Chapter 3

Interval-Valued Value Function and Its Application in Interval Optimization Problems

3.1 Introduction

The value function or the perturbation function is found in variational analysis which includes the problems based on well-posedness, sensitivity, and stability. In the 20th century, well-developed perturbation methods were first adopted to solve new problems due to the development of quantum mechanics in atomic and subatomic physics. In 1927, quantum perturbation theory was developed to estimate the changes observed in a particle with the emission of radioactive elements. Local perturbation analysis is also related to stability analysis of solution behaviour concerning changes in a given problem. Sensitivity and stability techniques have been used to obtain optimality conditions, solution algorithms, convergence and rate of convergence proofs, and duality results.

3.2 Motivation and Contribution

The development and computational implementation of standard accessible techniques for conducting nontrivial sensitivity analysis in nonlinear IOPs has not been widespread, and the IVFs subject to certain perturbations have not been studied yet. Also, if an IOP is modified by a small amount, then no conclusion has been drawn on the solution of the modified IOP. This provides a motivation to study the sensitivity of IOPs. Toward this, we briefly define the concept of the Lagrangian of IVFs, followed by a weak duality theorem for IVFs. Thereafter, we propose the notion of interval-valued value function and the characterization of its gH-subdifferential set for IOPs. After that, the characterization of the stability of a solution to an IOP with gH-subdifferential set of an interval-valued value function is given. A relation concerning the efficiency of a solution to an IOP with the help of gH-subdifferential set of an interval-valued value function under certain restrictions is given. Also, an example to show an application of interval-valued value function in a practical phenomenon is discussed.

3.3 Interval-Valued Lagrangian Function

In this section, we define the Lagrangian IVF, the Lagrangian dual of an IOP and discuss its saddle point analysis. In the sequel, an interrelation between saddle point criterion and KKT-type efficiency condition for IOP is also studied. The derived results are used later in Section 3.4 to derive the characterization of gH-subdifferential of interval-valued valued function for IOPs.

Definition 3.1 (Lagrangian IVF). *Consider the constrained IOP*

$$\left.\begin{aligned} &\inf_{y \in \mathcal{Y} \subseteq \mathbb{R}^n} && \boldsymbol{F}(y) \\ &\text{subject to} && \boldsymbol{G}_i(y) \preceq \boldsymbol{0},\ i = 1, 2, \ldots, m, \\ & && \boldsymbol{H}_j(y) = \boldsymbol{0},\ j = 1, 2, \ldots, p, \end{aligned}\right\} \tag{3.1}$$

where $\boldsymbol{F}$, $\boldsymbol{G}_i$, *and* $\boldsymbol{H}_j : \mathcal{Y} \to \overline{I(\mathbb{R})}$ *are extended IVFs. We define vector-valued IVFs* $\boldsymbol{G} : \mathcal{Y} \to I(\mathbb{R})^m$ *and* $\boldsymbol{H} : \mathcal{Y} \to I(\mathbb{R})^p$ *such that*

$$\boldsymbol{G}(y) = (\boldsymbol{G}_1(y), \boldsymbol{G}_2(y), \ldots, \boldsymbol{G}_m(y))^\top \text{ and } \boldsymbol{H}(y) = (\boldsymbol{H}_1(y), \boldsymbol{H}_2(y), \ldots, \boldsymbol{H}_p(y))^\top.$$

The Lagrangian IVF $\boldsymbol{L} : \mathcal{Y} \times \mathbb{R}^m \times \mathbb{R}^p \to I(\mathbb{R})$ *of (3.1) is defined as*

$$\boldsymbol{L}(y, u, v) = \boldsymbol{F}(y) \oplus u^\top \odot \boldsymbol{G}(y) \oplus v^\top \odot \boldsymbol{H}(y). \tag{3.2}$$

The Lagrangian dual IVF $\boldsymbol{D} : \mathbb{R}^m \times \mathbb{R}^p \to I(\mathbb{R})$ *of (3.1) is defined as*

$$\boldsymbol{D}(u, v) = \inf_{y \in \mathcal{Y}} \boldsymbol{L}(y, u, v). \tag{3.3}$$

The Lagrangian dual IOP of (3.1) is defined as

$$\left.\begin{array}{ll} \sup & \boldsymbol{D}(u, v) \\ \text{subject to} & u \geq 0, \end{array}\right\} \tag{3.4}$$

where u, v *are called Lagrange multipliers.*

Example 3.1 *Let* $\mathcal{Y} \subset \mathbb{R}$ *be the set* $\{y : -1 \leq y \leq 0\}$. *Consider the IOP*

$$\left.\begin{array}{ll} \inf\limits_{y \in \mathcal{Y}} & \boldsymbol{F}(y) = \left[\frac{1}{4}, 1\right] \odot y^2 \\ \text{subject to} & [y - 5, 4y] \preceq \boldsymbol{0}. \end{array}\right\} \tag{3.5}$$

The functions $\underline{f}(y) = \frac{y^2}{4}$ *and* $\overline{f}(y) = y^2$ *are depicted in Figure 3.1(a). The Lagrangian dual IVF of (3.5) is*

$$\boldsymbol{D}(u) = \inf_{y \in \mathcal{Y}} \boldsymbol{L}(y, u) = \inf_{y \in \mathcal{Y}} \left\{ \left[\frac{1}{4}, 1\right] \odot y^2 \oplus u \odot [y - 5, 4y] \right\}$$

$$= \inf_{y \in \mathcal{Y}} \left[\frac{1}{4} y^2 + uy - 5u, y^2 + 4uy \right].$$

Note that for any $u \in \mathbb{R}$*, the functions* $\underline{l}(y, u) = \frac{1}{4}y^2 + uy - 5u$ *and* $\overline{l}(y, u) = y^2 + 4uy$ *are convex and differentiable on* $\mathcal{Y}$*. Thus, by Lemma 1.5 and Remark 1.5, the IVF* $\boldsymbol{L}$ *is gH-differentiable in* $\mathcal{Y}$ *for any* $u \in \mathbb{R}$*.*

To find the possible efficient solutions of $\boldsymbol{L}$*, from Theorem 1.2, we must solve the relation:*

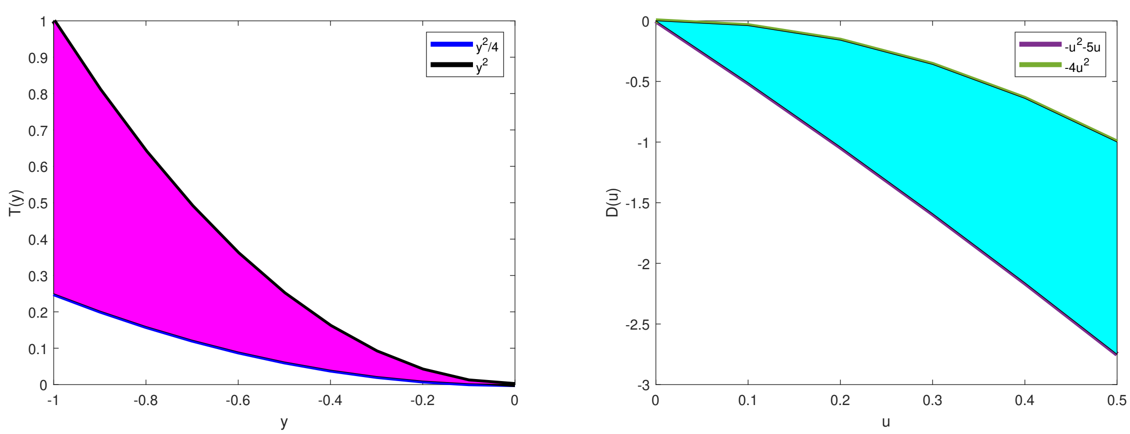
$$0 \in \nabla \boldsymbol{L}(y, u),$$

where, $\boldsymbol{L}(y, u) = \left[\frac{1}{4}, 1\right] \odot y^2 \oplus u \odot [y - 5, 4y]$*. Hence, we obtain* $y = -2u$*.*

Now the dual IVF $\boldsymbol{D}$ *with* $\underline{d}(u) = -u^2 - 5u$ *and* $\overline{d}(u) = -4u^2$ *is shown in Figure 3.1(b). The dual IOP of (3.5) is*

$$\left.\begin{array}{ll} \sup & \boldsymbol{D}(u) = [-u^2 - 5u, -4u^2] \\ \text{subject to} & 0 \leq u \leq \frac{1}{2}. \end{array}\right\} \tag{3.6}$$

It can be evident from Figure 3.1(b) that the efficient solution of IOP (3.6) is at $u = 0$*.*



(a) The IVF **F** (b) The IVF **D**

Figure 3.1: IVFs **F** and **D** of Example 3.1

Example 3.2 *Let $\mathcal{Y} \subset \mathbb{R}^2$ be the set $\{(y_1, y_2) : -3 \leq y_1 \leq 0, -7 \leq y_2 \leq 0\}$. Consider the IOP*

$$\left.\begin{array}{ll} \inf\limits_{y_1, y_2 \in \mathcal{Y}} & \boldsymbol{F}(y_1, y_2) = \left[\frac{(y_1+1)^2}{16} + y_2^2 - 2, (y_1+1)^2 + 2y_2^2\right] \\ \text{subject to} & \left[\frac{y_1}{8} - 7, 2y_1\right] \preceq \boldsymbol{0}, \\ & [y_2 - 7, 2y_2] \preceq \boldsymbol{0}. \end{array}\right\} \tag{3.7}$$

In Figure 3.2(a), the functions $\underline{f}(y_1, y_2) = \frac{(y_1+1)^2}{16} + y_2^2 - 2$ and $\overline{f}(y_1, y_2) = (y_1+1)^2 + 2y_2^2$ are shown with green and orange regions, respectively. The Lagrangian dual IVF of (3.7) is

$$\begin{aligned} &\boldsymbol{D}(u) \\ &= \inf_{y_1, y_2 \in \mathcal{Y}} \boldsymbol{L}(y_1, y_2, u_1, u_2) \\ &= \inf_{y_1, y_2 \in \mathcal{Y}} \left\{ \left[\frac{(y_1+1)^2}{16} + y_2^2 - 2, (y_1+1)^2 + 2y_2^2\right] \oplus u_1 \odot \left[\frac{y_1}{8} - 7, 2y_1\right] \oplus u_2[y_2 - 7, 2y_2] \right\} \\ &= \inf_{y_1, y_2 \in \mathcal{Y}} \left\{ \left[\frac{(y_1+1)^2}{16} + y_2^2 + \frac{u_1 y_1}{8} + u_2 y_2 - 16, (y_1+1)^2 + 2y_2^2 + 2u_1 y_1 + 2u_2 y_2\right] \right\}. \end{aligned}$$

Note that for any $u_1, u_2 \in \mathbb{R}$, the functions $\underline{l}(y_1, y_2, u_1, u_2) = \frac{(y_1+1)^2}{16} + y_2^2 + \frac{u_1 y_1}{8} + u_2 y_2 - 16$ and $\bar{l}(y_1, y_2, u_1, u_2) = (y_1+1)^2 + 2y_2^2 + 2u_1 y_1 + 2u_2 y_2$ are convex and differentiable on $\mathcal{Y}$. Thus, by Lemma 1.5 and Remark 1.5, the IVF $\boldsymbol{L}$ is gH-differentiable on $\mathcal{Y}$ for any $u_1, u_2 \in \mathbb{R}$.

To find the possible efficient solutions of $\boldsymbol{L}$, from Theorem 1.2, we must solve the relation:

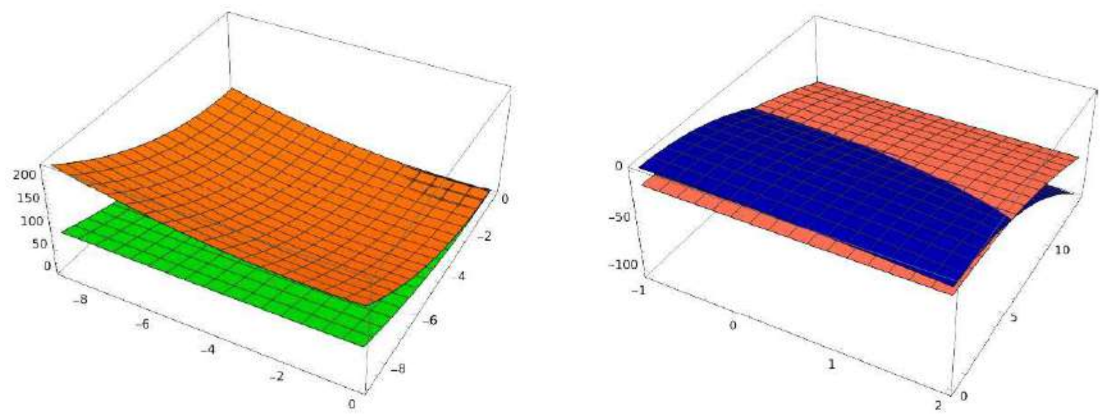
$$0 \in \nabla \boldsymbol{L}(y_1, y_2, u_1, u_2),$$

where, $\boldsymbol{L}(y_1, y_2, u_1, u_2) = \left[\frac{(y_1+1)^2}{16} + y_2^2 - 2, (y_1+1)^2 + 2y_2^2\right] \oplus u_1 \odot \left[\frac{y_1}{8} - 7, 2y_1\right] \oplus u_2[y_2 - 7, 2y_2]$. Hence, we obtain $y_1 = -u_1 - 1$ and $y_2 = -\frac{u_2}{2}$.

Now the dual IVF $\boldsymbol{D}$ *with* $\underline{d}(u_1, u_2) = -\frac{u_1^2}{16} - \frac{u_1}{8} - \frac{u_2^2}{4} - 16$ *and* $\overline{d}(u_1, u_2) = -u_1^2 - 2u_1 - \frac{u_2^2}{2}$ *are shown with pink and blue regions, respectively, in Figure 3.2(b). The dual IOP of (3.7) is*

$$\left.\begin{array}{ll} \sup & \boldsymbol{D}(u_1, u_2) = \left[-\frac{u_1^2}{16} - \frac{u_1}{8} - \frac{u_2^2}{4} - 16, -u_1^2 - 2u_1 - \frac{u_2^2}{2}\right] \\ \text{subject to} & -1 \leq u_1 \leq 2 \textit{ and } 0 \leq u_2 \leq 14. \end{array}\right\} \quad (3.8)$$

It can be observed that the dual efficient solution of (3.8) is at $u_1 = -1$ *and* $u_2 = 0$ *as shown in Figure 3.2(b).*



(a) The IVF **F** (b) The IVF **D**

Figure 3.2: IVFs **F** and **D** of Example 3.2

Theorem 3.1 (Concavity of Lagrangian dual IVF). *Let* $\mathcal{Y}$ *be a nonempty subset of* $\mathbb{R}^n$ *and* $\boldsymbol{D}$ *be the dual IVF corresponding to the IOP (3.1). Then, the dual IVF* $\boldsymbol{D}$ *is concave over* $\mathbb{R}^m \times \mathbb{R}^p$.

Proof: Let $(u_1, v_1), (u_2, v_2) \in \text{dom } \mathbf{D}$. Then, for any $\beta \in [0, 1]$, we have

$$\mathbf{D}(\beta(u_1, v_1) + (1 - \beta)(u_2, v_2))$$

$$
\begin{aligned}
&= \mathbf{D}(\beta u_1 + (1-\beta)u_2, \beta v_1 + (1-\beta)v_2) \\
&= \inf_{y \in \mathcal{Y}} \{\mathbf{F}(y) \oplus (\beta u_1 + (1-\beta)u_2)^\top \odot \mathbf{G}(y) \oplus (\beta v_1 + (1-\beta)v_2)^\top \odot \mathbf{H}(y)\} \\
&\succeq \inf_{y \in \mathcal{Y}} \{\beta \odot (\mathbf{F}(y) \oplus u_1^\top \odot \mathbf{G}(y) \oplus v_1^\top \odot \mathbf{H}(y)) \oplus (1-\beta) \odot (\mathbf{F}(y) \oplus u_2^\top \odot \mathbf{G}(y) \oplus \\
&\qquad\qquad v_2^\top \odot \mathbf{H}(y))\} \text{ from (ii) of Lemma 1.3 and (i) of Lemma 1.7} \\
&\succeq \inf_{y \in \mathcal{Y}} \{\beta \odot (\mathbf{F}(y) \oplus u_1^\top \odot \mathbf{G}(y) \oplus v_1^\top \odot \mathbf{H}(y))\} \oplus \inf_{y \in \mathcal{Y}} \{(1-\beta) \odot (\mathbf{F}(y) \oplus u_2^\top \odot \mathbf{G}(y) \oplus \\
&\qquad\qquad v_2^\top \odot \mathbf{H}(y))\} \text{ from (i) of Lemma 1.6} \\
&= \beta \odot \inf_{y \in \mathcal{Y}} \{(\mathbf{F}(y) \oplus u_1^\top \odot \mathbf{G}(y) \oplus v_1^\top \odot \mathbf{H}(y))\} \oplus (1-\beta) \odot \inf_{y \in \mathcal{Y}} \{(\mathbf{F}(y) \oplus u_2^\top \odot \mathbf{G}(y) \oplus \\
&\qquad\qquad v_2^\top \odot \mathbf{H}(y))\} \text{ from (iii) of Lemma 1.7.}
\end{aligned}
$$

Thus, $\beta \odot \mathbf{D}(u_1, v_1) \oplus (1-\beta) \odot \mathbf{D}(u_2, v_2) \preceq \mathbf{D}(\beta(u_1, v_1) + (1-\beta)(u_2, v_2))$.

Hence, the IVF $\mathbf{D}$ is concave. □

Theorem 3.2 (Weak duality theorem for IOPs). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\bar{y} \in \mathcal{Y}$ be a feasible solution to the IOP (3.1) and $(\bar{u}, \bar{v})$ be a feasible solution to the dual IOP (3.4). Then,*

$$\boldsymbol{D}(\bar{u}, \bar{v}) \preceq \boldsymbol{F}(\bar{y}).$$

Proof: For any $y \in \mathcal{Y}$, and $\bar{u} \in \mathbb{R}^m_+$, we have

$$
\begin{aligned}
\mathbf{D}(\bar{u}, \bar{v}) = \inf_{y \in \mathcal{Y}} \mathbf{L}(y, \bar{u}, \bar{v}) &\preceq \mathbf{F}(\bar{y}) \oplus \bar{u}^\top \odot \mathbf{G}(\bar{y}) \oplus \bar{v}^\top \odot \mathbf{H}(\bar{y}) \text{ for some } \bar{y} \in \mathcal{Y} \\
&= \mathbf{F}(\bar{y}) \oplus \bar{u}^\top \odot \mathbf{G}(\bar{y}) \quad \text{since } \bar{y} \text{ is feasible} \\
&\preceq \mathbf{F}(\bar{y}) \quad \text{from (iv) of Lemma 1.4,}
\end{aligned}
$$

which is the required relation. □

Definition 3.2 (Saddle point for IVF). *Let $\boldsymbol{L}$ be the Lagrangian IVF corresponding to the IOP (3.1). Then, a point $(y^*, \bar{u}, \bar{v})$ is called a saddle point of the Lagrangian IVF*

([3.2](#)) if for $y^* \in \mathcal{Y}, \bar{u} \in \mathbb{R}^m_+$, and $\bar{v} \in \mathbb{R}^p$, we have

$$\boldsymbol{L}(y^*, u, v) \preceq \boldsymbol{L}(y^*, \bar{u}, \bar{v}) \preceq \boldsymbol{L}(y, \bar{u}, \bar{v}) \text{ for all } y \in \mathcal{Y}, u \in \mathbb{R}^m_+, v \in \mathbb{R}^p.$$

Theorem 3.3 *Let $\boldsymbol{L}(y, u, v)$ be the Lagrangian IVF corresponding to the IOP (3.1). Then, a point $(y^*, \bar{u}, \bar{v})$ with $y^* \in \mathcal{Y}$ be a saddle point of (3.2) if and only if the following conditions are satisfied:*

(i) $\boldsymbol{L}(y^*, \bar{u}, \bar{v}) = \boldsymbol{D}(\bar{u}, \bar{v})$,

(ii) $\boldsymbol{G}(y^*) \preceq \boldsymbol{0}$, $\boldsymbol{H}(y^*) = \boldsymbol{0}$, *and*

(iii) $\bar{u}^\top \odot \boldsymbol{G}(y^*) = \boldsymbol{0}$.

Furthermore, $(y^*, \bar{u}, \bar{v})$ is a saddle point of (3.2) if and only if y^* and $(\bar{u}, \bar{v})$ are the efficient solutions of (3.1) and (3.4), respectively, with $\boldsymbol{F}(y^*) = \boldsymbol{D}(\bar{u}, \bar{v})$.

Proof: Let $(y^*, \bar{u}, \bar{v})$ be the saddle point of (3.2). Then, for each $y \in \mathcal{Y}$, $u \in \mathbb{R}^m_+$ and $v \in \mathbb{R}^p$,

$$\mathbf{L}(y^*, \bar{u}, \bar{v}) \preceq \mathbf{L}(y, \bar{u}, \bar{v}).$$

This implies that $\mathbf{L}(y^*, \bar{u}, \bar{v})$ is a lower bound of $\mathbf{L}(y, \bar{u}, \bar{v})$. Therefore, we have

$$\mathbf{L}(y^*, \bar{u}, \bar{v}) \preceq \inf_{y \in \mathcal{Y}} \mathbf{L}(y, \bar{u}, \bar{v}) = \mathbf{D}(\bar{u}, \bar{v}). \tag{3.9}$$

Also, we have

$$\begin{aligned} \mathbf{D}(\bar{u}, \bar{v}) &= \inf_{y \in \mathcal{Y}} \{\mathbf{F}(y) \oplus \bar{u}^\top \odot \mathbf{G}(y) \oplus \bar{v}^\top \odot \mathbf{H}(y)\} \\ &\preceq \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \oplus \bar{v}^\top \odot \mathbf{H}(y^*) \text{ for some } y^* \in \mathcal{Y} \\ &= \mathbf{L}(y^*, \bar{u}, \bar{v}). \end{aligned} \tag{3.10}$$

Therefore, from (3.9) and (3.10), we have $\mathbf{L}(y^*, \bar{u}, \bar{v}) = \mathbf{D}(\bar{u}, \bar{v})$. Hence, (i) is true. Also, for any u, v with $u \in \mathbb{R}^m_+$ and $v \in \mathbb{R}^p$, we have

$$\begin{aligned} &\mathbf{L}(y^*, u, v) \preceq \mathbf{L}(y^*, \bar{u}, \bar{v}) \\ \text{or, } &\mathbf{F}(y^*) \oplus u^\top \odot \mathbf{G}(y^*) \oplus v^\top \odot \mathbf{H}(y^*) \preceq \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \oplus \bar{v}^\top \odot \mathbf{H}(y^*). \end{aligned} \tag{3.11}$$

This implies that $\mathbf{G}(y^*) \preceq \mathbf{0}$ and $\mathbf{H}(y^*) = \mathbf{0}$. Otherwise, (3.11) can be violated by making a component of u or v sufficiently large. Hence, (ii) is true.

Now, take $u = 0$ in (3.11). We get $\mathbf{0} \preceq \bar{u}^\top \odot \mathbf{G}(y^*)$. Also, $\bar{u} \in \mathbb{R}^m_+$ and $\mathbf{G}(y^*) \preceq \mathbf{0}$. Hence, $\bar{u}^\top \odot \mathbf{G}(y^*) = \mathbf{0}$. This implies (iii) is true.

To prove the reverse inequality, we assume a point $(y^*, \bar{u}, \bar{v})$ with $\bar{u} \in \mathbb{R}^m_+$ such that conditions (i)-(iii) are satisfied. Then, from (i) we obtain

$$\mathbf{L}(y^*, \bar{u}, \bar{v}) = \mathbf{D}(\bar{u}, \bar{v}) \preceq \mathbf{L}(y, \bar{u}, \bar{v}) \text{ for all } y \in \mathcal{Y}.$$

Note that

$$\begin{aligned} \mathbf{L}(y^*, u, v) &= \mathbf{F}(y^*) \oplus u^\top \odot \mathbf{G}(y^*) \oplus v^\top \odot \mathbf{H}(y^*) \\ &\preceq \mathbf{F}(y^*) \text{ from (ii) and (iv) of Lemma (1.4)} \\ &= \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \oplus \bar{v}^\top \odot \mathbf{H}(y^*) \text{ from conditions (ii) and (iii)} \\ &= \mathbf{L}(y^*, \bar{u}, \bar{v}). \end{aligned}$$

Therefore, $(y^*, \bar{u}, \bar{v})$ is a saddle point of (3.2).

Next we assume that $(y^*, \bar{u}, \bar{v})$ is a saddle point of (3.2) such that conditions (i)-(iii) are satisfied. Then, from (ii), y^* is feasible to (3.1) and $\bar{u} \in \mathbb{R}^m_+$. Thus, $(\bar{u}, \bar{v})$ is feasible to (3.4). In view of conditions (i)-(iii),

$$\mathbf{D}(\bar{u}, \bar{v}) = \mathbf{L}(y^*, \bar{u}, \bar{v}) = \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \oplus \bar{v}^\top \odot \mathbf{H}(y^*) = \mathbf{F}(y^*).$$

Thus, y^* and $(\bar{u}, \bar{v})$ are efficient solutions of (3.1) and (3.4), respectively, with $\mathbf{F}(y^*) = \mathbf{D}(\bar{u}, \bar{v})$.

Conversely, we assume that y^* and $(\bar{u}, \bar{v})$ are efficient solutions of (3.1) and (3.4), respectively, with $\mathbf{F}(y^*) = \mathbf{D}(\bar{u}, \bar{v})$. This implies that, y^* and $(\bar{u}, \bar{v})$ are feasible. Thus, $\mathbf{G}(y^*) \preceq \mathbf{0}$, $\mathbf{H}(y^*) = \mathbf{0}$, and $\bar{u} \in \mathbb{R}^m_+$. Also,

$$\begin{aligned}\mathbf{D}(\bar{u}, \bar{v}) = \inf_{y \in \mathcal{Y}} \mathbf{L}(y, \bar{u}, \bar{v}) &\preceq \mathbf{L}(y^*, \bar{u}, \bar{v}) \text{ for some } y^* \in \mathcal{Y} \\ &= \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \oplus \bar{v}^\top \odot \mathbf{H}(y^*) \\ &= \mathbf{F}(y^*) \oplus \bar{u}^\top \odot \mathbf{G}(y^*) \\ &\preceq \mathbf{F}(y^*) \text{ from (iv) of Lemma 1.4.}\end{aligned}$$

Since we have assumed that $\mathbf{D}(\bar{u}, \bar{v}) = \mathbf{F}(y^*)$, therefore

$$\mathbf{L}(y^*, \bar{u}, \bar{v}) = \mathbf{D}(\bar{u}, \bar{v}) \text{ and } \bar{u}^\top \odot \mathbf{G}(y^*) = \mathbf{0}.$$

Hence, $(y^*, \bar{u}, \bar{v})$ is a saddle point of (3.2). □

Theorem 3.4 (Interrelation between saddle point for IVF and KKT-type efficiency condition for IOP). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$. Consider the constrained IOP*

$$\left.\begin{aligned} &\inf_{y \in \mathcal{Y}} && \boldsymbol{F}(y) \\ &\text{subject to} && g_i(y) \leq 0,\ i = 1, 2, \ldots, m, \end{aligned}\right\} \tag{3.12}$$

where $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ be a gH-differentiable convex IVF. Let A be any finite set of indices such that $A(\bar{y}) = \{i : g_i(\bar{y}) = 0\}$. For each $i \in A$, $g_i : \mathcal{Y} \to \mathbb{R}$ are real-valued convex functions which are differentiable on $\mathcal{Y}$. Let $\bar{y} \in \mathcal{Y}$ satisfy the KKT condition, i.e., there exist $\bar{u}_i \geq 0,\ i = 1, 2, \ldots, m$, such that

$$\nabla \boldsymbol{F}(\bar{y}) \oplus \bar{u}^\top \nabla g(\bar{y}) = \boldsymbol{0} \tag{3.13}$$

$$\bar{u}^\top g(\bar{y}) = 0, \ \ i = 1, 2, \ldots, m. \tag{3.14}$$

Then, $(\bar{y}, \bar{u})$ is a saddle point of (3.2).

Proof: Let $(\bar{y}, \bar{u})$ with $\bar{u}_i \geq 0$ satisfies the conditions (3.13) and (3.14). Since $\mathbf{F}$ and g_i's are convex and gH-differentiable, then from Theorem 1.3 , we have

$$(y - \bar{y})^\top \odot \nabla \mathbf{F}(\bar{y}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y}) \tag{3.15}$$

$$(y - \bar{y})^\top \nabla g_i(\bar{y}) \preceq g_i(y) - g_i(\bar{y}) \ \ i \in A(\bar{y}). \tag{3.16}$$

For simplicity, let us assume $A(\bar{y}) = \{i : i = 1, 2, \ldots, k\}$ for some $k \in \{1, 2, \ldots, m\}$. Now inserting $\bar{u}_i \geq 0$ in (3.16), and then adding it with (3.15), we get

$$(y - \bar{y})^\top \odot \nabla \mathbf{F}(\bar{y}) \oplus_{i=1}^k \bar{u}_i (y - \bar{y})^\top \nabla g_i(\bar{y}) \preceq (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y})) \oplus_{i=1}^k \bar{u}_i (g_i(y) - g_i(\bar{y}))$$

or, $(y - \bar{y})^\top \odot (\nabla \mathbf{F}(\bar{y}) \oplus_{i=1}^k \bar{u}_i \nabla g_i(\bar{y})) \preceq (\mathbf{F}(y) \oplus_{i=1}^k \bar{u}_i g_i(y)) \ominus_{gH} (\mathbf{F}(\bar{y}) \oplus_{i=1}^k \bar{u}_i g_i(\bar{y}))$

from (i) of Lemma 1.3 and (iii) of Lemma 1.4

or, $\mathbf{0} \preceq (\mathbf{F}(y) \oplus_{i=1}^k \bar{u}_i g_i(y)) \ominus_{gH} (\mathbf{F}(\bar{y}) \oplus_{i=1}^k \bar{u}_i g_i(\bar{y}))$ by (3.13)

or, $\mathbf{F}(\bar{y}) \oplus_{i=1}^k \bar{u}_i g_i(\bar{y}) \preceq \mathbf{F}(y) \oplus_{i=1}^k \bar{u}_i g_i(y)$

or, $\mathbf{L}(\bar{y}, \bar{u}) \preceq \mathbf{L}(y, \bar{u}).$ (3.17)

Also, $g_i(\bar{y}) \leq 0$ with $\bar{u}^\top g(\bar{y}) = 0$, we have

$$\begin{aligned} \mathbf{L}(\bar{y}, u) &= \mathbf{F}(\bar{y}) \oplus u^\top g(\bar{y}) \\ &\preceq \mathbf{F}(\bar{y}) \text{ from (iv) of Lemma 1.4} \quad (3.18) \\ &\preceq \mathbf{F}(\bar{y}) \oplus \bar{u}^\top g(\bar{y}) = \mathbf{L}(\bar{y}, \bar{u}). \quad (3.19) \end{aligned}$$

From (3.17) and (3.18), $(\bar{y}, \bar{u})$ is a saddle point for (3.2). □

The following theorem reveals that sometimes it is not easy to estimate the dual of certain IVFs on the entire domain. In this regard, we can decompose the given domain into a union of disjoint sub-domains. Finally, on taking the infimum of acquired solutions of dual IVFs will lead to the required solution.

Theorem 3.5 *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F}$ be an IVF on $\mathcal{Y}$. Let the domain $\mathcal{Y}$ in (3.1) be decomposed into a union of disjoint sub-domains, $\mathcal{Y} = \bigcup_{k=1}^{p} \mathcal{Y}^k$. Let*

$$\begin{aligned}
&\mathcal{S} = \{y \in \mathcal{Y} : \boldsymbol{G}(y) \preceq \boldsymbol{0},\ \boldsymbol{H}(y) = \boldsymbol{0}\},\\
&\mathcal{S}_k = \{y \in \mathcal{Y}^k : \boldsymbol{G}(y) \preceq \boldsymbol{0},\ \boldsymbol{H}(y) = \boldsymbol{0}\},\\
&\boldsymbol{D}(\beta, \delta) = \inf_{y\in\mathcal{Y}} \boldsymbol{L}(y, \beta, \delta),\ \textit{and}\\
&\boldsymbol{D}_k(\beta, \delta) = \inf_{y\in\mathcal{Y}_k} \boldsymbol{L}(y, \beta, \delta),\ k = 1, 2, \ldots, p.
\end{aligned}$$

Let us assume that (β^*, δ^*) be a solution of the dual IOP (3.4) on $\mathcal{Y}$ and (β_k^*, δ_k^*) be a solution to the dual IOPs on $\mathcal{Y}^k$, $k = 1, 2, \ldots, p$. Then,

$$\boldsymbol{D}(\beta^*, \delta^*) \preceq \inf_{1\leq k\leq p} \boldsymbol{D}_k(\beta_k^*, \delta_k^*) \preceq \inf_{y\in\mathcal{S}} \boldsymbol{F}(y).$$

Proof: Since $\mathcal{Y}_k \subseteq \mathcal{Y}$, then from (i) of Lemma 1.7

$$\begin{aligned}
&\mathbf{D}(\beta, \delta) \preceq \mathbf{D}_k(\beta, \delta) \text{ for all } \beta \in \mathbb{R}^m_+,\ k = 1, 2, \ldots, p\\
\text{or, }&\mathbf{D}(\beta^*, \delta^*) \preceq \mathbf{D}_k(\beta_k^*, \delta_k^*) \text{ for all } k = 1, 2, \ldots, p\\
\text{or, }&\mathbf{D}(\beta^*, \delta^*) \preceq \inf_{1\leq k\leq p} \mathbf{D}_k(\beta_k^*, \delta_k^*).
\end{aligned}$$

In view of Theorem 3.2, we have

$$\begin{aligned}
&\mathbf{D}_k(\beta_k^*, \delta_k^*) \preceq \inf_{y\in\mathcal{S}_k} \mathbf{F}(y)\\
\text{or, }&\inf_{1\leq k\leq p} \mathbf{D}_k(\beta_k^*, \delta_k^*) \preceq \inf_{1\leq k\leq p} \inf_{y\in\mathcal{S}_k} \mathbf{F}(y) = \inf_{y\in\mathcal{S}} \mathbf{F}(y),
\end{aligned}$$

which is the required result. □

3.4 Interval-Valued Value Function

In this section, we define the interval-valued value function for IOPs and discuss its characteristics. Thereafter, we explore the gH-subdifferential of the interval-valued value function and use this in characterizing the stability of a solution to an IOP.

Definition 3.3 (The interval-valued value function). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$. Consider the IOP*

$$\left.\begin{array}{ll}\inf\limits_{y \in \mathcal{Y}} & \boldsymbol{F}(y) \\ \text{subject to} & \boldsymbol{G}_i(y) \preceq \boldsymbol{0},\ i = 1, 2, \ldots, m, \\ & \boldsymbol{H}_j(y) = \boldsymbol{0},\ j = 1, 2, \ldots, p. \end{array}\right\} \tag{3.20}$$

where $\boldsymbol{F}$, $\boldsymbol{G}_i$ and $\boldsymbol{H}_j$ are extended IVFs from $\mathcal{Y}$ to $\overline{I(\mathbb{R})}$. Define the vector-valued IVFs $\boldsymbol{G} : \mathcal{Y} \to I(\mathbb{R})^m$ and $\boldsymbol{H} : \mathcal{Y} \to I(\mathbb{R})^p$ by

$$\boldsymbol{G}(y) = (\boldsymbol{G}_1(y), \boldsymbol{G}_2(y), \ldots, \boldsymbol{G}_m(y))^\top \text{ and } \boldsymbol{H}(y) = (\boldsymbol{H}_1(y), \boldsymbol{H}_2(y), \ldots, \boldsymbol{H}_p(y))^\top.$$

Therefore, the IOP (3.20) can be written more compactly as

$$\inf_{y \in \mathcal{Y}} \left\{ \boldsymbol{F}(y) : \boldsymbol{G}(y) \preceq \boldsymbol{0},\ \ \boldsymbol{H}(y) = \boldsymbol{0} \right\}. \tag{3.21}$$

The interval-valued value function associated with the IOP (3.21) is the IVF $\boldsymbol{V} : \mathbb{R}^m \times \mathbb{R}^p \to \overline{I(\mathbb{R})}$, which is given by

$$\boldsymbol{V}(s, r) = \inf_{y \in \mathcal{Y}} \{ \boldsymbol{F}(y) : \boldsymbol{G}(y) \preceq [s, s],\ \ \boldsymbol{H}(y) = [r, r] \}. \tag{3.22}$$

Definition 3.4 (Feasible set of the interval-valued value function). *Let $\mathcal{Y}$ be a nonempty* *subset of* $\mathbb{R}^n$ *and* $\boldsymbol{V}$ *be an interval-valued value function associated with the IOP (3.20). Then, the feasible set of the IOP (3.22) is denoted by* $C(s,r)$*, defined by*

$$C(s,r) = \{y \in \mathcal{Y} : \boldsymbol{G}(y) \preceq [s,s],\ \boldsymbol{H}(y) = [r,r]\}.$$

The interval-valued value function associated with the feasible set is

$$\boldsymbol{V}(s,r) = \inf\{\boldsymbol{F}(y) : y \in C(s,r)\}.$$

Example 3.3 *In this example, we calculate the interval-valued function corresponding to the variations in constraints* $\boldsymbol{G}_i$ *of an IOP. Let* $\mathcal{Y} = \{(y_1,y_2) \in \mathbb{R}^2 : y_1, y_2 \geq 0\} \subset \mathbb{R}^2$ *and consider an IOP*

$$\left.\begin{array}{ll} \inf\limits_{y_1,y_2\in\mathcal{Y}} & \boldsymbol{F}(y_1,y_2) = [1,2] \odot e^{-\sqrt{y_1y_2}} \\ \text{subject to} & [1,2] \odot y_1 \preceq \boldsymbol{0}. \end{array}\right\} \tag{3.23}$$

In Figure 3.3, the functions $\underline{f}(y_1,y_2) = e^{-\sqrt{y_1y_2}}$ *and* $\overline{f}(y_1,y_2) = 2e^{-\sqrt{y_1y_2}}$ *are depicted with brown and blue regions, respectively. The interval-valued value function of (3.23) is given by*

$$\boldsymbol{V}(s) = \inf_{y_1,y_2\in\mathcal{Y}}\{[1,2]\odot e^{-\sqrt{y_1y_2}} : [1,2]\odot y_1 \leq s\} = \begin{cases} [1,2], & \text{if } s = 0 \\ \boldsymbol{0}, & \text{if } s > 0 \\ +\infty, & \text{if } s < 0. \end{cases}$$

Lemma 3.1 (Monotonicity of the interval-valued value function). *Let* $\mathcal{Y}$ *be a nonempty subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F}$, $\boldsymbol{G}_i$ *and* $\boldsymbol{H}_j$ *are extended IVFs from* $\mathcal{Y}$ *to* $\overline{I(\mathbb{R})}$, $i = 1,2,\ldots,m,\ j = 1,2,\ldots,p$. *Let* $\boldsymbol{V}$ *be an interval-valued value function for the IOP (3.20). Then, for*

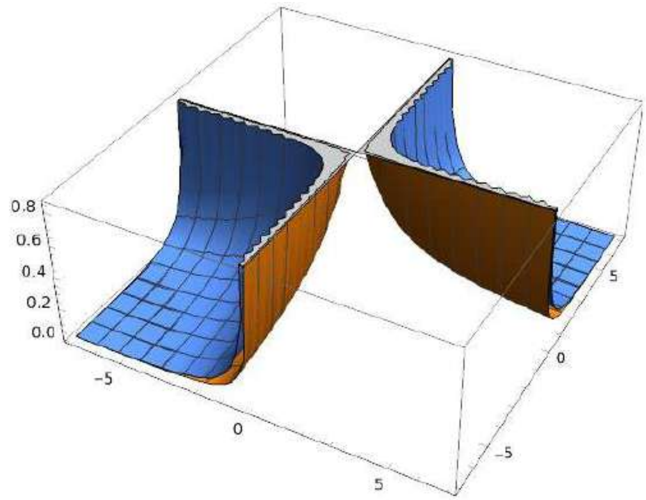


Figure 3.3: The IVF **F** of Example 3.3

$s \leq w$, *we have*

$$\boldsymbol{V}(w, r) \preceq \boldsymbol{V}(s, r) \text{ for all } s, w \in \mathbb{R}^m \text{ and } r \in \mathbb{R}^p.$$

Proof: The inequality $s \leq w$ implies that $C(s, r) \subseteq C(w, r)$. Thus, from (i) of Lemma 1.7, we have

$$\mathbf{V}(w, r) \preceq \mathbf{V}(s, r) \text{ and } r \in \mathbb{R}^p,$$

which is the required relation. $\square$

Theorem 3.6 (Convexity of the interval-valued value function). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F}$, $\boldsymbol{G}_i$, and $\boldsymbol{H}_j : \mathcal{Y} \to \overline{I(\mathbb{R})}$ are extended convex IVFs , $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, p$. Then, the interval-valued value function in (3.22) is convex over* $\text{dom}(\mathbf{V}) \subseteq \mathbb{R}^m \times \mathbb{R}^p$.

Proof: Let $(s, r), (w, t) \in \text{dom}(\mathbf{V})$ and $\beta \in [0, 1]$. We need to show that

$$\mathbf{V}(\beta s + (1 - \beta)w, \beta r + (1 - \beta)t) \preceq \beta \odot \mathbf{V}(s, r) \oplus (1 - \beta) \odot \mathbf{V}(w, t).$$

Consider sequences $\{x_k\} \in C(s,r),\ \{y_k\} \in C(w,t)$ such that $\mathbf{F}(x_k)$ and $\mathbf{F}(y_k)$ converging to $\mathbf{V}(s,r)$ and $\mathbf{V}(w,t)$, respectively as $k \to \infty$. Thus, we have

$$\mathbf{G}_i(x_k) \preceq [s,s] \text{ and } \mathbf{G}_i(y_k) \preceq [w,w],\ i = 1,2,\ldots,m.$$

Therefore, by convexity of the components of $\mathbf{G}_i$, and $\mathbf{H}_j$, we have

$$\mathbf{G}_i(\beta x_k + (1-\beta)y_k) \preceq \beta \odot \mathbf{G}_i(x_k) \oplus (1-\beta) \odot \mathbf{G}_i(y_k) \preceq \beta \odot [s,s] + (1-\beta) \odot [w,w] \tag{3.24}$$

and $\mathbf{H}_j(\beta x_k + (1-\beta)y_k) = \beta \odot \mathbf{H}_j(x_k) \oplus (1-\beta) \odot \mathbf{H}_j(y_k) = \beta \odot [r,r] \oplus (1-\beta) \odot [t,t]$. (3.25)

Combining (3.24) and (3.25), we conclude that

$$\beta x_k + (1-\beta)y_k \in C(\beta s + (1-\beta)w, \beta r + (1-\beta)t).$$

Since $\mathbf{F}$ is convex IVF, then we have

$$\mathbf{F}(\beta x_k + (1-\beta)y_k) \preceq \beta \odot \mathbf{F}(x_k) \oplus (1-\beta) \odot \mathbf{F}(y_k). \tag{3.26}$$

As, $\mathbf{F}(x_k),\ \mathbf{F}(y_k)$ converging to $\mathbf{V}(s,r), \mathbf{V}(w,t)$, respectively as $k \to \infty$. Therefore, (3.26) implies that

$$\liminf_{k\to\infty} \mathbf{F}(\beta x_k + (1-\beta)y_k) \preceq \beta \odot \mathbf{V}(s,r) \oplus (1-\beta) \odot \mathbf{V}(w,t). \tag{3.27}$$

In view of Definition 3.3, we have

$$\mathbf{V}(\beta s + (1-\beta)w, \beta r + (1-\beta)t) \preceq \mathbf{F}(\beta x_k + (1-\beta)y_k)$$
$$\text{or, } \mathbf{V}(\beta s + (1-\beta)w, \beta r + (1-\beta)t) \preceq \liminf_{k\to\infty} \mathbf{F}(\beta x_k + (1-\beta)y_k).$$

Thus, by combining the above relation with (3.27), we obtain

$$\mathbf{V}(\beta s + (1-\beta)w, \beta r + (1-\beta)t) \preceq \beta \odot \mathbf{V}(s,r) \oplus (1-\beta) \odot \mathbf{V}(w,t).$$

Hence, the interval-valued value function $\mathbf{V}$ is convex. □

Theorem 3.7 (Saddle point efficiency interpretation of the interval-valued value function). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\bar{y}$ is an efficient solution to the IOP (3.1). Let $\boldsymbol{V}$ be the interval-valued value function given in (3.22). If $(\bar{y}, \bar{u}, \bar{v})$ is a saddle point for the Lagrangian IVF (3.2), then*

$$\boldsymbol{V}(0,0) \ominus_{gH} (\bar{u}^\top, \bar{v}^\top)(s,r)^\top \preceq \boldsymbol{V}(s,r) \ \ \textit{for all } s \in \mathbb{R}^m,\ r \in \mathbb{R}^p. \tag{3.28}$$

Proof: Let $(\bar{y}, \bar{u}, \bar{v})$ is a saddle point of (3.2). Then, for all $s \in \mathbb{R}^m,\ r \in \mathbb{R}^p$ and from Theorem 3.3, we have

$$\begin{aligned}
&\mathbf{V}(0,0)\\
&= \inf_{y\in\mathcal{Y}} \Big(\mathbf{F}(y) \oplus \bar{u}^\top \odot \mathbf{G}(y) \oplus \bar{v}^\top \odot \mathbf{H}(y)\Big)\\
&= \inf_{y\in\mathcal{Y}} \Big(\mathbf{F}(y) \oplus_{i=1}^{m} \bar{u}_i^\top \odot (\mathbf{G}_i(y) \ominus_{gH} [s_i, s_i]) \oplus_{j=1}^{p} \bar{v}_j^\top \odot (\mathbf{H}_j(y) \ominus_{gH} [r_j, r_j])\Big)\\
&\qquad \oplus (\bar{u}^\top, \bar{v}^\top)(s,r)^\top. \qquad (3.29)
\end{aligned}$$

Applying Lemma 3.2 to the value function, we obtain

$$\begin{aligned}
&\inf_{y\in\mathcal{Y}} \{\mathbf{F}(y) \oplus \bar{u}^\top \odot (\mathbf{G}(y) \ominus_{gH} [s,s]) \oplus \bar{v}^\top \odot (\mathbf{H}(y) \ominus_{gH} [r,r])\} \preceq \mathbf{V}(s,r)\\
\text{or, } &\inf_{y\in\mathcal{Y}} \{\mathbf{F}(y) \oplus \bar{u}^\top \odot (\mathbf{G}(y) \ominus_{gH} [s,s]) \oplus \bar{v}^\top \odot (\mathbf{H}(y) \ominus_{gH} [r,r])\} \oplus (\bar{u}^\top, \bar{v}^\top)(s,r)^\top\\
&\qquad \preceq \mathbf{V}(s,r) \oplus (\bar{u}^\top, \bar{v}^\top)(s,r)^\top\\
\text{or, } &\mathbf{V}(0,0) \preceq \mathbf{V}(s,r) \oplus (\bar{u}^\top, \bar{v}^\top)(s,r)^\top \text{ by (3.29)}\\
\text{or, } &\mathbf{V}(0,0) \ominus_{gH} (\bar{u}^\top, \bar{v}^\top)(s,r)^\top \preceq \mathbf{V}(s,r) \ \text{ for all } s \in \mathbb{R}^m,\ r \in \mathbb{R}^p.
\end{aligned}$$

□

Theorem 3.8 (Characterization of gH-subdifferential of interval-valued value function). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F}$, $\boldsymbol{G}_i : \mathcal{Y} \to \overline{I(\mathbb{R})}$, $i = 1, 2, \ldots, m$ are extended convex IVFs. Let $\boldsymbol{V}$ be the interval-valued value function given in (3.22). Assume that $\boldsymbol{F}_{\text{eff}}$ and $\boldsymbol{D}_{\text{eff}}$ are the set of efficient points of (3.20) and (3.4), respectively, which is nonempty and coincides. Then, $(\bar{u}, \bar{v})$ is an efficient solution of (3.4) if and only if $-(\bar{u}, \bar{v}) \in \partial \boldsymbol{V}(0, 0)$.*

Proof: Let $(\bar{u}, \bar{v})$ be an efficient solution of the dual IOP (3.4).

Note that $\mathbf{V}(0, 0) = \mathbf{F}_{\text{eff}}$. Thus,

$$\mathbf{V}(0,0) = \mathbf{F}_{\text{eff}} = \mathbf{D}_{\text{eff}} = \mathbf{D}(\bar{u}, \bar{v}) = \inf_{w \in \mathcal{Y}} \mathbf{L}(w, \bar{u}, \bar{v}) \preceq \mathbf{L}(y, \bar{u}, \bar{v}) \text{ for all } y \in \mathcal{Y}.$$

Therefore, for any feasible point $y \in C(s, r), s \in \mathbb{R}^m, r \in \mathbb{R}^p$, we have

$$\begin{aligned}
&\mathbf{V}(0,0) \oplus (-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \\
\preceq\ &\mathbf{L}(y, \bar{u}, \bar{v}) \oplus (-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \\
=\ &\mathbf{F}(y) \oplus \bar{u}^\top \odot \mathbf{G}(y) \oplus \bar{v}^\top \odot \mathbf{H}(y) \oplus (-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \\
=\ &\mathbf{F}(y) \oplus \bar{u}^\top \odot (\mathbf{G}(y) \oplus (-s)) \oplus \bar{v}^\top \odot (\mathbf{H}(y) \oplus (-r)) \text{ from (iii) of Lemma 1.4} \\
\preceq\ &\mathbf{F}(y) \text{ from (iv) of Lemma 1.4.}
\end{aligned}$$

Thus, we obtain a bound of $\mathbf{F}$, i.e.,

$$\mathbf{V}(0,0) \oplus (-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \preceq \mathbf{F}(y) \text{ for all } y \in C(s, r).$$

Since infimum of $\mathbf{F}$ over $C(s, r)$ is $\mathbf{V}(s, r)$, then

$$\mathbf{V}(0,0) \oplus (-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \preceq \mathbf{V}(s, r)$$

$$(-\bar{u}^\top, -\bar{v}^\top)(s, r)^\top \preceq \mathbf{V}(s, r) \ominus_{gH} \mathbf{V}(0, 0).$$

Hence, $-(\bar{u}, \bar{v}) \in \partial \mathbf{V}(0, 0)$.

In order to prove the reverse inequality, we assume that $-(\bar{u}, \bar{v}) \in \partial \mathbf{V}(0, 0)$. Then, for $(s, r) \in \mathbb{R}^m \times \mathbb{R}^p$, we have

$$(-\bar{u}^\top, -\bar{v}^\top)(s, r)^\top \preceq \mathbf{V}(s, r) \ominus_{gH} \mathbf{V}(0, 0). \tag{3.30}$$

For $y \in \mathcal{Y}$, take $s = \max\{\underline{g}(y), \overline{g}(y)\}$ and $r = \max\{\underline{h}(y), \overline{h}(y)\}$. Then, $C(0, 0) \subseteq C(s, r)$. Therefore,

$$\begin{aligned}
& \mathbf{V}(s, r) \preceq \mathbf{V}(0, 0) = \mathbf{F}_{\text{eff}} \preceq \mathbf{F}(y) \text{ from Lemma 3.1} \\
\Longrightarrow \quad & \mathbf{V}(s, r) \preceq \mathbf{F}(y) \\
\Longrightarrow \quad & \mathbf{V}(s, r) \ominus_{gH} \mathbf{V}(0, 0) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{V}(0, 0) \\
\Longrightarrow \quad & -\bar{u}^\top s - \bar{v}^\top r \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{V}(0, 0) \text{ by (3.30)} \\
\Longrightarrow \quad & \mathbf{0} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{V}(0, 0) \oplus \bar{u}^\top s \oplus \bar{v}^\top r \\
\Longrightarrow \quad & \mathbf{0} \preceq \mathbf{L}(y, \bar{u}, \bar{v}) \ominus_{gH} \mathbf{V}(0, 0) \text{ from (iii) of Lemma 1.4} \\
\Longrightarrow \quad & \mathbf{V}(0, 0) \preceq \mathbf{L}(y, \bar{u}, \bar{v}) \\
\Longrightarrow \quad & \mathbf{V}(0, 0) \preceq \inf_{y \in \mathcal{Y}} \mathbf{L}(y, \bar{u}, \bar{v}) \text{ since } y \text{ is arbitrary} \\
\Longrightarrow \quad & \mathbf{F}_{\text{eff}} \preceq \mathbf{D}(\bar{u}, \bar{v}).
\end{aligned} \tag{3.31}$$

For any $j \in \{1, 2, \ldots, m\}$, taking $s = e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ with 1 at jth place and $r = 0$ in (3.30). We obtain

$$-u_j \preceq \mathbf{V}(e_j, 0) \ominus_{gH} \mathbf{V}(0, 0) \text{ and } \mathbf{V}(e_j, 0) \preceq \mathbf{V}(0, 0) \text{ from Lemma 3.1},$$

which is true for any j. Therefore, $\bar{u} \geq 0$ and $(\bar{u}, \bar{v})$ is a feasible solution of $\mathbf{D}$. Thus, from (3.31) we get

$$\mathbf{D}_{\text{eff}} = \mathbf{F}_{\text{eff}} = \mathbf{V}(0,0) \preceq \mathbf{D}(\bar{u},\bar{v}) \preceq \mathbf{D}_{\text{eff}}.$$

This shows that $(\bar{u},\bar{v})$ is an efficient solution of (3.4) and $\mathbf{D}(\bar{u},\bar{v}) = \mathbf{D}_{\text{eff}}$. □

Definition 3.5 (Stability of a solution to an IOP). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and (3.20) and (3.22) be corresponding IOP and interval-valued value function, respectively. Then, the IOP (3.20) is said to be stable if $\partial\, \boldsymbol{V}(0,0)$ is nonempty.*

Example 3.4 *In this example, we calculate the interval-valued value function of the IOP (3.32). After that, we check the stability of the solution to this IOP. Consider the IOP*

$$\left.\begin{aligned} &\inf_{y\in\mathcal{Y}} && \boldsymbol{F}(y) = [-2,-1]\odot\sqrt{y}\\ &\text{subject to} && [1,2]\odot y \preceq \boldsymbol{0},\\ & && \mathcal{Y} = \{y\in\mathbb{R} : y \geq 0\}. \end{aligned}\right\} \tag{3.32}$$

The interval-valued value function of (3.32) is given by

$$\boldsymbol{V}(s) = \inf_{y\in\mathcal{Y}}\{[-2,-1]\odot\sqrt{y} : [1,2]\odot y \leq s\} = \begin{cases} [-2,-1]\odot\sqrt{s}, & \text{if } s \geq 0\\ +\infty, & \text{if } s < 0. \end{cases}$$

The functions $\underline{v}(s) = -2\sqrt{s}$ and $\overline{v}(s) = -\sqrt{s}$ are depicted in Figure 3.4. It is evident from Figure 3.4 that the IVF $\boldsymbol{V}$ is not gH-subdifferentiable at $s = 0$. To show this, assume contrarily that there exists $\boldsymbol{G} \in I(\mathbb{R})$ such that $\boldsymbol{G} \in \partial\, \boldsymbol{V}(0)$. Then, for any $s \geq 0$

$$\begin{aligned} & (s-0)\odot\boldsymbol{G} \preceq \boldsymbol{V}(s)\ominus_{gH}\boldsymbol{V}(0)\\ \implies\quad & s\odot\boldsymbol{G} \preceq [-2,-1]\odot\sqrt{s}\\ \implies\quad & [s\underline{g}, s\overline{g}] \preceq [-2\sqrt{s}, -\sqrt{s}] \end{aligned}$$

$$\implies \quad s\underline{g} \leq -2\sqrt{s} \text{ and } s\overline{g} \leq -\sqrt{s}.$$

It can be observed that for $s > 0$, we have

$$\underline{g} < 0 \text{ and } \overline{g} < 0 \text{ (i.e., in particular } \boldsymbol{G} \prec 0).$$

Also, in particular, for any $\boldsymbol{G} \prec 0$, take $s = \frac{1}{2\overline{g}^2} > 0$, we get

$$s\overline{g} \leq -\sqrt{s} \implies \frac{1}{2\overline{g}} \leq -\sqrt{\frac{1}{2\overline{g}^2}} \implies \frac{1}{4\overline{g}^2} \geq \frac{1}{2\overline{g}^2},$$

which is not possible. Therefore, the $\partial \boldsymbol{V}(0)$ is empty. Thus, the solution of the IOP (3.32) is not stable at $s = 0$.

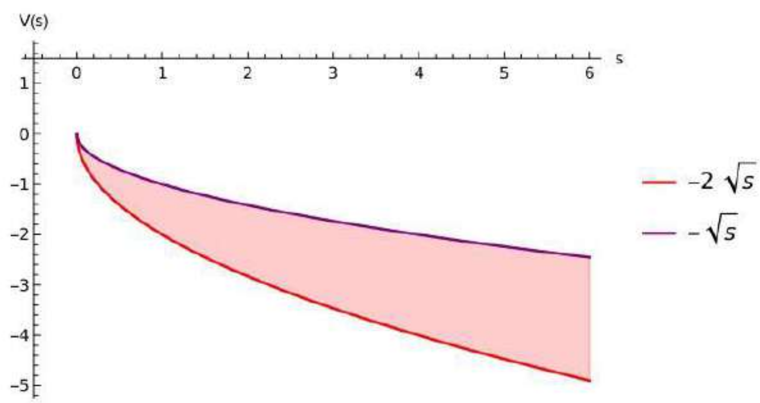


Figure 3.4: The IVF **V** of Example 3.4

Theorem 3.9 (Characterization of stability of a solution to an IOP). *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F}$, $\boldsymbol{G}_i(y) : \mathcal{Y} \to \overline{I(\mathbb{R})}$, $i = 1, 2, \ldots, m$ are extended convex IVFs. Let $\boldsymbol{V}$ be an interval-valued value function in (3.22) which is finite at $(\bar{s}, \bar{r})$. If*

$\partial \boldsymbol{V}(\bar{s},\bar{r})$ is nonempty, then there exists an $M \geq 0$ such that

$$\frac{1}{\|(s,r)-(\bar{s},\bar{r})\|} \odot (\, \boldsymbol{V}(\bar{s},\bar{r}) \ominus_{gH} \ \boldsymbol{V}(s,r)) \preceq [M,M],$$

for all $s \in \mathbb{R}^m,\ r \in \mathbb{R}^p,\ (s,r) \neq (\bar{s},\bar{r})$.

Proof: Let $\partial \mathbf{V}(\bar{s},\bar{r})$ is nonempty and assume $\widehat{\mathbf{K}} \in \partial \mathbf{V}(\bar{s},\bar{r})$. Then, from Definition 1.24, for all $s \in \mathbb{R}^m,\ r \in \mathbb{R}^p$,

$$\begin{aligned} & ((s,r)-(\bar{s},\bar{r}))^\top \odot \widehat{\mathbf{K}} \preceq \mathbf{V}(s,r) \ominus_{gH} \mathbf{V}(\bar{s},\bar{r}) \\ \implies \quad & \mathbf{V}(\bar{s},\bar{r}) \ominus_{gH} \mathbf{V}(s,r) \preceq ((\bar{s},\bar{r})-(s,r))^\top \odot \widehat{\mathbf{K}}. \end{aligned}$$

Let $\|\widehat{\mathbf{K}}\| = M$. Then,

$$\begin{aligned} \frac{1}{\|(s,r)-(\bar{s},\bar{r})\|} \odot (\mathbf{V}(\bar{s},\bar{r}) \ominus_{gH} \mathbf{V}(s,r)) \ &\preceq\ \frac{1}{\|(s,r)-(\bar{s},\bar{r})\|} \odot (((\bar{s},\bar{r})-(s,r))^\top \odot \widehat{\mathbf{K}}) \\ &\leq\ \frac{\|\widehat{\mathbf{K}}\| \|((\bar{s},\bar{r})-(s,r))\|}{\|(s,r)-(\bar{s},\bar{r})\|} \leq M, \end{aligned}$$

which is the required result. □

3.5 Application of gH-Subdifferential of Interval-Valued Value Function

In this section, we consider an interval-valued optimization problem (3.20) which satisfy certain condition given in (3.33). After that, we apply Theorem 3.8 to it to observe the efficiency of a solution to the modified IOP. Furthermore, an example to show an application of interval-valued value function in a practical phenomenon is discussed.

Theorem 3.10 *Let $\mathcal{Y}$ be a nonempty subset of $\mathbb{R}^n$ and $\boldsymbol{F},\ \boldsymbol{G}_i$ and $\boldsymbol{H}_j : \mathcal{Y} \to \overline{I(\mathbb{R})},\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,p$ are extended convex IVFs. Let $\boldsymbol{V}$ be the interval-valued value* *function given in (3.22). Let* $\boldsymbol{F}_{\text{eff}}$ *and* $\boldsymbol{D}_{\text{eff}}$ *are the set of efficient points of (3.1) and (3.4), respectively. Let* $(\bar{u}, \bar{v})$ *be an efficient solution of (3.4) and for an* $\tilde{y} \in \mathcal{Y}$,

$$\boldsymbol{M}(\tilde{y}) \equiv \boldsymbol{F}(\tilde{y}) \ominus_{gH} \boldsymbol{F}_{\text{eff}} \oplus \rho_1 \|\boldsymbol{G}(\tilde{y})\| \oplus \rho_2 \|\boldsymbol{H}(\tilde{y})\| \preceq [\delta, \delta], \tag{3.33}$$

where $\delta > 0$ *and* ρ_1, ρ_2 *satisfy* $\rho_1 \leq 2\|\bar{u}\|,\ \rho_2 \leq 2\|\bar{v}\|$. *Then,*

$$\boldsymbol{F}(\tilde{y}) \ominus_{gH} \boldsymbol{F}_{\mathit{eff}} \preceq [\delta, \delta], \tag{3.34}$$

$$\|\boldsymbol{G}(\tilde{y})\| \leq \frac{2}{\rho_1}\delta \ \textit{and} \ \|\boldsymbol{H}(\tilde{y})\| \leq \frac{2}{\rho_2}\delta. \tag{3.35}$$

Proof: Since $\rho_1 \geq 2\|\bar{u}\|$ and $\rho_2 \geq 2\|\bar{v}\|$ are nonnegative in (3.33), it can be observed that

$$\mathbf{F}(\tilde{y}) \ominus_{gH} \mathbf{F}_{\text{eff}} \preceq [\delta, \delta].$$

As $(\bar{u}, \bar{v})$ is an efficient solution of (3.4), from Theorem 3.8, for any $(s, r) \in \text{dom}(\mathbf{V})$, we get

$$(-\bar{u}^\top s) \oplus (-\bar{v}^\top r) \preceq \mathbf{V}(s, r) \ominus_{gH} \mathbf{V}(0, 0). \tag{3.36}$$

In particular, take $s = \bar{s} = \max\{|\underline{g}(\tilde{y})|, |\overline{g}(\tilde{y})|\}$ and $r = \bar{r} = \max\{|\underline{h}(\tilde{y})|, |\overline{h}(\tilde{y})|\}$. Then, $C(0, 0) \subseteq C(\bar{s}, \bar{r})$. Therefore, from Lemma 3.1, we have

$$\mathbf{V}(\bar{s}, \bar{r}) \preceq \mathbf{V}(0, 0) = \mathbf{F}_{\text{eff}} \preceq \mathbf{F}(\tilde{y}). \tag{3.37}$$

Now we consider the following expression

$$\begin{aligned}
& (\rho_1 - \|\bar{u}\|)\|\bar{s}\| + (\rho_2 - \|\bar{v}\|)\|\bar{r}\| \\
= \ & -\|\bar{u}\|\|\bar{s}\| - \|\bar{v}\|\|\bar{r}\| + \rho_1\|\bar{s}\| + \rho_2\|\bar{r}\| \\
\leq \ & -\bar{u}^\top \bar{s} - \bar{v}^\top \bar{r} + \rho_1\|\bar{s}\| + \rho_2\|\bar{r}\| \text{ from inner product of real variables}
\end{aligned}$$

$$\preceq \mathbf{V}(\bar{s}, \bar{r}) \ominus_{gH} \mathbf{V}(0,0) \oplus \rho_1 \|\bar{s}\| \oplus \rho_2 \|\bar{r}\| \text{ by } (3.36)$$

$$\preceq \mathbf{F}(\tilde{y}) \ominus_{gH} \mathbf{F}_{\text{eff}} \oplus \rho_1 \|\bar{s}\| \oplus \rho_2 \|\bar{r}\| \text{ by } (3.37)$$

$$\preceq [\delta, \delta] \text{ by } (3.33).$$

Since both the expressions $(\rho_1 - \|\bar{u}\|)\|\bar{s}\|$ and $(\rho_2 - \|\bar{v}\|)\|\bar{r}\|$ are nonnegative, it follows that

$$(\rho_1 - \|\bar{u}\|)\|\bar{s}\| \leq \delta \text{ and } (\rho_2 - \|\bar{v}\|)\|\bar{r}\| \leq \delta.$$

Using the assumption $\rho_1 \geq 2\|\bar{u}\|, \rho_2 \geq 2\|\bar{v}\|$, we have

$$\|\mathbf{G}(\tilde{y})\| = \|\bar{s}\| \leq \frac{\delta}{\rho_1 - \|\bar{u}\|} \leq \frac{2}{\rho_1}\delta \text{ and } \|\mathbf{H}(\tilde{y})\| = \|\bar{r}\| \leq \frac{\delta}{\rho_2 - \|\bar{v}\|} \leq \frac{2}{\rho_2}\delta,$$

which is the required result. □

Remark 3.1 *From Theorem 3.10, it can be noted that the expressions* $\boldsymbol{F}(\tilde{y}) \ominus_{gH} \boldsymbol{F}(y)$ *as well as* $\|\boldsymbol{G}(\tilde{y})\|$ *and* $\|\boldsymbol{H}(\tilde{y})\|$ *need not be too small such that* $\boldsymbol{\delta}$ *is dominated by (3.33). However, if* $\rho_1,\ \rho_2$ *are chosen to be large enough, then with the help of Theorem 3.8 such conclusion can be drawn.*

Example 3.5 *In this example, we exemplify Theorem 3.10. For this, we consider an IOP which satisfies the conditions given in Theorem 3.10. Let us consider the IOP (3.5) in Example 3.1:*

$$\left.\begin{array}{ll} \inf & \boldsymbol{F}(y) = \left[\frac{1}{4}, 1\right] \odot y^2 \\ \text{subject to} & \boldsymbol{G}(y) = [y-5, 4y] \preceq \boldsymbol{0} \\ & -\frac{1}{2} \leq y \leq 0. \end{array}\right\} \quad (3.38)$$

It can be seen that $\bar{y} = 0$ *is an efficient point of the IOP (3.38), and* $\boldsymbol{F}_{\mathit{eff}} = \boldsymbol{F}(\bar{y}) = \mathbf{0}$.

The dual IOP of (3.38) is given by (refer to Example 3.1)

$$\left.\begin{array}{ll} \sup & \boldsymbol{D}(u) = [-u^2 - 5u, -4u^2] \\ \text{subject to} & 0 \leq u \leq 1, \end{array}\right\} \tag{3.39}$$

which has an efficient solution at $\bar{u} = 0$.

Next, we aim to find possible values of δ *and* ρ *for which the IOP (3.38) satisfies the relation (3.33):*

$$\begin{aligned} &\boldsymbol{M}(y,\rho) = \left[\tfrac{1}{4}, 1\right] \odot y^2 \ominus_{gH} [0,0] \oplus \rho \|[y-5, 4y]\| \preceq [\delta, \delta] \text{ and } \rho \geq 2\|\bar{u}\| \\ \implies &\left[\tfrac{1}{4}, 1\right] \odot y^2 \oplus \rho \|[y-5, 4y]\| \preceq [\delta, \delta] \text{ and } \rho \geq 0 \\ \implies &\tfrac{1}{4}y^2 + \rho(\max\{|y-5|, |4y|\}) \leq \delta, y^2 + \rho(\max\{|y-5|,\ |4y|\}) \leq \delta,\ \text{ and } \rho \geq 0 \end{aligned} \tag{3.40}$$

$$\implies \rho \leq \frac{\delta - \frac{1}{4}y^2}{|y-5|},\ \rho \leq \frac{\delta - y^2}{|y-5|},\ \text{ and } \rho \geq 0. \tag{3.41}$$

From (3.40) and (3.41), we observe that

$$\begin{aligned} \rho \geq 0 \implies &\delta \geq \max\left\{\tfrac{1}{4}y^2, y^2\right\} = y^2 \text{ for any } y \in \left[-\tfrac{1}{2}, 0\right] \\ \implies &\delta \geq \tfrac{1}{4} \text{ for any } y \in \left[-\tfrac{1}{2}, 0\right]. \end{aligned}$$

Also, for any $y \in \left[-\frac{1}{2}, 0\right]$, *we get*

$$\rho \leq \tfrac{\delta - y^2}{|y-5|} \leq \tfrac{\delta - \frac{1}{4}y^2}{|y-5|} \leq \tfrac{\delta}{5}.$$

Therefore, for any $\delta \geq \frac{1}{4}$, *we can choose* $\rho \leq \frac{\delta}{5}$ *such that the relation (3.33) holds. Now, we show that these values of* δ *and* ρ *satisfies the relations (3.34) and (3.35). To this,* *observe that*

$$\boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}_{\mathit{eff}} = \left[\tfrac{1}{4}, 1\right] \odot y^2 \preceq [\delta, \delta] \text{ (for any } \delta \geq \tfrac{1}{4})$$
$$\text{and} \quad \|\boldsymbol{G}(y)\| = \|[y-5, 4y]\| = |y-5| \leq \tfrac{\delta - y^2}{|y-5|} \leq \tfrac{\delta}{\rho} \leq \tfrac{2}{\rho}\delta \text{ (from (3.41))}.$$

Thus, we can conclude that for suitable values of δ and ρ which satisfies relation (3.33), the relations (3.34) and (3.35) holds.

Example 3.6 (An application of interval-valued value function in estimating the range of changes in profit in a firm due to variation in utilizing resources).

To show an application of interval-valued value functions in a practical phenomenon, we consider an IOP to maximizing the profit ($-\boldsymbol{F}(y)$) of a firm subject to the constraint $\boldsymbol{A} \odot y = b$ (a constant), where $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ is a gH-differentiable IVF on $\mathcal{Y} \subseteq \mathbb{R}^n$. We suppose that an optimal solution y^* to this problem exists.

The constant b in the constraint reflects that the firm has utilized the resources of quantity $[b, b]$. The firm is interested in determining the optimal profit change due to a slight alteration of the resource-utilization level from $[b, b]$ to $[b+r, b+r]$. Thus, the firm needs to determine the change in $\boldsymbol{F}(y^*)$ resulting from a small change r in b.

Towards this, let us consider the following interval-valued value function IOP:

$$\left.\begin{array}{ll} \inf\limits_{y \in \mathcal{Y}} & \boldsymbol{F}(y) \\ \text{subject to} & \boldsymbol{A} \odot y = [b+r, b+r]. \end{array}\right\} \tag{3.42}$$

Suppose $y^*(r) = y^* + \Delta y^*$ be an optimal solution to this problem. Notice that $y^*(0) = y^*$.

The change in the value of $\boldsymbol{F}(y^*)$ *is given by*

$$\begin{aligned} & \Delta \boldsymbol{F}(y^*) && (3.43) \\ = \; & \boldsymbol{F}(y^*(r)) \ominus_{gH} \boldsymbol{F}(y^*) \\ = \; & [\min\{\underline{f}(y^*(r)) - \underline{f}(y^*), \overline{f}(y^*(r)) - \overline{f}(y^*)\}, \max\{\underline{f}(y^*(r)) - \underline{f}(y^*), \overline{f}(y^*(r)) - \overline{f}(y^*)\}] \\ = \; & [\min\{\Delta\underline{f}(y^*), \Delta\overline{f}(y^*)\}, \max\{\Delta\underline{f}(y^*), \Delta\overline{f}(y^*)\}]. && (3.44) \end{aligned}$$

To find the possible candidates for efficient solutions to (3.42)*, according to Theorem* 1.2*, we must solve the following inclusion relation:*

$$0 \in \nabla \boldsymbol{L}(y, v),$$

where $\boldsymbol{L}(y, v) = \boldsymbol{F}(y) \oplus v \odot ((\boldsymbol{A} \odot y) \ominus_{gH} [b + r, b + r])$*. Therefore, we have*

$$0 \in \nabla \boldsymbol{F}(y^*) \oplus v \odot (\boldsymbol{A}) \implies -v \odot \boldsymbol{A} \in \nabla \boldsymbol{F}(y^*). \tag{3.45}$$

Since $\boldsymbol{A} \odot y^* = [b, b]$ *and* $\boldsymbol{A} \odot y^*(r) = [b + r, b + r]$*, therefore*

$$\boldsymbol{A} \odot \Delta y^* = \boldsymbol{A} \odot (y^*(r) - y^*) = r. \tag{3.46}$$

First order approximation for real-valued functions gives

$$\Delta f(y^*) = \nabla f(y^*)^\top \Delta y^* + o(\|\Delta y^*\|). \tag{3.47}$$

Thus, in view of relation (3.47) in (3.43), we get

$$\begin{aligned} \Delta \boldsymbol{F}(y^*) &= [\min\{\Delta\underline{f}(y^*), \Delta\overline{f}(y^*)\}, \max\{\Delta\underline{f}(y^*), \Delta\overline{f}(y^*)\}] \\ &= [\min\{\nabla\underline{f}(y^*)\Delta y^*, \nabla\overline{f}(y^*)\Delta y^*\}, \max\{\nabla\underline{f}(y^*)\Delta y^*, \nabla\overline{f}(y^*)\Delta y^*\}] \end{aligned}$$

$$= [\min\{\nabla \underline{f}(y^*), \nabla \overline{f}(y^*)\}, \max\{\nabla \underline{f}(y^*), \nabla \overline{f}(y^*)\}] \odot \Delta y^*. \tag{3.48}$$

Therefore, from (3.45) and (3.48), we obtain the possible range of $\Delta \boldsymbol{F}$ as

$$\{(-v \odot \boldsymbol{A} \odot \Delta y^*) : v \in \mathbb{R}\}.$$

Thus, the range of the rate of change in optimal value corresponding to the change in level b is given by

$$(-v \odot \boldsymbol{A} \odot \Delta y^*) \oslash r = (-v) \in (\Delta \boldsymbol{F}(y^*) \oslash r) \text{ by (3.46).}$$

Therefore, it is observed that the problem that we have considered in this example can be modeled by a new function, namely, interval-valued value function. The efficient solutions of this interval-valued value function, further, gives information on the change in the optimal solution to (3.42).

3.6 Conclusion

In this chapter, the concept of interval-valued value function (Definition 3.3) has been studied. Also, an interrelation of saddle point efficiency of interval-valued value function has been derived (Theorem 3.7). Further, we have proposed the characterization of gH-subdifferential of interval-valued value function (Theorem 3.8). To develop these concepts, we have first proposed the concept of a Lagrangian IVFs (Definition 3.1) followed by a weak duality theorem of IVFs (Theorem 3.2) and a saddle point characterization for Lagrangian IVF (Theorem 3.3). We have further defined the stability criterion for an IOP (Definition 3.5) and presented a characterization of stability of a solution to an IOP (Theorem 3.9). Moreover, the stability characterization have been applied to an IOP with certain restrictions to observe the efficiency of a solution to an

IOP (Theorem 3.10). In the last, we have discussed an example to show an application of interval-valued value function in a practical phenomenon (Example 3.6).

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