - 0.5e_6 + 0.4e_7, & i = j \\ -0.1e_o - 0.2e_1 + 0.3e_2 - 0.5e_3 - 0.1e_4 + 0.2e_5 + 0.3e_6 - 0.5e_7, & i > j \end{cases}, \tag{4.47}$$

$$g_i^{(b)}(z) = 10\left(\frac{|z+2| - |z+1|}{2}\right), \quad \tau_j(t) = 1, \tag{4.48}$$

where $i, j = 1, 2, ..., 72$ *and* $b = 0, 1, 2, ..., 7$. *Now, the color image pattern "V" is stored by using RGB color model. A color in such model is represented by indicating how much of each of the red, green, and blue is included. The color is expressed as an RGB triplet* $(\cdot, \cdot, \cdot)$. *In order to store the color image pattern "V", the equilibrium point of the above designed OVNNs should be of the form*

$$x = (x_1, x_2, ..., x_{72}) \in \mathbb{O}^{72},$$

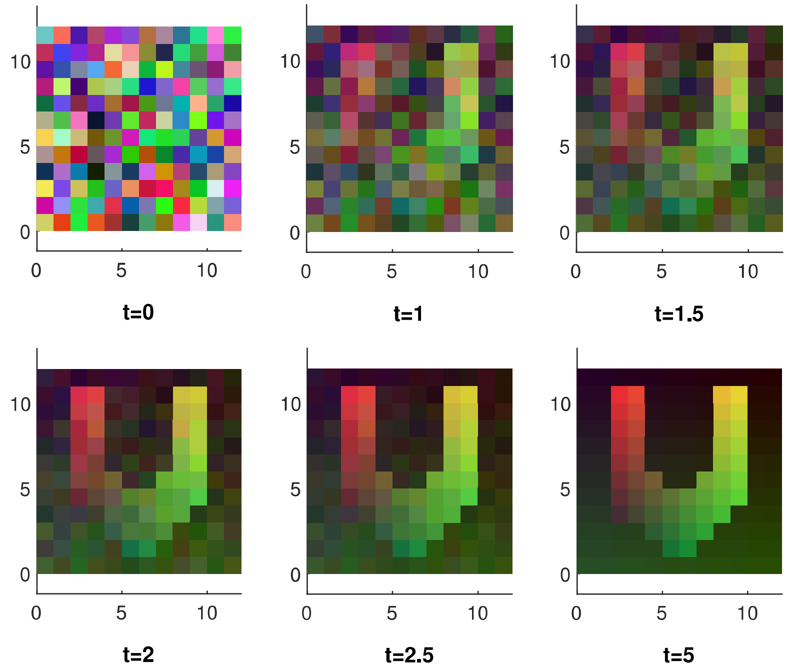

FIGURE 4.5: Plots of numerical simulation for retrieving the pattern "V" with random initial values at time, t = 0, 1, 1.5, 2, 2.5 and 5.

where $x_1 = (0e_0 + 0.15e_1 + 0.3e_2 + 0.15e_3 + 0e_4 + 0.15e_5 + 0.27e_6 + 0.15e_7), x_2 = (0e_0 + 0.15e_1 + 0.3e_2 + 0.14e_3 + 0e_4 + 0.15e_5 + 0.27e_6 + 0.14e_7), ..., x_{72} = (0e_0 + 0.15e_1 + 0.03e_2 + 0e_3 + 0e_4 + 0.15e_5 + 0e_6 + 0e_7)$, *which correspond with the color* $\{(0.15, 0.3, 0.15)$ *and* $(0.15, 0.27, 0.15)\}, \{(0.15, 0.3, 0.14)$ *and* $(0.15, 0.27, 0.14)\}, ...,$ $\{(0.15, 0.03, 0)$ *and* $(0.15, 0, 0)\}$ *of the pixels in the color image pattern "V". Note that for each* $i = 1, 2, ..., 72$, x_i*'s store the RGB values of two pixels. According to the above equilibrium point* x, *we have the external input as*

$$k = (k_1, k_2, ..., k_{72}) \in \mathbb{O}^{72}, \tag{4.49}$$

where $k_1 = (-48.2e_0 + 35.95e_1 - 119.5e_2 + 109.2e_3 - 49.4e_4 - 274.1e_5 + 22.07e_6 + 106.8e_7), k_2 = (-48e_0 + 35.95e_1 - 117.9e_2 + 107.3e_3 - 49.2e_4 - 270.5e_5 + 20.47e_6 + 105.3e_7), ..., k_{72} = (-34e_0 + 35.95e_1 - 6.17e_2 - 18.8e_3 - 35.2e_4 - 18.45e_5 - 91.8e_6 + 7.2e_7)$.

Due to page limitations, only three elements of x and k are mentioned. By doing a simulation with random initial values as depicted in Fig. 4.5, it can be concluded that the designed OVNNs with parameters (4.45)-(4.49) *have the ability to recall the above pattern "V" reliably.*

Remark 4.4. For remembering and retrieving true colored images, the approach proposed in [44], has designed CVNNs with 432 neurons to store a 12×12 pixel figure. Similarly, as discussed in [7], 144 neurons are required to store 12×12 while using QVNNs. However, in the Example 4.2, the number of neurons is reduced by half to 72 by using OVNNs, to store the 12×12 pixel colored image, which is significantly less as compared for the cases of CVNNs and QVNNS.

Remark 4.5. It is noted that the Example 4.2 is designed only for one pattern and in OVNNs, there is no existing result for multiple patterns yet. For multiple patterns of QVNNs, a system is designed in [53].

4.5 Conclusion

In this chapter, the multistability analysis is studied for n-dimensional octonion valued neural networks (OVNNs) with time-varying delays for a general class of activation functions. Firstly, OVNNs are decomposed into eight real-valued systems, and then based on geometrical properties of activation functions, 3^{8n} disjoint regions are constructed in $\mathbb{O}^n$. Then, by using the inequality technique and Brouwer's fixed point theorem, several sufficient conditions are obtained to ensure the existence of 3^{8n} equilibrium points of the system, each of which is located in one of the regions, and 2^{8n} of those are locally exponentially stable. Moreover positively invariant sets are also estimated in this scientific contribution. Two numerical examples are provided to illustrate the effectiveness of the obtained results. Especially, the numerical

Example 2.2 demonstrates that the designed OVNNs work efficiently on storing and retrieving the truecolor images. The present work can be extended further with mixed delay terms and also with different types of activation functions viz., non-monotonic piece-wise nonlinear activation functions and discontinuous activation functions.

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