Chapter 1

Introduction

While performing design and analysis of nonlinear dynamical systems in electrical, mechanical, control and other engineering disciplines, one needs to deal with nonlinear systems analysis tools. In the past many years, Lyapunov theory and contraction analysis have gained significant attention for the development of the stability theory of the nonlinear dynamical systems. Despite the extensive research in these areas, in most of the cases, the literature keeps using scalar energy-like or distance function which makes these analysis somewhat restrictive particularly for large-scale systems. So, it is important to think of a vector framework that relaxes these restrictions and presents a simplified structure to perform stability analysis.

1.0.1 Why do we need vectors?

Let us consider that you have different types of fruits like apples, oranges and bananas, etc. Suppose, if you want to add 10 kg amount of quantity to every fruit, then it is quite elegant if one can go for vector addition. Let v be the vector of fruits:

$$v \text{ (Fruits)} = \begin{bmatrix} v_1 \text{ (Apples)} \\ v_2 \text{ (Oranges)} \\ v_3 \text{ (Bananas)} \end{bmatrix}$$

where v_1 is the first component of v representing 1 kg of apples, v_2 is the second component of v representing 1 kg of oranges and v_3 is the third component of v representing 1 kg of bananas. Then, if we add 10 kg to a vector v, this 10 kg is automatically added to every component of v, that is,

$$v + 10 = \begin{bmatrix} v_1 + 10 \\ v_2 + 10 \\ v_3 + 10 \end{bmatrix}$$

In this way, when we have more than one quantity or variable, then addition or multiplication using vectors is pretty easy.

1.0.2 Why vector energy-like function (Vector Lyapunov function)?

Scalar Lyapunov function is an interesting tool that converts a given complicated higher-dimensional system into a relatively simpler scalar dynamical system and thus it is enough to analyze this simpler dynamical system. However, it can be noticed that, the more complex the dynamical system is, the more difficult it is to find a scalar Lyapunov function for the analysis of stability, since the construction of a suitable scalar Lyapunov function with the restricted assumption of positive definiteness is not easy as there is no specified method to construct it. Moreover, to prove that the derivative of the Lyapunov function should be negative or negative semi-definite to guarantee asymptotic stability or stability about the origin is also a difficult assumption to be verified. Thus, it is relevant to apply the approach which transforms a complex large-scale dynamical system into several simple dynamical systems of reduced dimension. Also, there is a great need to weaken the hypothesis of Lyapunov stability to enlarge the class of Lyapunov theory. Thus, there comes a notion of vector Lyapunov functions (VLF) for which each component gives information about a part of the dynamics being analyzed.

Moreover, it is well known that employing vector Lyapunov functions instead of a single scalar Lyapunov function is more advantageous as it offers a flexible framework that each function satisfies less strict conditions to guarantee stability. These vector Lyapunov functions have potential applications in guaranteeing the stability of the large-scale systems as these systems can be decomposed into several subsystems and scalar Lyapunov function corresponding to each subsystem can be used to guarantee the stability of the whole system.

Rated Stability

For nonlinear systems, various notions of stability, such as asymptotic and exponential stability, are used to describe the convergence of their trajectories to an equilibrium point in infinite time duration. However, as can be noted in the industrial and engineering sectors, several essential applications require a convergence of the trajectories to the equilibrium in finite/fixed-time or a pre-specified time. Researchers studied the finite-time stability of autonomous systems using continuous Holder energy functions [1]. In most of these finite-time problems, the time of convergence primarily depends on initial conditions, which is indeed a crucial feature. The notion of fixed-time convergence to the equilibrium point has been introduced to overcome finite-time stability limitations. Fixed-time stability of systems has been investigated in particular in [2, 3], in which uniformity relative to the initial conditions is required for computing the upper bound of the settling time and this bound depends explicitly on system parameters. As a matter of fact, in various applications, it is advantageous to show the convergence of the trajectories in the time of our wish, this is the case, for example, for differentiators and missile guidance. It can be noted that literature keeps using scalar Lyapunov functions as the key tools of the proofs for arbitrary-time convergent systems. It is quite appreciated if one can perform analysis using vector Lyapunov functions that relax certain strict conditions of scalar Lyapunov functions [4–6].

1.0.3 Why vector distance function (Vector Contraction analysis)?

Trouble comes in some situations that how to find a suitable Lyapunov function candidate. One other restriction is also observed in Lyapunov methods, they provide stability relative to some attractor or equilibrium point. Therefore, to minimize troubles, an alternative theory to Lyapunov stability, inspired from fluid mechanics to prove convergence (stability in differential framework) known as contraction theory was first popularized in [10]. In general, stability is viewed with respect to some specific attractor or equilibrium point, however contraction theory is based on slightly different view of what the stability is. In fact, stability can be analyzed differentially which reveals whether neighboring trajectories converge with respect to each other rather than with respect to some specific attractor as in the case of Lyapunov theory. This form of stability is called as convergence to remove any ambiguity. As an illustration, consider the $n-$dimensional differential system

$$\Sigma : \quad \dot{x} = f(t,x), \qquad x(t_0) = x_0 \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear vector field and $t \in \mathbb{R}_{\geq 0}$ is the time. From the perspective of the fluid dynamics, x and $\dot{x}$ denote the $n-$dimensional position and velocity of the fluid flow. We assume that the system Σ has a unique solution $\psi(t, x_0)$. Let δx be the virtual displacement between any two neighboring trajectories of the vector field of (1.1). Then, the squared scalar distance between these two trajectories is $\|\delta x\|^2 = \delta x^\top \delta x$ (see [10]). Let x is displaced by δx displacement, then the dynamics Σ becomes:

$$(\dot{x} + \delta\dot{x}) = f(t, x + \delta x) \tag{1.2}$$

Now, substracting (1.1) from (1.2), we get,

$$\delta\dot{x} = f(t, x + \delta x) - f(t,x). \tag{1.3}$$

As f is smooth, on applying Taylor's series expansion, $f(t, x+\delta x) = f(t,x) + \frac{\partial f(t,x)}{\partial x}\delta x + \cdots$, ignoring higher order terms, we get the variational system as

$$\Sigma_\delta : \ \delta\dot{x} \ = \ \frac{\partial f(t, \psi(t, x_0))}{\partial x}\delta x \ = \ h(t, x, \delta x), \tag{1.4}$$

where $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\delta x \in \mathbb{R}^n$ and t is time. To denote the solution to Σ_δ, we use $\delta\psi(t, x_0, \delta x_0)$ from the initial state δx_0 at time t along $\psi(t, x_0)$. Thus, the rate of change of $\delta x^\top \delta x$ is given by

$$\frac{d}{dt}(\delta x^\top \delta x) = 2\delta x^\top \delta\dot{x} = 2\delta x^\top \frac{\partial f}{\partial x}\delta x.$$

Let $\lambda_{\max}(t,x)$ be the largest eigenvalue of the symmetric part of the Jacobian matrix $\frac{\partial f}{\partial x}$, i.e., of $\frac{1}{2}\left(\frac{\partial f}{\partial x} + (\frac{\partial f}{\partial x})^\top\right)$. Then, any infinitesimal distance $\|\delta x\|$ converges exponentially to zero as $t \to \infty$, if $\lambda_{\max}(t,x)$ is uniformly negative definite (see [10]).

It can be noted that the calculation of $\lambda_{\max}(t,x)$ is not easy for large-scale nonlinear time-varying systems, since the Jacobian matrix $\frac{\partial f}{\partial x}$ becomes quite large consisting of nonlinear time-varying terms. Certainly, it is important to search for another convergence analysis technique that relaxes these restrictions and provides the convergence in a relatively simpler way. For this, the vector distance function is employed to develop a generalized theory known as vector contraction theory. This approach relaxes the negative definite derivative condition of standard contraction analysis. Specifically, the convergence analysis is performed with the help of a comparison system by comparing the relative distances of the trajectories of the original nonlinear dynamical system with that of the comparison system. This theory is further exploited to tackle the following mentioned problems.

Controller and Observer Design

The interest in recursive design-based controller design approaches (see [11–13]) and observer design techniques [14–16] for nonlinear systems such as chaotic systems has been growing over the past many years. A large number of these well-known techniques (see [17–25]) are based on the formulation of a suitable Lyapunov function to derive stability results, for example, the backstepping technique of control design requires the construction of a Lyapunov function to prove the stability of each subsystem.

In contrast, the theory of contraction [10] is well suited for controller and observer design, since in these types of problems, one is interested in the convergence of trajectories with respect to each other (like convergence of estimated and actual trajectory in the case of observer design) rather than to a particular trajectory. This theory provides significant advantages over the Lyapunov stability approach in the essence that the former does not need to know the specific attractor and does not require a difficult task of choosing appropriate Lyapunov function without any explanation of the structure of the selected function. However, this theory also witnesses some restrictions: it requires a complex procedure to prove Jacobian matrix to be negative definite (as it employs distance function between trajectories as a scalar function), since the matrix contains nonlinear terms particularly for the class of large scale highly nonlinear systems. The vector-based contraction approach is able to effectively design controller and observer for nonlinear systems particularly chaotic systems. It simplifies the problem with the help of a much simpler comparison system acquiring specific properties. In fact, results are derived by comparing the relative distances of the comparison system with that of the auxiliary system. The comparison system is chosen with the property that it is quasi-monotone nondecreasing and contracting. The analysis avoids the calculation of the Jacobian matrix which consists of highly nonlinear terms and solves the problem through a Jacobian matrix containing mainly linear terms.

Interval Observer Design

For the purpose of feedback control, all the states of a dynamical system are not measurable due to a fewer number of sensors. Thus, to reproduce all the information of these systems, that is, to estimate the unmeasured states, observers ([26]) are used. Basically, there is no need of state information in transient periods for the problems of stabilization and tracking. However, in the case of monitoring purposes and other essential practical applications, there is a great need of state estimation at all times. This type of knowledge of state estimation at all times cannot be available from classical observers like Luenberger observer ([27]). Moreover, classical observers do not provide guaranteed exact estimates in the presence of perturbations and uncertainties. To overcome these shortcomings and provide guaranteed state estimations at all times in the presence of perturbations and uncertainties, there comes the notion of interval observers ([28]). They play an important role in fulfilling the practical demands. In fact, they have a major advantage over classical observers in the sense that they can easily cope up with large uncertainties and have been successfully applied in biological fields ([29]) where large uncertainties are present. The structure of interval observer comprises of two dynamical equations that estimate upper and lower bounds (interval) of the state vector at all times when the initial conditions are unknown and there are uncertainties present in the system, however, these initial conditions and uncertainties are bounded.

In the analysis and design of interval observers, two important conditions need to be satisfied. The first is the framer property, and the other one is the convergence property. To prove the framer property one needs to have a positive dynamical system, that is, for example, for the linear systems case, the system matrix should be a Metzler matrix. This property is necessary to ensure the partial ordering of the state trajectories. In other words, if the initial conditions are bounded between two values (known), then the real state trajectory is bounded by the trajectories originating from these bounds. Vector based contraction theory is well suited to prove the convergence property of the interval observer through a *comparison system* with specified properties. It provides the advantage over other existing approaches in the way that it does not require the formulation of error dynamics and need not require the Lyapunov candidate function to show convergence.

Multi-agent systems

Many complicated systems in nature can be delineated as Multi-Agent Systems (MAS) whose topology is represented by a graph with nodes associated with individual agents communicating information over the edges reporting their interconnections. Generally speaking, the major challenge for these types of systems is to derive control algorithms in order to bring the states of all the agents to some constant value (consensus) with the assumption that each agent can share information with its neighbors. A particularly stronger theory than Lyapunov stability to assess convergence of trajectories towards each other is contraction analysis. However, contraction analysis usually adopted in the literature presents some difficulties such as it requires eigenvalues or matrix measures calculations to show the convergence of the agents, which is indeed a difficult task especially for large-scale nonlinear multi-agent systems. The extended results on contraction theory using the framework of vector distances are able to synchronize the motion of agents in multi-agent systems. The proposed results reduce the complexity of proving negative definiteness property of the largest eigenvalue or measure of the Jacobian matrix of the complex multi-agent systems. Specifically, it provides convergence analysis through a *comparison system*. As a result, the convergence analysis problem of the complicated MAS system simplifies into the convergence analysis of a much simpler *comparison system* and also the controller design for consensus is done using the convergence conditions of the *comparison system*. Hence, the procedure of designing consensus controls for multi-agent systems remains simple despite nonlinearities as compared to the existing literature.

1.1 Motivation

Recent technological demands require the analysis and control design of increasingly complex large-scale nonlinear dynamical systems such as aerospace systems, network systems, power systems and structural systems, etc. The examples can also be simply cited from our daily lives like suppose we have to estimate the research performance of some large educational Institute like IIT (BHU), then the main difficulty one faces is how to take data from such a large Institute to estimate performance. In these scenarios, it is quite desirable to treat the overall system (or Institute) as a collection of interconnected subsystems (departments). The framework of vectors offers a flexible frame in these situations. Vectors play an important role in almost every field like physics, mathematics, engineering and industries, etc, in essence that computation (addition or multiplication) using vectors is pretty easy when we have more than one system or quantity.

When we talk about control theory, the scalar Lyapunov function has been a key tool for analyzing the stability of the nonlinear dynamical systems. However, it can be noted that scalar Lyapunov functions always yield a first-order comparison system, that may represent a drastic cut down, as stability or instability of which provides the same property for the initial dynamical system. Thus, to reduce the loss of information, it is interesting to use several (vector) Lyapunov functions instead of a single (scalar) Lyapunov function. Also, there is no unified procedure of constructing scalar Lyapunov function for analyzing large-scale system stability. The more complex the dynamical system is, the more difficult it is to find a scalar Lyapunov function for the analysis of stability. Thus, it is relevant to apply the approach which transforms a complex large-scale dynamical system into several simple dynamical systems of reduced dimension. Also, there is a great need to weaken the hypothesis of Lyapunov stability to enlarge the class of Lyapunov theory. Thus, there comes a notion of vector Lyapunov function (VLF) for which each component gives information about a part of the dynamics being analyzed. Vector Lyapunov functions are more advantageous as they offer a flexible framework that each function satisfies less strict conditions to guarantee stability. We found that not much research on stability analysis and control design was available using these vector Lyapunov functions particularly for arbitrary-time stable systems.

Furthermore, we notice that trouble comes in some situations that how to find a suitable Lyapunov function candidate. Moreover, one other restriction is also observed in Lyapunov methods, they provide stability relative to some attractor or equilibrium point. These restrictions encouraged the research to think of an alternative theory to Lyapunov theory for the stability analysis of nonlinear dynamical systems. As a consequence, a theory inspired from fluid dynamics known as contraction theory is developed, which is based on a slightly different form of what the stability is, motivated by the fact that stability does not require one to have some specific attractor or equilibrium point. Intuitively, a system is said to be stable if nearby trajectories converge with respect to each other, thus, stability can also be analyzed differentially rather than through some transformations as in feedback linearization or through constructing some motion integral as in Lyapunov methods. This form of stability is termed as convergence to avoid any ambiguity. Despite the immense research in this area, we notice that in contraction analysis, literature keeps using a scalar distance function between any pair of system trajectories that witnesses a restriction that the largest eigenvalue of the symmetric part of the Jacobian matrix should be negative definite. The utilization of this scalar distance function makes the contraction analysis somewhat restrictive, particularly for a large-scale nonlinear system, since it is quite complicated to check negative definiteness property or its mild variations of the Jacobian matrix for a large-scale nonlinear system.

This motivates us to think of vector distance functions that provide a resilient framework to deal with convergence analysis of these large-scale dynamical systems, hence this theory is termed as vector contraction analysis. We wanted to provide an ease to the user to utilize the well-established structure of the vector-valued norm to perform convergence analysis without thinking too much about its construction. We also drew motivation and encouragement from the fact that designing controllers, observers, interval observers and consensus controls need not require the knowledge of some specific attractor, rather one is interested in finding the convergence of system trajectories with respect to each other. That led us to go ahead and solve these realistic problems by the exploitation of vector contraction analysis.

At last, we note that several linear and sliding mode control based designs based on scalar Lyapunov functions were implemented on an experimental Helicopter setup and a two-link manipulator, they provide asymptotic or finite-time convergence of the states. This fact motivates us to design a nonlinear control scheme using vector framework based stability analysis and implement it on these practical systems.

In the nutshell, the thesis work tried to answer the questions: (1) Whether it might be more advantageous to use vector Lyapunov functions to obtain arbitrary time stability conditions for large-scale dynamical systems? (2) Is it is more flexible to use vector distance functions than that of scalar distance functions in standard contraction analysis? (3) Is it is possible to design real-world control problems such as controller and observer design, interval observer design and consensus problems in multi-agent systems using vector framework based contraction analysis? (4) Is it is possible to implement the proposed results on experimental setups and practical systems?

1.2 Literature Review

1.2.1 Stabilization in arbitrary time

The rate of convergence is an essential attribute for a given control strategy, for attaining asymptotic, finite-time or fixed-time convergence to the equilibrium. A vast variety of research has been done on finite and fixed-time convergence and their stability conditions were obtained by exploiting mainly scalar Lyapunov functions. Finite-time stability was explored for several families of systems which include, in particular, homogeneous systems [31], switched systems with uncertainty [32], and multi-agent systems [33]. This stability was further investigated for higher-order systems [34] and using output feedback [35] as well. Finite-time estimation issues have been considered, e.g., in [36, 37]. Fixed-time stability of systems has been investigated in particular in [3] in which uniformity relative to the initial conditions is required for computing the upper bound of the settling time. Moreover, in the problems studied in [38, 39], the fixed-time of convergence depends on system parameters. Another notion of convergence that overcomes fixed-time problems' design constraints utilizing time-varying functions is prescribed time [40]. However, it lacks simplicity because of the utilization of different scaling functions. Hence, in most of the problems, convergence time depends on initial conditions and system parameters.

On the other hand, discontinuous controllers guaranteeing finite-time convergence were also developed in the literature [41]. Nevertheless, they result in chattering due to uncertainties or unmodeled dynamics in practical applications. As a matter of fact, in various applications, it is advantageous to show the convergence of the trajectories in the time of our wish, this is the case, for example, for differentiators and missile guidance. Hence, arbitrary time convergent systems have been studied in [42, 43], where scalar Lyapunov functions are the key tool of the proofs. These systems possess continuous dynamics up to the arbitrary time, and then switching occurs. The key feature of these systems is that the state and its derivative converge to zero as the time approaches the arbitrary time, independent of any initial condition and system parameters. Vector Lyapunov functions were first introduced in [44] to relax certain strict conditions of scalar Lyapunov functions [4–6]. Several works exploiting vector Lyapunov functions were reported in the literature in particular in [7–9].

1.2.2 Controller and Observer design

A large number of controller and observer design techniques in particular ([11,14,17–19]) are based on the formulation of a suitable Lyapunov function to derive stability results. On the contrary, the theory of contraction [10] is extensively used to analyze the convergence of system trajectories with respect to each other. Enormous practices of contraction theory have been done in the field of control and synchronization of chaotic systems [45–50] and coupled oscillators [52–54]. Adaptive synchronization approaches have been proposed for the chaotic circuits problems [45].

However, the contraction approach requires a complex procedure to prove the Jacobian matrix to be negative definite (as it employs distance function between trajectories as a scalar function), since the matrix contains nonlinear terms particularly for the class of large scale highly nonlinear systems. Hence, vector framework based contraction approach would be helpful for the controller and observer design and other applications with relaxed above mentioned restrictions.

1.2.3 Interval Observer design

For the design and analysis of interval observers, two important conditions need to be satisfied. The first is the framer condition, and the other one is the convergence condition. To prove the framer condition, one needs to have a positive dynamical system, that is, for example, for the linear systems case, the system matrix should be a Metzler matrix. Furthermore, if the system is a non-positive system, there are various methods available in the literature [63] for linear systems in particular [55], in which by using time-varying transformations, the non-positive system gets transformed into a positive system. In [56], an internal positive realization scheme has been used to design interval observers. Moreover, several works were reported in the literature for the design of interval observers for nonlinear systems and discrete-time systems as well (see [57–62]). The second important property for the design of interval observers is the convergence property. One needs this convergence property to show that the norm of the error between the state bounds converges to zero when the disturbance acting on the system is zero. Several researchers (see [64–66]) proved this convergence property using Lyapunov stability analysis.

But still, in these types of convergence problems, there is a need to find the convergence of the estimated state bounds and the real state trajectory with respect to each other, rather than their convergence to a particular point or set. To show convergence of trajectories with respect to each other, a particularly stronger theory than Lyapunov stability, known as contraction theory ([10]) exists in the literature. This theory differs from Lyapunov stability in the sense that it does not require a specific attractor to show stability and does not require the construction of a Lyapunov candidate function without having any knowledge of its structure. This theory analyzes the stability in terms of a differential framework, that is, a system is said to be stable in some region if its final behavior is independent of initial conditions and temporary perturbations, and hence, state trajectories converge with respect to one another, therefore, this form of stability is called convergence, to remove any ambiguity. However, contraction analysis usually utilized in the literature requires a complex procedure to solve matrix measure or largest eigenvalue of the symmetric part of the Jacobian matrix of large scale nonlinear systems to show convergence since it considers a scalar distance function. In such cases, it is desirable to prove convergence property using vector distance functions and comparison principles.

1.2.4 Multi-agent systems

The consensus problems in multi-agent systems have allured the attention of various researchers [67,68]. The different application areas of consensus problems are robotics [69], flocking problem [70], rendezvous problem [71], and networked systems [72], etc. The objective of all these problems is to drive the agents at a common location in space (where their velocities and accelerations are zero in case of third-order dynamics). Different methods have been designed in the past [73–76] to find the solution to these types of problems. Typically, such problems are generally analyzed by Lyapunov stability analysis [77], which shows the stability of some equilibrium point or invariant set. But still, in these types of problems, we are interested in finding the conditions for the convergence of the nodes in a network with respect to each other. Therefore, it is better to find the convergence of all the solutions with respect to each other instead of finding their convergence to a particular trajectory or set.

A particularly stronger theory than Lyapunov stability to assess convergence of trajectories towards each other is contraction analysis which was first popularized in [10]. Numerous practices of contraction analysis have been done for the rendezvous problem [78], cooperative control of robots [79], coordination of switched complex networks [80], and modular coordination of MAS systems with switching topologies [81], etc. On the other hand, there are various works in the literature for consensus of second-order [82] and third-order dynamics multi-agent systems, however, third-order consensus with acceleration constraints have not been analyzed for a particular interest. The constrained acceleration problem has attracted the attention of many researchers [83], due to its practical significance in surface-to-air guided missiles. Besides the acceleration constraint, control input constraint is another commonly existing restriction because of the finite power of actuators in practical systems [84]. Apart from this, the consensus for heterogeneous systems is of keen interest as in various practical applications, agents with different dynamics are coupled with each other because of various strict conditions or the common goals with mixed agents, however, this consensus problem is limited to the composition of first-order and second-order dynamics [85]. The composition of third-order and second-order dynamics under communication imperfections has great practical significance (robotics and aircraft), since most of the practical systems are of second and third-order dynamics and the communication (interaction) among them is also not perfect (continuous). Further, many consensus problems deal with time-invariant couplings in networked multi-agent systems in particular in [86], however, in practice, the coupling between agents may be time-varying due to the time-varying interactions between them.

In the end, we noticed that several control designs based on scalar Lyapunov functions were implemented on a Helicopter laboratory set up and a robotic system (two-link manipulator) in particular in [87–93], however they provide asymptotic or finite-time convergence of the states. Our work exploits vector framework based stability analysis tools, in particular, vector Lyapunov functions to design and implement control on these practical systems to achieve arbitrary time convergence.

1.3 Contributions

The main contributions of this thesis are summarized as follows:

- General results of arbitrary time stability of nonlinear systems using vector Lyapunov functions are developed for the first time. In fact, a vector comparison system, which is arbitrary time convergent, is constructed, and after that the stability of the original dynamical system is proved using differential inequalities and comparison principles. Moreover, the robust arbitrary time controllers for large-scale systems using vector control Lyapunov functions (VCLF) are designed. Also, aggregation of comparison systems is done to reduce their dimensionality in order to effectively apply the derived results on a class of underactuated systems. The theoretical results are implemented on an underactuated system.

- New results for the convergence analysis of nonlinear systems by extending contraction theory in the framework of vector distances are developed. A new tool, vector contraction analysis utilizing a new notion of the vector-valued norm is introduced, which evidently induces a vector distance between any pair of trajectories of the system. A level of mathematical sophistication is exploited in defining and validating the properties of a new notion, vector-valued norm. The proposed tool offers an amenable framework as each component of the vector-valued norm function satisfies fewer strict conditions than that of standard contraction analysis. Particularly, every element of vector-valued norm derivative need not be strictly negative definite for convergence of any pair of trajectories of the system. Moreover, vector-valued norm derivative satisfies a componentwise inequality employing some comparison system. In fact, the convergence analysis is performed by comparing the relative distances between any pair of the trajectories of the original nonlinear system with that of the comparison system. Comparison results are derived by utilizing the concepts of quasi-monotonicity property of the function and vector differential inequalities. Moreover, the results are also derived in the framework of the cone ordering instead of utilizing componentwise inequalities between vectors. Furthermore, the proposed theory is exploited to solve the most ubiquitous problems in control theory: controller and state observer design problems for nonlinear systems. In fact, the control input and observer gain are estimated in such a way that the comparison system thus obtained follows the indicated properties. The comparison system is chosen with the property that it is quasi-monotone non-decreasing and contracting. The analysis avoids the calculation of the Jacobian matrix consisting of highly nonlinear terms and solves the problem through a Jacobian matrix containing mainly linear terms. Implementation of the derived results is done on chaotic systems with simulation outcomes.

- One other real-world problem of interval observer design for nonlinear systems having disturbances, inputs and outputs is solved in a very efficient way by the exploitation of the proposed theory. An elegant feature is that it does not require the formulation of error dynamics to show the convergence properties and need not require the construction of a Lyapunov candidate function without any idea of the structure of the function. Specifically, it performs the convergence analysis through a *comparison system* that has specified properties. Furthermore, this theory is exploited to design dynamic output feedback control through the obtained state bounds from the constructed interval observer and the system outputs, to make the interval observer to be globally asymptotically stable. Examples with simulation outcomes are provided to validate the theoretical results.

- Finally, the vector-based contraction approach is utilized to synchronize the motion of agents in multi-agent systems. The convergence analysis of the agents using this approach provides a flexible frame as the components of a vector distance norm follow less stringent conditions than classical contraction analysis. The key feature is that it encompasses the formulation of some auxiliary (comparison) system, as a result, the problem of convergence analysis of a complex multi-agent system reduces to the convergence analysis of a much simpler auxiliary system. Hence, despite the nonlinearities, the methods of designing consensus controls remain simple as compared to the existing literature. Four different types of realistic problems are addressed: 1) consensus of third-order dynamics multi-agent systems subjected to acceleration and input constraints; 2) consensus of multi-agent systems with heterogeneous nodes (second and third-order dynamics have taken together) under communication imperfections; 3) synchronization of multi-agent systems with connected and disconnected switching topologies; 4) synchronization in networked systems with time-varying couplings. Moreover, an effective synchronization control scheme for a class of multi-agent systems with underactuated agent dynamics is developed using the method of aggregation of comparison systems.

- In the end, we demonstrate the implementation of the proposed results, in particular, vector control Lyapunov functions, to solve stabilization and regulation problem of the practical systems such as 2 DOF Helicopter model and two link manipulator (robotic system). Firstly, results are demonstrated experimentally on a 2 DOF Helicopter model with the aim that pitch and yaw positions of the Helicopter regulate at the desired positions in the set arbitrary time. After that, we implement it on a two-link manipulator and provide simulation results to validate the developed theoretical results.

1.4 Organization of the Thesis

The thesis comprises of eight chapters.

Chapter 1 gives a brief idea of the work in the thesis. It provides the motivation behind the work, literature survey to visualize the research gap and at last the main contributions.

In Chapter 2, mathematical notations and preliminaries with important definitions are entitled that are used throughout the thesis.

In Chapter 3, we derive results of the arbitrary time stability of nonlinear systems by the exploitation of vector Lyapunov functions. Then, robust universal arbitrary time controllers that are robust to bounded disturbances are designed for large-scale systems. In addition, we discuss the aggregation procedure of comparison systems in order to apply the derived results effectively on underactuated systems. An illustrative example of an underactuated surface vessel with the simulation results is also provided.

In Chapter 4, firstly, the generalized convergence analysis theory known as vector contraction analysis is introduced with the definition of a new notion, the vector-valued norm. Then, the main comparison results are provided that connect the solution of the relative distances of the trajectories of the original nonlinear dynamical system and the comparison system using the framework of vector inequalities and comparison principles. Moreover, the results are also derived in the frame of the cone ordering instead of component-wise inequalities between vectors. Furthermore, the implementation of these proposed results is shown in controller and observer design applications. Simulation results are also provided by considering examples of chaotic systems.

In Chapter 5, we exploit the proposed generalized contraction theory to design interval observer for nonlinear systems having disturbances, inputs and outputs. After that, this theory is used to design dynamic output feedback control through the obtained state bounds from the constructed interval observer and the system outputs, to make the interval observer to be globally asymptotically stable. Several academic and practical examples with respective simulations are also presented to validate the mathematical results.

In Chapter 6, firstly the motivation behind considering consensus problems is emphasized. Then, the main results are derived to solve four different types of consensus problems: 1) consensus of third-order dynamics multi-agent systems subjected to acceleration and input constraints; 2) consensus of multi-agent systems with heterogeneous nodes (second and third-order dynamics have taken together) under communication imperfections; 3) synchronization of multi-agent systems with connected and disconnected switching topologies; 4) synchronization in networked systems with time-varying couplings. Moreover, an effective synchronization control scheme for a class of multi-agent systems with under-actuated agent dynamics is developed using the method of aggregation of comparison systems. Examples with simulation outcomes are also illustrated at the end.

In Chapter 7, we demonstrate the implementation of the proposed results, in particular, vector control Lyapunov functions based stabilization in arbitrary time, on an experimental Helicopter set up to validate the simulation and theoretical results. Then, we consider a two-link manipulator model for implementation and provide simulation results to show the efficacy of the theoretical results.

Finally, in Chapter 8, conclusions are drawn with the scope of the future research investigation.