

MULTISTABILITY ANALYSIS AND
SYNCHRONIZATION OF ARTIFICIAL NEURAL
NETWORKS



*The thesis submitted in partial fulfilment
for the Award of Degree
DOCTOR OF PHILOSOPHY*

by

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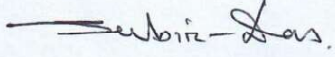
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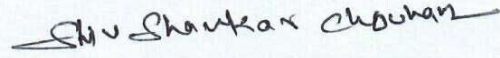
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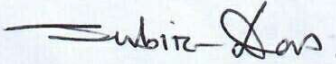
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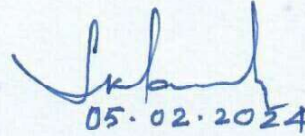
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Contents

List of Figures	xii
List of Tables	xiii
Preface	xv
1 Introduction	1
1.1 Biological Neuron	2
1.1.1 Artificial Neural Network	4
1.1.2 Activation Functions	6
1.2 Architectures of an Artificial Neural Network	8
1.2.1 Hopfield Neural Networks.	11
1.2.2 Associative Memories	12
1.3 Training Process or Learning Algorithm of Artificial Neural Network .	13
1.3.1 Supervised Learning	13
1.3.2 Unsupervised Learning	14
1.3.3 Reinforcement Learning	14
1.4 Delay Differential Equations	14
1.4.1 Types of Delays	15
1.5 Stability	16
1.5.1 Equilibrium Point	16
1.5.2 Stability and Attractivity	17
1.5.3 Asymptotically Stability	17
1.5.4 Exponentially Stability	17
1.5.5 μ -Stability	18
1.5.6 Positively Invariant Set	18
1.5.7 Finite Time Stability	19
1.5.8 Fixed Time Stability	19
1.5.9 Stability Analysis by Lyapunov Functional Method	20
1.6 Synchronization	21
1.7 Types of Neural Network	22

1.8	Quaternion Algebra	22
1.9	Octonion Algebra	24
2	Coexistence of locally multistable equilibrium points for n-neuron delayed quaternion-valued neural networks with continuous piecewise nonlinear activation functions	27
2.1	Introduction	27
2.2	Model Description	29
2.3	Main Results	34
2.3.1	Existence of multiple stable EPs and positively invariant sets for the DQVNNs (2.1)	35
2.4	Numerical example	58
2.5	Conclusions	63
3	Multiple μ-stability analysis of time-varying delayed quaternion-valued neural networks	65
3.1	Introduction	65
3.2	Model Description and Preliminaries	66
3.3	Main Results	72
3.3.1	Coexistence of multiple equilibria of the system (3.1)	73
3.4	Numerical Examples	93
3.5	Conclusions	97
4	Multistability Analysis of Octonion-Valued Neural Networks with Time-Varying Delays	99
4.1	Introduction	99
4.2	Model Description and Preliminaries	100
4.3	Main Results	102
4.4	Numerical Example	123
4.5	Conclusion	130
5	Fixed time synchronization of octonion valued neural networks with time varying delays	133
5.1	Introduction	133
5.2	Problem Formulation and Preliminaries	134
5.3	Main results	138
5.4	Numerical Example	145
5.5	Conclusion	149
6	Summary of the thesis and future work	151
6.1	Summary	151
6.2	Future work	154

Bibliography	157
List of Publications	167

List of Figures

1.1	Structure of a biological neuron.	3
1.2	Visualization of synaptic connections of interconnected neurons.	4
1.3	McCulloch-Pitts model of neural network.	5
1.4	(a) A Unit step function, (b) Piecewise linear function, (c) Sigmoid activation function for $\beta = 2, 3$, and 4, (d) tanh function, and (e) Relu function.	6
1.5	A feedforward network without hidden layer.	10
1.6	Multilayer feedforward neural network with one hidden layer.	10
1.7	Feedback neural network with n neurons.	11
1.8	Diagram of Hopfield neural networks.	12
1.9	(a): Original pattern of image the “V”, (b): Noisy sample of the original image “V”.	12
2.1	Configuration of piecewise nonlinear activation function $g_q^{(b)}(r)$ defined in equation (2.2)	30
2.2	Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (2.1), respectively.	59
2.3	Plot of the original color image of pattern “#”.	61
2.4	Simulation of restoring the pattern “#” with random initial values at $t = 0, 1, 1.5, 2$ and 2.5.	62
3.1	Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ for the function in Class \mathcal{A}	76
3.2	Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$ for the function in Class \mathcal{B}	76
3.3	Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (3.1) with $\tau_{ij} = 1$ ($i, j = 1, 2$), respectively.	94
3.4	Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (3.1) with $\tau_{ij} = 0.8t$ ($i, j = 1, 2$), respectively.	96
4.1	Configurations for $\hat{G}_i^{(b)}$ and $\check{G}_i^{(b)}$	111
4.2	Figures (a)-(d) demonstrate state trajectories of the 1st part-4th part of the system (4.1), respectively.	125
4.3	Figures (a)-(d) demonstrate state trajectories of the 5th part-8th part of the system (4.1), respectively.	126
4.4	Plot of the original color image of pattern “V”.	128

4.5	Plots of numerical simulation for retrieving the pattern “V” with random initial values at time, $t = 0, 1, 1.5, 2, 2.5$ and 5	129
5.1	Figures (a), (b), (c) and (d), respectively demonstrate the state trajectories of $p_1^{(0)}$ and $s_1^{(0)}$, $p_1^{(1)}$ and $s_1^{(1)}$, $p_1^{(2)}$ and $s_1^{(2)}$, $p_1^{(3)}$ and $s_1^{(3)}$ of the drive-response systems (5.3) and (5.4) without control.	147
5.2	Figures (a), (b), (c) and (d), respectively demonstrate the state trajectories of $p_1^{(0)}$ and $s_1^{(0)}$, $p_1^{(1)}$ and $s_1^{(1)}$, $p_1^{(2)}$ and $s_1^{(2)}$, $p_1^{(3)}$ and $s_1^{(3)}$ of the drive-response systems (5.3) and (5.4) with control.	148

List of Tables

1.1 Table of multiplication between octonion units.	24
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