Chapter 4

Synchronization of hypercomplex neural networks with mixed time-varying delays

4.1 Introduction

This chapter delves into the fixed-time synchronization (FTS) concept applied to HCNNs featuring mixed time-varying delays. Unlike finite-time synchronization (FNTS), which depends on initial conditions, FTS offers the advantage of adjustable settling times tailored to specific requirements. In HCNNs, where state vectors, weight matrices, activation functions, and input vectors are all represented as hypercomplex numbers, traditional techniques used in CVNNs and QVNNs cannot be directly applied due to their limitation to dimensions of eight or fewer. The approach begins with a decomposition method to divide HCNNs into $(n+1)$ RVNNs, utilizing the distributive law to address non-commutativity and non-associativity. A nonlinear controller is then developed to synchronize the master-response systems of HCNNs, with the stability of the error system proven using the Lyapunov-based method. FTS of mixed time-varying delayed HCNNs is achieved through applying a suitable lemma, the Lipschitz condition, the construction of an appropriate Lyapunov functional, and the design of suitable controllers. Two distinct algebraic criteria for settling time are derived using two separate lemmas, with empirical evidence indicating that the settling time derived from Lemma 1 yields more precise results than that from Lemma 2. Furthermore, three numerical examples of CVNNs, QVNNs and OVNNs illustrate the efficacy and effectiveness of the proposed theoretical framework.

4.2 The mathematical model

In this chapter, the following HCNN model with mixed time-varying delays is considered as

$$\begin{aligned} \dot{w}_\mu(\mathbf{t}) = &- c_\mu w_\mu(\mathbf{t}) + \sum_{\nu=1}^{N} a_{\mu\nu} g_\nu(w_\nu(\mathbf{t})) + \sum_{\nu=1}^{N} b_{\mu\nu} g_\nu(w_\nu(\mathbf{t} - \phi(\mathbf{t}))) \\ &+ \sum_{\nu=1}^{N} d_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_\nu(w_\nu(r))\, dr + R_\mu(\mathbf{t}), \end{aligned} \tag{4.1}$$

where N corresponds to the number of neurons, $w_\mu(t) \in \mathbb{H}$ represents the state of the μth neuron at time t and $w = (w_1, w_2, \ldots, w_N)^T \in \mathbb{H}^N$, $c_\mu \in \mathbb{R}$ with $c_\mu > 0$ is diagonal matrix. The connection weight matrix without time-varying delay is denoted by $A = (a_{\mu\nu})_{N\times N} \in \mathbb{H}^{N\times N}$. Again $B = (b_{\mu\nu})_{N\times N}, D = (b_{\mu\nu})_{N\times N} \in \mathbb{H}^{N\times N}$ are the connection weight matrices with discrete and distributed time-varying delays, respectively. $g(.) = (g_1(.), g_2(.), \ldots,\ g_N(.))^T : \mathbb{H}^N \longrightarrow \mathbb{H}^N$ is the hypercomplex valued activation function. The external input vector is given by $R_\mu \in \mathbb{H}$, and $\phi(\mathbf{t}), \psi(\mathbf{t})$ are discrete and distributed time-varying delays, respectively.

Assumption 4.2.1. The activation function is defined as

$$\begin{aligned} g_\nu(w_\nu(\mathbf{t})) =& \sum_{\rho=0}^{n} g_\nu^{(\rho)}(w_\nu^{(\rho)}(\mathbf{t}))\gamma_\rho \\ =& g_\nu^{(0)}(w_\nu^{(0)}(\mathbf{t}))\gamma_0 + g_\nu^{(1)}(w_\nu^{(1)}(\mathbf{t}))\gamma_1 + \ldots + g_\nu^{(n)}(w_\nu^{(n)}(\mathbf{t}))\gamma_n, \end{aligned} \tag{4.2}$$

where $\nu = 1, 2, \ldots, N$.

Assumption 4.2.2. For $y_1, y_2 \in \mathbb{R}$, there exists $l_\nu \in \mathbb{R}$ s.t. the activation functions $g_\nu^{(\rho)}(.)$ satisfy the following inequalities.

$$|g_\nu^{(\rho)}(y_1) - g_\nu^{(\rho)}(y_2)| \leq l_\nu |y_1 - y_2|, \tag{4.3}$$

where $\rho = 0, 1, 2, \ldots, n$ and $\nu = 1, 2, \ldots, N$.

Remark 4.2.1. The key elements those affect the dynamic properties of the constructed NNs are the activation functions. Since the activation functions are continuous, therefore the existence and uniqueness of the system (4.1) can be ensured based on Assumption 4.2.2 [98]. The following activation functions such as piecewise linear function, Logistic sigmoid function, and hyperbolic tangent function satisfy the requirements of the assumption of the Lipschitz condition.

Now, by using (1.65) and Assumption 4.2.1, the system (4.1) can be seprated into $(n+1)$ real-valued systems of equations with following expressions.

$$\dot{w}_\mu^{(0)}(\mathbf{t}) = -c_\mu w_\mu^{(0)}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a_{\mu\nu}^{(0)} g_\nu^{(0)}(w_\nu^{(0)}(\mathbf{t})) + \sum_{\alpha,\beta=1}^{n} a_{\mu\nu}^{(\alpha)} g_\nu^{(\beta)}(w_\nu^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,0} \Big)$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t})))+\sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,0}\Big)\\
&+\sum_{\nu=1}^{N}d_{\mu\nu}^{(0)}\Big(\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(w_{\nu}^{(0)}(r))\,dr+\sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(r))\\
&\times\kappa_{\alpha\beta,0}\,dr\Big)+R_{\mu}^{(0)}(\mathbf{t}),\\
\dot{w}_{\mu}^{(1)}(\mathbf{t})=&-c_{\mu}w_{\mu}^{(1)}(\mathbf{t})+\sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(1)}(w_{\nu}^{(1)}(\mathbf{t}))+\sum_{\nu=1}^{N}a_{\mu\nu}^{(1)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}))\\
&+\sum_{\alpha,\beta=1}^{n}a_{\mu\nu}^{(\alpha)}sg_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,1}\Big)+\sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(1)}(w_{\nu}^{(1)}(\mathbf{t}-\phi(\mathbf{t})))\\
&+b_{\mu\nu}^{(1)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t})))+\sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,1}\Big)\\
&+\sum_{\nu=1}^{N}\Big(d_{\mu\nu}^{(0)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(1)}(w_{\nu}^{(1)}(r))\,dr+d_{\mu\nu}^{(1)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(w_{\nu}^{(0)}(r))\,dr\\
&+\sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,1}\,dr\Big)+R_{\mu}^{(1)}(\mathbf{t}),\\
\dot{w}_{\mu}^{(2)}(\mathbf{t})=&-c_{\mu}w_{\mu}^{(2)}(\mathbf{t})+\sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(2)}(w_{\nu}^{(2)}(\mathbf{t}))+\sum_{\nu=1}^{N}a_{\mu\nu}^{(2)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}))\\
&+\sum_{\alpha,\beta=1}^{n}a_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,2}\Big)+\sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(2)}(w_{\nu}^{(2)}(\mathbf{t}-\phi(\mathbf{t})))\\
&+b_{\mu\nu}^{(2)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t})))+\sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,2}\Big)\\
&+\sum_{\nu=1}^{N}\Big(d_{\mu\nu}^{(0)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(2)}(w_{\nu}^{(2)}(r))\,dr+d_{\mu\nu}^{(2)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(w_{\nu}^{(0)}(r))\,dr\\
&+\sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,2}\,dr\Big)+R_{\mu}^{(2)}(\mathbf{t}),\\
&\qquad\qquad\vdots
\end{aligned}
$$

$$\begin{aligned}\dot{w}_{\mu}^{(n)}(\mathbf{t}) =& -c_{\mu}w_{\mu}^{(n)}(\mathbf{t}) + \sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(n)}(w_{\nu}^{(n)}(\mathbf{t})) + \sum_{\nu=1}^{N}a_{\mu\nu}^{(n)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}))\\&+\sum_{\alpha,\beta=1}^{n}a_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,n}\Big) + \sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(n)}(w_{\nu}^{(n)}(\mathbf{t}-\phi(\mathbf{t})))\\&+b_{\mu\nu}^{(n)}g_{\nu}^{(0)}(w_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,n}\Big)\\&+\sum_{\nu=1}^{N}\Big(d_{\mu\nu}^{(0)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(n)}(w_{\nu}^{(n)}(r))\,dr + d_{\mu\nu}^{(n)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(w_{\nu}^{(0)}(r))\,dr\\&+\sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,n}\,dr\Big) + R_{\mu}^{(n)}(\mathbf{t}).\end{aligned} \tag{4.4}$$

The response system corresponding to the master system (4.1) is illustrated as

$$\begin{aligned}\dot{z}_{\mu}(\mathbf{t}) =& -c_{\mu}z_{\mu}(\mathbf{t}) + \sum_{\nu=1}^{N}a_{\mu\nu}g_{\nu}(z_{\nu}(\mathbf{t})) + \sum_{\nu=1}^{N}b_{\mu\nu}g_{\nu}(z_{\nu}(\mathbf{t}-\phi(\mathbf{t})))\\&+\sum_{\nu=1}^{N}d_{\mu\nu}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}(z_{\nu}(r))\,dr + R_{\mu}(\mathbf{t}) + L(\mathbf{t}),\end{aligned} \tag{4.5}$$

where $z = (z_1, z_2, \ldots, z_N)^T \in \mathbb{H}^N$, is the state vector of NNs with N neurons and $L(\mathbf{t})$ is the controller.

The $(n+1)$ real components of (4.5) are as follows.

$$\begin{aligned}\dot{z}_{\mu}^{(0)}(\mathbf{t}) =& -c_{\mu}z_{\mu}^{(0)}(\mathbf{t}) + \sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t})) + \sum_{\alpha,\beta=1}^{n}a_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,0}\Big)\\&+\sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,0}\Big)\\&+\sum_{\nu=1}^{N}d_{\mu\nu}^{(0)}\Big(\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(z_{\nu}^{(0)}(r))\,dr + \sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(r))\\&\times\kappa_{\alpha\beta,0}\,dr\Big) + R_{\mu}^{(0)}(\mathbf{t}) + L_{\mu}^{(0)}(\mathbf{t}),\\ \dot{z}_{\mu}^{(1)}(\mathbf{t}) =& -c_{\mu}z_{\mu}^{(1)}(\mathbf{t}) + \sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(1)}(z_{\nu}^{(1)}(\mathbf{t})) + \sum_{\nu=1}^{N}a_{\mu\nu}^{(1)}g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t}))\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\alpha,\beta=1}^{n} a_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,1}\Big) + \sum_{\nu=1}^{N} \Big(b_{\mu\nu}^{(0)} g_{\nu}^{(1)}(z_{\nu}^{(1)}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b_{\mu\nu}^{(1)} g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,1}\Big) \\
&+ \sum_{\nu=1}^{N} \Big(d_{\mu\nu}^{(0)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(1)}(z_{\nu}^{(1)}(r))\,dr + d_{\mu\nu}^{(1)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(0)}(z_{\nu}^{(0)}(r))\,dr \\
&+ \sum_{\alpha,\beta=1}^{n} d_{\mu\nu}^{(\alpha)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,1}\,dr\Big) + R_{\mu}^{(1)}(\mathbf{t}) + L_{\mu}^{(1)}(\mathbf{t}),
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_{\mu}^{(2)}(\mathbf{t}) =& - c_{\mu} z_{\mu}^{(2)}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a_{\mu\nu}^{(0)} g_{\nu}^{(2)}(z_{\nu}^{(2)}(\mathbf{t})) + \sum_{\nu=1}^{N} a_{\mu\nu}^{(2)} g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t})) \\
&+ \sum_{\alpha,\beta=1}^{n} a_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,2}\Big) + \sum_{\nu=1}^{N} \Big(b_{\mu\nu}^{(0)} g_{\nu}^{(2)}(z_{\nu}^{(2)}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b_{\mu\nu}^{(2)} g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,2}\Big) \\
&+ \sum_{\nu=1}^{N} \Big(d_{\mu\nu}^{(0)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(2)}(z_{\nu}^{(2)}(r))\,dr + d_{\mu\nu}^{(2)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(0)}(z_{\nu}^{(0)}(r))\,dr \\
&+ \sum_{\alpha,\beta=1}^{n} d_{\mu\nu}^{(\alpha)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,2}\,dr\Big) + R_{\mu}^{(2)}(\mathbf{t}) + L_{\mu}^{(2)}(\mathbf{t}),
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{z}_{\mu}^{(n)}(\mathbf{t}) =& - c_{\mu} z_{\mu}^{(n)}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a_{\mu\nu}^{(0)} g_{\nu}^{(n)}(z_{\nu}^{(n)}(\mathbf{t})) + \sum_{\nu=1}^{N} a_{\mu\nu}^{(n)} g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t})) \\
&+ \sum_{\alpha,\beta=1}^{n} a_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,n}\Big) + \sum_{\nu=1}^{N} \Big(b_{\mu\nu}^{(0)} g_{\nu}^{(n)}(z_{\nu}^{(n)}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b_{\mu\nu}^{(n)} g_{\nu}^{(0)}(z_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b_{\mu\nu}^{(\alpha)} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,n}\Big) \\
&+ \sum_{\nu=1}^{N} \Big(d_{\mu\nu}^{(0)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(n)}(z_{\nu}^{(n)}(r))\,dr + d_{\mu\nu}^{(n)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(0)}(z_{\nu}^{(0)}(r))\,dr \\
&+ \sum_{\alpha,\beta=1}^{n} d_{\mu\nu}^{(\alpha)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,n}\,dr\Big) + R_{\mu}^{(n)}(\mathbf{t}) + L_{\mu}^{(n)}(\mathbf{t}). \qquad (4.6)
\end{aligned}
$$

Now defining the error function as $\epsilon(\mathbf{t}) = z(\mathbf{t}) - w(\mathbf{t})$, where $\epsilon(\mathbf{t}) = (\epsilon_1(\mathbf{t}), \epsilon_2(\mathbf{t}), \ldots, \epsilon_N(\mathbf{t}))^T \in \mathbb{H}^N$, and using the equations (4.4) and (4.6), we obtain

$$
\begin{aligned}
\dot{\epsilon}^{(0)}_{\mu}(\mathbf{t}) =& - c_\mu \epsilon^{(0)}_{\mu}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a^{(0)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t})) + \sum_{\alpha,\beta=1}^{n} a^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}))\kappa_{\alpha\beta,0} \Big) \\
&+ \sum_{\nu=1}^{N} \Big(b^{(0)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,0} \Big) \\
&+ \sum_{\nu=1}^{N} d^{(0)}_{\mu\nu} \Big(\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(r))\, dr + \sum_{\alpha,\beta=1}^{n} d^{(\alpha)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(r)) \\
&\times \kappa_{\alpha\beta,0}\, dr \Big) + L^{(0)}_{\mu}(\mathbf{t}), \\
\dot{\epsilon}^{(1)}_{\mu}(\mathbf{t}) =& - c_\mu \epsilon^{(1)}_{\mu}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a^{(0)}_{\mu\nu} g^{(1)}_{\nu}(\epsilon^{(1)}_{\nu}(\mathbf{t})) + \sum_{\nu=1}^{N} a^{(1)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t})) \\
&+ \sum_{\alpha,\beta=1}^{n} a^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}))\kappa_{\alpha\beta,1} \Big) + \sum_{\nu=1}^{N} \Big(b^{(0)}_{\mu\nu} g^{(1)}_{\nu}(\epsilon^{(1)}_{\nu}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b^{(1)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,1} \Big) \\
&+ \sum_{\nu=1}^{N} \Big(d^{(0)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(1)}_{\nu}(\epsilon^{(1)}_{\nu}(r))\, dr + d^{(1)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(r))\, dr \\
&+ \sum_{\alpha,\beta=1}^{n} d^{(\alpha)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(r))\kappa_{\alpha\beta,1}\, dr \Big) + L^{(1)}_{\mu}(\mathbf{t}), \\
\dot{\epsilon}^{(2)}_{\mu}(\mathbf{t}) =& - c_\mu \epsilon^{(2)}_{\mu}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a^{(0)}_{\mu\nu} g^{(2)}_{\nu}(\epsilon^{(2)}_{\nu}(\mathbf{t})) + \sum_{\nu=1}^{N} a^{(2)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t})) \\
&+ \sum_{\alpha,\beta=1}^{n} a^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}))\kappa_{\alpha\beta,2} \Big) + \sum_{\nu=1}^{N} \Big(b^{(0)}_{\mu\nu} g^{(2)}_{\nu}(\epsilon^{(2)}_{\nu}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b^{(2)}_{\mu\nu} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b^{(\alpha)}_{\mu\nu} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,2} \Big) \\
&+ \sum_{\nu=1}^{N} \Big(d^{(0)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(2)}_{\nu}(\epsilon^{(2)}_{\nu}(r))\, dr + d^{(2)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(0)}_{\nu}(\epsilon^{(0)}_{\nu}(r))\, dr \\
&+ \sum_{\alpha,\beta=1}^{n} d^{(\alpha)}_{\mu\nu} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g^{(\beta)}_{\nu}(\epsilon^{(\beta)}_{\nu}(r))\kappa_{\alpha\beta,2}\, dr \Big) + L^{(2)}_{\mu}(\mathbf{t}),
\end{aligned}
$$

$$\vdots$$

$$\begin{aligned}
\dot{\epsilon}_{\mu}^{(n)}(\mathbf{t}) =& -c_{\mu}\epsilon_{\mu}^{(n)}(\mathbf{t}) + \sum_{\nu=1}^{N}\Big(a_{\mu\nu}^{(0)}g_{\nu}^{(n)}(\epsilon_{\nu}^{(n)}(\mathbf{t})) + \sum_{\nu=1}^{N}a_{\mu\nu}^{(n)}g_{\nu}^{(0)}(\epsilon_{\nu}^{(0)}(\mathbf{t})) \\
&+ \sum_{\alpha,\beta=1}^{n}a_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(\epsilon_{\nu}^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,n}\Big) + \sum_{\nu=1}^{N}\Big(b_{\mu\nu}^{(0)}g_{\nu}^{(n)}(\epsilon_{\nu}^{(n)}(\mathbf{t}-\phi(\mathbf{t}))) \\
&+ b_{\mu\nu}^{(n)}g_{\nu}^{(0)}(\epsilon_{\nu}^{(0)}(\mathbf{t}-\phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n}b_{\mu\nu}^{(\alpha)}g_{\nu}^{(\beta)}(\epsilon_{\nu}^{(\beta)}(\mathbf{t}-\phi(\mathbf{t})))\kappa_{\alpha\beta,n}\Big) \\
&+ \sum_{\nu=1}^{N}\Big(d_{\mu\nu}^{(0)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(n)}(\epsilon_{\nu}^{(n)}(r))\,dr + d_{\mu\nu}^{(n)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(0)}(\epsilon_{\nu}^{(0)}(r))\,dr \\
&+ \sum_{\alpha,\beta=1}^{n}d_{\mu\nu}^{(\alpha)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}g_{\nu}^{(\beta)}(\epsilon_{\nu}^{(\beta)}(r))\kappa_{\alpha\beta,n}\,dr\Big) + L_{\mu}^{(n)}(\mathbf{t}), \qquad (4.7)
\end{aligned}$$

where $g_{\nu}^{(\rho)}(\epsilon_{\nu}^{(\rho)}(\mathbf{t})) = g_{\nu}^{(\rho)}(z_{\nu}^{(\rho)}(\mathbf{t})) - g_{\nu}^{(\rho)}(w_{\nu}^{(\rho)}(\mathbf{t}))$, $g_{\nu}^{(\beta)}(\epsilon_{\nu}^{(\beta)}(\mathbf{t})) = g_{\nu}^{(\beta)}(z_{\nu}^{(\beta)}(\mathbf{t})) - g_{\nu}^{(\beta)}(w_{\nu}^{(\beta)}(\mathbf{t}))$, $g_{\nu}^{(\rho)}(\epsilon_{\nu}^{(\rho)}(\mathbf{t}-\phi(\mathbf{t}))) = g_{\nu}^{(\rho)}(z_{\nu}^{(\rho)}(\mathbf{t}-\phi(\mathbf{t}))) - g_{\nu}^{(\rho)}(w_{\nu}^{(\rho)}(\mathbf{t}-\phi(\mathbf{t})))$, and $g_{\nu}^{(\rho)}(\epsilon_{\nu}^{(\rho)}(r)) = g_{\nu}^{(\rho)}(z_{\nu}^{(\rho)}(r)) - g_{\nu}^{(\rho)}(w_{\nu}^{(\rho)}(r))$, $\rho = 0, 1, 2, \ldots, n$.

Remark 4.2.2. After reducing the error system using the decomposition method in the $(n+1)$ system of equations, we will aim to synchronize the systems (4.1) and (4.5) within a fixed-time by constructing appropriate Lyapunov functional and designing controllers, and then using two different Lemmas stated in the next section.

4.2.1 Definition and lemmas

In this subsection, some key definitions and lemmas are discussed which will be needed to derive the main findings of the chapter.

Definition 4.2.1. [31] Let y_0 is the initial value of system (4.7), then the origin of the system (4.7) is stated to have finite-time stability (FNTS*) if $\exists$ settling time function $\mathcal{T} : \mathbb{R}^N \longrightarrow \mathbb{R}$ s.t. $\lim_{t\to\mathcal{T}(y_0)} \|y(\mathbf{t})\| = 0$ and $y(\mathbf{t}) = 0 \;\forall\; t \geq \mathcal{T}(\epsilon_0)$.

Definition 4.2.2. [31] If the origin of the system (4.7) is FNTS* and the settling time function is bounded, i.e. $\exists\, \mathcal{B} > 0$ s.t. $\mathcal{T}(y_0) < \mathcal{B}\ \forall\ y_0 \in \mathbb{H}^N$, then the origin of the system (4.7) is said to be fixed-time stable (FTS*).

Remark 4.2.3. The settling time in FNTS* is dependent on the initial conditions, i.e., we will have various settling time expressions for each change in initial conditions, as can be observed from the definitions of FNTS* and FTS*. The settling time is invariant of initial conditions in FTS*, which means the settling time expression is independent of initial conditions.

Lemma 4.2.1. [99] Supoose $V(.) : \mathbb{R}^N \longrightarrow \mathbb{R}_+ \cup 0$ be a radially unbounded function, which is continuous. Then the system's origin is FTS* if $\epsilon(\mathbf{t})$ is any solution of equation (4.7), s.t.

(i) $V(\epsilon(\mathbf{t})) = 0$ iff $\epsilon(\mathbf{t}) = 0$;

(ii) for $a_1, a_2, a_3 > 0$,

$$\dot{V}(\mathbf{t}) \leq -a_1 V^{\lambda_1}(\epsilon(\mathbf{t})) - a_2 V^{\lambda_2}(\epsilon(\mathbf{t})) - a_3 V(\epsilon(\mathbf{t})),\ 0 < \lambda_1 < 1 \text{ and } \lambda_2 > 1,$$

The expression for the settling time is $\mathcal{T}^{(1)}_{\text{set}} = \frac{1}{a_3(1-\lambda_1)} \ln\Big(1 + \frac{a_3}{a_1}\Big) + \frac{1}{a_3(\lambda_2-1)} \ln\Big(1 + \frac{a_3}{a_2}\Big)$.

Lemma 4.2.2. [100] Supoose $V(.) : \mathbb{R}^N \longrightarrow \mathbb{R}_+$ be a radially unbounded function that is continuous. Then the system's origin is FTS* if $\epsilon(\mathbf{t})$ is any solution of equation (4.7), s.t.

(i) $V(\epsilon(\mathbf{t})) = 0$ iff $\epsilon(\mathbf{t}) = 0$,

(ii) for $a_1, a_2 > 0$,

$$\dot{V}(\mathbf{t}) \leq -a_1 V^{\lambda_1}(\epsilon(\mathbf{t})) - a_2 V^{\lambda_2}(\epsilon(\mathbf{t})),\ 0 < \lambda_1 < 1 \text{ and } \lambda_2 > 1,$$

The expression for the settling time is $\mathcal{T}_{\text{set}}^{(2)} = \frac{1}{a_1(1-\lambda_1)} + \frac{1}{a_2(\lambda_2-1)}$.

Lemma 4.2.3. [101] Suppose $s_\mu \geq 0$ for $\mu = 1, 2, \ldots, N$, then

$$\sum_{\mu=1}^{N} s_\mu^{\lambda_1} \geq \Big(\sum_{\mu=1}^{N} s_\mu\Big)^{\lambda_1}, \ \sum_{\mu=1}^{N} s_\mu^{\lambda_2} \geq N^{1-\lambda_2}\Big(\sum_{\mu=1}^{N} s_\mu\Big)^{\lambda_2},$$

where $0 < \lambda_1 \leq 1, \ \lambda_2 > 1$.

4.3 Lyapunov functional method to achieve FTS

To achieve FTS of the master–response systems (4.1) and (4.5), some sufficient criteria are derived in this section through designing suitable controllers. The controllers listed below are designed to achieve FTS of the master-response systems.

$$\begin{aligned} L_\mu^{(\rho)} =& - m_{1\mu}\epsilon_\mu^{(\rho)}(\mathbf{t}) - sgn(\epsilon_\mu^{(\rho)}(\mathbf{t}))\Big[m_{2\mu}^{(\rho)}|\epsilon_\mu^{(\rho)}(\mathbf{t}-\phi(\mathbf{t}))| + m_{3\mu}^{(\rho)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(\rho)}(r)|\,dr \\ &+ m_{4\mu}|\epsilon_\mu^{(\rho)}(\mathbf{t})|^{\lambda_1} + m_{5\mu}|\epsilon_\mu^{(\rho)}(\mathbf{t})|^{\lambda_2}\Big], \end{aligned} \tag{4.8}$$

where $0 < \lambda_1 < 1, \ \lambda_2 > 1$, and the parameters $m_{4\mu}, m_{5\mu} > 0$, $m_{1\mu}, m_{2\mu}, m_{3\mu}$ for $\mu = 1, 2, \ldots, N$ are designed later. The controller given in (4.8) is nonlinear, involving the signum function, absolute values, integrals, and power-law dependencies. The first term directly counters the current error, providing immediate corrective action. The signum function is used to apply the appropriate direction of correction regardless of the error's sign. The terms involving the absolute values of the delayed error and its integral account for the history and accumulation of errors, ensuring that the controller can correct both recent and past deviations. The power-law terms, allow the controller to adapt to varying magnitudes of errors, providing more robust and precise control. This combination of features makes the controller versatile and effective in managing different error dynamics to achieve the desired stability.

Remark 4.3.1. The controller is developed to include both linear and nonlinear terms. The rate of synchronization is influenced by the nonlinear term. In addition, the settling time achieved is unaffected by delays. In this thesis, the Lyapunov-based method is used to achieve FTS. The Lyapunov-based approach focuses on the construction of the Lyapunov function that proves the stability of a system. We can obtain less conservative stability results by carefully choosing these functions compared to the algebraic method. Lyapunov functions can capture the system's dynamics more accurately and may lead to less conservative stability conditions.

Theorem 4.3.1. If the system (4.7) with Assumption 4.2.2 and controllers (4.8) meets the following criteria, then it achieves FTS*.

$$
\begin{aligned}
e_\mu^{(0)} =& c_\mu + m_{1\mu} - \sum_{\nu=1}^{N} \big(|a_{\nu\mu}^{(0)}| + |a_{\nu\mu}^{(1)}| + |a_{\nu\mu}^{(2)}| + \ldots + |a_{\nu\mu}^{(n)}|\big) l_\mu > 0, \\
e_\mu^{(1)} =& c_\mu + m_{1\mu} - \sum_{\nu=1}^{N} \Big(\big(|a_{\nu\mu}^{(1)}||\kappa_{11,0}| + |a_{\nu\mu}^{(2)}||\kappa_{21,0}| + \ldots + |a_{\nu\mu}^{(n)}||\kappa_{n1,0}|\big) + \big(|a_{\nu\mu}^{(0)}| \\
&+ |a_{\nu\mu}^{(1)}||\kappa_{11,1}| + |a_{\nu\mu}^{(2)}||\kappa_{21,1}| + \ldots + |a_{\nu\mu}^{(n)}||\kappa_{n1,1}|\big) + \big(|a_{\nu\mu}^{(1)}||\kappa_{11,2}| + |a_{\nu\mu}^{(2)}| \\
&\times |\kappa_{21,2}| + \ldots + |a_{\nu\mu}^{(n)}||\kappa_{n1,2}|\big) + \ldots + \big(|a_{\nu\mu}^{(1)}||\kappa_{11,n}| + |a_{\nu\mu}^{(2)}||\kappa_{21,n}| + \ldots \\
&+ |a_{\nu\mu}^{(n)}||\kappa_{n1,n}|\big)\Big) l_\mu > 0, \\
&\vdots \\
e_\mu^{(n)} =& c_\mu + m_{1\mu} - \sum_{\nu=1}^{N} \Big[\Big(\big(|a_{\nu\mu}^{(1)}||\kappa_{1n,0}| + |a_{\nu\mu}^{(2)}||\kappa_{2n,0}| + \ldots + |a_{\nu\mu}^{(n)}||\kappa_{nn,0}|\big) + \big(|a_{\nu\mu}^{(1)}| \\
&\times |\kappa_{1n,1}| + |a_{\nu\mu}^{(2)}||\kappa_{2n,1}| + \ldots + |a_{\nu\mu}^{(n)}||\kappa_{nn,1}|\big) + \big(|a_{\nu\mu}^{(1)}||\kappa_{1n,2}| + |a_{\nu\mu}^{(2)}||\kappa_{2n,2}| \\
&+ \ldots + |a_{\nu\mu}^{(n)}||\kappa_{nn,2}|\big) + \ldots + \big(|a_{\nu\mu}^{(0)}| + |a_{\nu\mu}^{(1)}||\kappa_{1n,n}| + |a_{\nu\mu}^{(2)}||\kappa_{2n,n}| + \ldots
\end{aligned}
$$

$$+ |a^{(n)}_{\nu\mu}||\kappa_{nn,n}|\big)\Big)l_\mu > 0,$$

$$m^{(0)}_{2\mu} - \sum_{\nu=1}^{N} \big(|b^{(0)}_{\nu\mu}| + |b^{(1)}_{\nu\mu}| + |b^{(2)}_{\nu\mu}| + \ldots + |b^{(n)}_{\nu\mu}|\big)l_\mu > 0,$$

$$m^{(1)}_{2\mu} - \sum_{\nu=1}^{N} \Big(\big(|b^{(1)}_{\nu\mu}||\kappa_{11,0}| + |b^{(2)}_{\nu\mu}||\kappa_{21,0}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{n1,0}|\big) + \big(|b^{(0)}_{\nu\mu}| + |b^{(1)}_{\nu\mu}||\kappa_{11,1}|$$

$$+ |b^{(2)}_{\nu\mu}||\kappa_{21,1}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{n1,1}|\big) + \big(|b^{(1)}_{\nu\mu}||\kappa_{11,2}| + |b^{(2)}_{\nu\mu}||\kappa_{21,2}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{n1,2}|\big)$$

$$+ \ldots + \big(|b^{(1)}_{\nu\mu}||\kappa_{11,n}| + |b^{(2)}_{\nu\mu}||\kappa_{21,n}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{n1,n}|\big)\Big)l_\mu > 0,$$

$$\vdots$$

$$m^{(n)}_{2\mu} - \sum_{\nu=1}^{N} \Big[\Big(\big(|b^{(1)}_{\nu\mu}||\kappa_{1n,0}| + |b^{(2)}_{\nu\mu}||\kappa_{2n,0}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{nn,0}|\big) + \big(|b^{(1)}_{\nu\mu}||\kappa_{1n,1}|$$

$$+ |b^{(2)}_{\nu\mu}||\kappa_{2n,1}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{nn,1}|\big) + \big(|b^{(1)}_{\nu\mu}||\kappa_{1n,2}| + |b^{(2)}_{\nu\mu}||\kappa_{2n,2}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{nn,2}|\big)$$

$$+ \ldots + \big(|b^{(0)}_{\nu\mu}| + |b^{(1)}_{\nu\mu}||\kappa_{1n,n}| + |b^{(2)}_{\nu\mu}||\kappa_{2n,n}| + \ldots + |b^{(n)}_{\nu\mu}||\kappa_{nn,n}|\big)\Big)l_\mu > 0,$$

$$m^{(0)}_{3\mu} - \sum_{\nu=1}^{N} \big(|d^{(0)}_{\nu\mu}| + |d^{(1)}_{\nu\mu}| + |d^{(2)}_{\nu\mu}| + \ldots + |d^{(n)}_{\nu\mu}|\big)l_\mu > 0,$$

$$m^{(1)}_{3\mu} - \sum_{\nu=1}^{N} \Big(\big(|d^{(1)}_{\nu\mu}||\kappa_{11,0}| + |d^{(2)}_{\nu\mu}||\kappa_{21,0}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{n1,0}|\big) + \big(|d^{(0)}_{\nu\mu}| + |d^{(1)}_{\nu\mu}|$$

$$\times |\kappa_{11,1}| + |d^{(2)}_{\nu\mu}||\kappa_{21,1}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{n1,1}|\big) + \big(|d^{(1)}_{\nu\mu}||\kappa_{11,2}| + |d^{(2)}_{\nu\mu}||\kappa_{21,2}|$$

$$+ \ldots + |d^{(n)}_{\nu\mu}||\kappa_{n1,2}|\big) + \ldots + \big(|d^{(1)}_{\nu\mu}||\kappa_{11,n}| + |d^{(2)}_{\nu\mu}||\kappa_{21,n}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{n1,n}|\big)\Big)l_\mu > 0,$$

$$\vdots$$

$$m^{(n)}_{3\mu} - \sum_{\nu=1}^{N} \Big[\Big(\big(|d^{(1)}_{\nu\mu}||\kappa_{1n,0}| + |d^{(2)}_{\nu\mu}||\kappa_{2n,0}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{nn,0}|\big) + \big(|d^{(1)}_{\nu\mu}||\kappa_{1n,1}|$$

$$+ |d^{(2)}_{\nu\mu}||\kappa_{2n,1}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{nn,1}|\big) + \big(|d^{(1)}_{\nu\mu}||\kappa_{1n,2}| + |d^{(2)}_{\nu\mu}||\kappa_{2n,2}| + \ldots + |d^{(n)}_{\nu\mu}|$$

$$\times |\kappa_{nn,2}|\big) + \ldots + \big(|d^{(0)}_{\nu\mu}| + |d^{(1)}_{\nu\mu}||\kappa_{1n,n}| + |d^{(2)}_{\nu\mu}||\kappa_{2n,n}| + \ldots + |d^{(n)}_{\nu\mu}||\kappa_{nn,n}|\big)\Big)l_\mu > 0,$$

The predicted settling time is calculated using $\mathcal{T}^{(1)}_{\text{set}} = \frac{1}{e(1-\lambda_1)} \ln\big(1 + \frac{e}{\min_\mu(m_{4\mu})}\big) + \frac{1}{e(\lambda_2-1)} \ln\big(1 + \frac{e}{\min_\mu(m_{5\mu})(N(n+1))^{1-\lambda_2}}\big)$, where $e = \min_{1\le\mu\le N}\{e^{(0)}_\mu, e^{(1)}_\mu, \ldots, e^{(n)}_\mu\}$.

Proof. Set the Lyapunov functional as

$$V(\mathbf{t}) = \sum_{\rho=0}^{n} V^{(\rho)}(\mathbf{t}), \tag{4.9}$$

where $V^{(\rho)}(\mathbf{t}) = \sum_{\mu=1}^{N} |\epsilon_\mu^{(\rho)}(\mathbf{t})|, \ \rho = 0, 1, \ldots, n.$

Then, along the trajectories of the proposed system, the dinni derivative of $V^{(0)}(\mathbf{t})$ is given by

$$\begin{aligned}
\dot{V}^{(0)}(\mathbf{t}) =& \sum_{\mu=1}^{N} sgn(\epsilon_\mu^{(0)}(\mathbf{t}))\dot{\epsilon}_\mu^{(0)}(\mathbf{t}) \\
=& \sum_{\mu=1}^{N} sgn(\epsilon_\mu^{(0)}(\mathbf{t}))\Big[- c_\mu \epsilon_\mu^{(0)}(\mathbf{t}) + \sum_{\nu=1}^{N} \Big(a_{\mu\nu}^{(0)} g_\nu^{(0)}(\epsilon_\nu^{(0)}(\mathbf{t})) + \sum_{\alpha,\beta=1}^{n} a_{\mu\nu}^{(\alpha)} \\
& \times g_\nu^{(\beta)}(\epsilon_\nu^{(\beta)}(\mathbf{t}))\kappa_{\alpha\beta,0}\Big) + \sum_{\nu=1}^{N} \Big(b_{\mu\nu}^{(0)} g_\nu^{(0)}(\epsilon_\nu^{(0)}(\mathbf{t} - \phi(\mathbf{t}))) + \sum_{\alpha,\beta=1}^{n} b_{\mu\nu}^{(\alpha)} \\
& \times g_\nu^{(\beta)}(\epsilon_\nu^{(\beta)}(\mathbf{t} - \phi(\mathbf{t})))\kappa_{\alpha\beta,0}\Big) + \sum_{\nu=1}^{N} d_{\mu\nu}^{(0)} \Big(\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_\nu^{(0)}(\epsilon_\nu^{(0)}(r))\, dr \\
& + \sum_{\alpha,\beta=1}^{n} d_{\mu\nu}^{(\alpha)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} g_\nu^{(\beta)}(\epsilon_\nu^{(\beta)}(r))\kappa_{\alpha\beta,0}\, dr\Big) + L_\mu^{(0)}(\mathbf{t})\Big] \\
\leq& - \sum_{\mu=1}^{N} c_\mu |\epsilon_\mu^{(0)}(\mathbf{t})| + \sum_{\mu=1}^{N}\sum_{\nu=1}^{N} \Big(|a_{\mu\nu}^{(0)}| l_\nu |\epsilon_\nu^{(0)}(\mathbf{t})| + \sum_{\alpha,\beta=1}^{n} |a_{\mu\nu}^{(\alpha)}| l_\nu |\epsilon_\nu^{(\beta)}(\mathbf{t})||\kappa_{\alpha\beta,0}| \\
& + |b_{\mu\nu}^{(0)}| l_\nu |\epsilon_\nu^{(0)}(\mathbf{t} - \phi(\mathbf{t}))| + \sum_{\alpha,\beta=1}^{n} |b_{\mu\nu}^{(\alpha)}| l_\nu |\epsilon_\nu^{(\beta)}(\mathbf{t} - \phi(\mathbf{t}))||\kappa_{\alpha\beta,0}| + |d_{\mu\nu}^{(0)}| l_\nu \\
& \times \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\nu^{(0)}(r)|\, dr + \sum_{\alpha,\beta=1}^{n} |d_{\mu\nu}^{(\alpha)}| l_\nu |\kappa_{\alpha\beta,0}| \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\nu^{(\beta)}(r)|\, dr\Big) \\
& + \sum_{\mu=1}^{N} sgn(\epsilon_\mu^{(0)}(\mathbf{t}))\Big(- m_{1\mu}\epsilon_\mu^{(0)}(\mathbf{t}) - sgn(\epsilon_\mu^{(0)}(\mathbf{t}))\Big(m_{2\mu}^{(0)} |\epsilon_\mu^{(0)}(\mathbf{t} - \phi(\mathbf{t}))| \\
& + m_{3\mu}^{(0)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(0)}(r)|\, dr + m_{4\mu} |\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_1} + m_{5\mu} |\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_2}\Big)\Big)
\end{aligned}$$

$$\dot{V}^{(0)}(\mathbf{t}) \leq -\sum_{\mu=1}^{N}\big(c_\mu - \sum_{\nu=1}^{N}|a_{\nu\mu}^{(0)}|l_\mu + m_{1\mu}\big)|\epsilon_\mu^{(0)}(\mathbf{t})| + \sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\Big(\sum_{\alpha,\beta=1}^{n}|a_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t})|$$
$$\times|\kappa_{\alpha\beta,0}| + |b_{\nu\mu}^{(0)}|l_\mu|\epsilon_\mu^{(0)}(\mathbf{t}-\phi(\mathbf{t}))| + \sum_{\alpha,\beta=1}^{n}|b_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t}-\phi(\mathbf{t}))||\kappa_{\alpha\beta,0}|$$
$$+ |d_{\nu\mu}^{(0)}|l_\mu\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(0)}(r)|\,dr + \sum_{\alpha,\beta=1}^{n}|d_{\nu\mu}^{(\alpha)}|l_\mu|\kappa_{\alpha\beta,0}|\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(\beta)}(r)|\,dr\Big)$$
$$-\sum_{\mu=1}^{N}m_{2\mu}^{(0)}|\epsilon_\mu^{(0)}(\mathbf{t}-\phi(\mathbf{t}))| - \sum_{\mu=1}^{N}m_{3\mu}^{(0)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(0)}(r)|\,dr - \sum_{\mu=1}^{N}\Big(m_{4\mu}$$
$$\times|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_1} + m_{5\mu}|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_2}\Big).$$

Similarly,

$$\dot{V}^{(1)}(\mathbf{t}) \leq -\sum_{\mu=1}^{N}\big(c_\mu - \sum_{\nu=1}^{N}|a_{\nu\mu}^{(0)}|l_\mu + m_{1\mu}\big)|\epsilon_\mu^{(1)}(\mathbf{t})| + \sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\Big(|a_{\nu\mu}^{(1)}|l_\mu|\epsilon_\mu^{(0)}(\mathbf{t})|$$
$$+\sum_{\alpha,\beta=1}^{n}|a_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t})||\kappa_{\alpha\beta,1}| + |b_{\nu\mu}^{(0)}|l_\mu|\epsilon_\mu^{(1)}(\mathbf{t}-\phi(\mathbf{t}))| + |b_{\nu\mu}^{(1)}|l_\mu|\epsilon_\mu^{(0)}(t$$
$$-\phi(\mathbf{t}))| + \sum_{\alpha,\beta=1}^{n}|b_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t}-\phi(\mathbf{t}))||\kappa_{\alpha\beta,1}| + |d_{\nu\mu}^{(0)}|l_\mu\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(1)}(r)|\,dr$$
$$+ |d_{\nu\mu}^{(1)}|l_\mu\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(0)}(r)|\,dr + \sum_{\alpha,\beta=1}^{n}|d_{\nu\mu}^{(\alpha)}|l_\mu|\kappa_{\alpha\beta,1}|\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(\beta)}(r)|\,dr\Big)$$
$$-\sum_{\mu=1}^{N}m_{2\mu}^{(1)}|\epsilon_\mu^{(1)}(\mathbf{t}-\phi(\mathbf{t}))| - \sum_{\mu=1}^{N}m_{3\mu}^{(1)}\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(1)}(r)|\,dr - \sum_{\mu=1}^{N}\Big(m_{4\mu}$$
$$\times|\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_1} + m_{5\mu}|\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_2}\Big).$$

$$\dot{V}^{(2)}(\mathbf{t}) \leq -\sum_{\mu=1}^{N}\big(c_\mu - \sum_{\nu=1}^{N}|a_{\nu\mu}^{(0)}|l_\mu + m_{1\mu}\big)|\epsilon_\mu^{(2)}(\mathbf{t})| + \sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\Big(|a_{\nu\mu}^{(2)}|l_\mu|\epsilon_\mu^{(0)}(\mathbf{t})|$$
$$+\sum_{\alpha,\beta=1}^{n}|a_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t})||\kappa_{\alpha\beta,2}| + |b_{\nu\mu}^{(0)}|l_\mu|\epsilon_\mu^{(2)}(\mathbf{t}-\phi(\mathbf{t}))| + |b_{\nu\mu}^{(2)}|l_\mu|\epsilon_\mu^{(0)}(t$$
$$-\phi(\mathbf{t}))| + \sum_{\alpha,\beta=1}^{n}|b_{\nu\mu}^{(\alpha)}|l_\mu|\epsilon_\mu^{(\beta)}(\mathbf{t}-\phi(\mathbf{t}))||\kappa_{\alpha\beta,2}| + |d_{\nu\mu}^{(0)}|l_\mu\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon_\mu^{(2)}(r)|\,dr$$

$$
\begin{aligned}
&+ |d_{\nu\mu}^{(2)}| l_\mu \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(0)}(r)|\, dr + \sum_{\alpha,\beta=1}^{n} |d_{\nu\mu}^{(\alpha)}| l_\mu |\kappa_{\alpha\beta,2}| \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(\beta)}(r)|\, dr\Big) \\
&- \sum_{\mu=1}^{N} m_{2\mu}^{(2)} |\epsilon_\mu^{(2)}(\mathbf{t}-\phi(\mathbf{t}))| - \sum_{\mu=1}^{N} m_{3\mu}^{(2)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(2)}(r)|\, dr - \sum_{\mu=1}^{N} \Big(m_{4\mu} \\
&\times |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_1} + m_{5\mu} |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_2} \Big).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\dot{V}^{(n)}(\mathbf{t}) \leq &- \sum_{\mu=1}^{N} \big(c_\mu - \sum_{\nu=1}^{N} |a_{\nu\mu}^{(0)}| l_\mu + m_{1\mu}\big) |\epsilon_\mu^{(n)}(\mathbf{t})| + \sum_{\mu=1}^{N} \sum_{\nu=1}^{N} \Big(|a_{\nu\mu}^{(n)}| l_\mu |\epsilon_\mu^{(0)}(\mathbf{t})| \\
&+ \sum_{\alpha,\beta=1}^{n} |a_{\nu\mu}^{(\alpha)}| l_\mu |\epsilon_\mu^{(\beta)}(\mathbf{t})| |\kappa_{\alpha\beta,n}| + |b_{\nu\mu}^{(0)}| l_\mu |\epsilon_\mu^{(n)}(\mathbf{t}-\phi(\mathbf{t}))| + |b_{\nu\mu}^{(n)}| l_\mu |\epsilon_\mu^{(0)}(t \\
&- \phi(\mathbf{t}))| + \sum_{\alpha,\beta=1}^{n} |b_{\nu\mu}^{(\alpha)}| l_\mu |\epsilon_\mu^{(\beta)}(\mathbf{t}-\phi(\mathbf{t}))| |\kappa_{\alpha\beta,n}| + |d_{\nu\mu}^{(0)}| l_\mu \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(n)}(r)|\, dr \\
&+ |d_{\nu\mu}^{(n)}| l_\mu \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(0)}(r)|\, dr + \sum_{\alpha,\beta=1}^{n} |d_{\nu\mu}^{(\alpha)}| l_\mu |\kappa_{\alpha\beta,n}| \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(\beta)}(r)|\, dr\Big) \\
&- \sum_{\mu=1}^{N} m_{2\mu}^{(n)} |\epsilon_\mu^{(n)}(\mathbf{t}-\phi(\mathbf{t}))| - \sum_{\mu=1}^{N} m_{3\mu}^{(n)} \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(n)}(r)|\, dr - \sum_{\mu=1}^{N} \Big(m_{4\mu} \\
&\times |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_1} + m_{5\mu} |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_2} \Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{V}(\mathbf{t}) = &\sum_{\rho=0}^{n} \dot{V}^{(\rho)}(\mathbf{t}) \\
\leq &- \sum_{\mu=1}^{N} \big(c_\mu + m_{1\mu} - \sum_{\nu=1}^{N} \big(|a_{\nu\mu}^{(0)}| + |a_{\nu\mu}^{(1)}| + |a_{\nu\mu}^{(2)}| \ldots + |a_{\nu\mu}^{(n)}|\big) l_\mu) |\epsilon_\mu^{(0)}(\mathbf{t})|\big) \\
&- \sum_{\mu=1}^{N} \big(m_{2\mu}^{(0)} - \sum_{\nu=1}^{N} \big(|b_{\nu\mu}^{(0)}| + |b_{\nu\mu}^{(1)}| + |b_{\nu\mu}^{(2)}| \ldots + |b_{\nu\mu}^{(n)}|\big) l_\mu) |\epsilon_\mu^{(0)}(\mathbf{t}-\phi(\mathbf{t}))|\big) \\
&- \sum_{\mu=1}^{N} \big(m_{3\mu}^{(0)} - \sum_{\nu=1}^{N} \big(|d_{\nu\mu}^{(0)}| + |d_{\nu\mu}^{(1)}| + |d_{\nu\mu}^{(2)}| \ldots + |d_{\nu\mu}^{(n)}|\big) l_\mu) \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(0)}(r)|\, dr\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{\mu=1}^{N}\Big[\Big(c_\mu+m_{1\mu}-\sum_{\nu=1}^{N}\Big(\big(|a^{(1)}_{\nu\mu}||\kappa_{11,0}|+|a^{(2)}_{\nu\mu}||\kappa_{21,0}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{n1,0}|\big)\\
&+\big(|a^{(0)}_{\nu\mu}|+|a^{(1)}_{\nu\mu}||\kappa_{11,1}|+|a^{(2)}_{\nu\mu}||\kappa_{21,1}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{n1,1}|\big)+\big(|a^{(1)}_{\nu\mu}||\kappa_{11,2}|\\
&+|a^{(2)}_{\nu\mu}||\kappa_{21,2}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{n1,2}|\big)+\ldots+\big(|a^{(1)}_{\nu\mu}||\kappa_{11,n}|+|a^{(2)}_{\nu\mu}||\kappa_{21,n}|\\
&+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{n1,n}|\big)\Big)l_\mu\Big)|\epsilon^{(1)}_\mu(\mathbf{t})|\Big]-\sum_{\mu=1}^{N}\Big[\Big(m^{(1)}_{2\mu}-\sum_{\nu=1}^{N}\Big(\big(|b^{(1)}_{\nu\mu}||\kappa_{11,0}|\\
&+|b^{(2)}_{\nu\mu}||\kappa_{21,0}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{n1,0}|\big)+\big(|b^{(0)}_{\nu\mu}|+|b^{(1)}_{\nu\mu}||\kappa_{11,1}|+|b^{(2)}_{\nu\mu}||\kappa_{21,1}|\\
&+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{n1,1}|\big)+\big(|b^{(1)}_{\nu\mu}||\kappa_{11,2}|+|b^{(2)}_{\nu\mu}||\kappa_{21,2}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{n1,2}|\big)\\
&+\ldots+\big(|b^{(1)}_{\nu\mu}||\kappa_{11,n}|+|b^{(2)}_{\nu\mu}||\kappa_{21,n}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{n1,n}|\big)\Big)l_\mu\Big)|\epsilon^{(1)}_\mu(\mathbf{t}-\phi(\mathbf{t})|\Big]\\
&-\sum_{\mu=1}^{N}\Big[\Big(m^{(1)}_{3\mu}-\sum_{\nu=1}^{N}\Big(\big(|d^{(1)}_{\nu\mu}||\kappa_{11,0}|+|d^{(2)}_{\nu\mu}||\kappa_{21,0}|+\ldots+|d^{(n)}_{\nu\mu}||\kappa_{n1,0}|\big)\\
&+\big(|d^{(0)}_{\nu\mu}|+|d^{(1)}_{\nu\mu}||\kappa_{11,1}|+|d^{(2)}_{\nu\mu}||\kappa_{21,1}|+\ldots+|d^{(n)}_{\nu\mu}||\kappa_{n1,1}|\big)+\big(|d^{(1)}_{\nu\mu}||\kappa_{11,2}|\\
&+|d^{(2)}_{\nu\mu}||\kappa_{21,2}|+\ldots+|d^{(n)}_{\nu\mu}||\kappa_{n1,2}|\big)+\ldots+\big(|d^{(1)}_{\nu\mu}||\kappa_{11,n}|+|d^{(2)}_{\nu\mu}||\kappa_{21,n}|\\
&+\ldots+|d^{(n)}_{\nu\mu}||\kappa_{n1,n}|\big)\Big)l_\mu\Big)\int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}}|\epsilon^{(1)}_\mu(r)|\,dr\Big]\\
&\qquad\qquad\vdots\\
&-\sum_{\mu=1}^{N}\Big(c_\mu+m_{1\mu}-\sum_{\nu=1}^{N}\Big[\Big(\big(|a^{(1)}_{\nu\mu}||\kappa_{1n,0}|+|a^{(2)}_{\nu\mu}||\kappa_{2n,0}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{nn,0}|\big)\\
&+\big(|a^{(1)}_{\nu\mu}||\kappa_{1n,1}|+|a^{(2)}_{\nu\mu}||\kappa_{2n,1}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{nn,1}|\big)+\big(|a^{(1)}_{\nu\mu}||\kappa_{1n,2}|+|a^{(2)}_{\nu\mu}|\\
&\times|\kappa_{2n,2}|+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{nn,2}|\big)+\ldots+\big(|a^{(0)}_{\nu\mu}|+|a^{(1)}_{\nu\mu}||\kappa_{1n,n}|+|a^{(2)}_{\nu\mu}||\kappa_{2n,n}|\\
&+\ldots+|a^{(n)}_{\nu\mu}||\kappa_{nn,n}|\big)\Big)l_\mu\Big]|\epsilon^{(n)}_\mu(\mathbf{t})|+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\Big[\Big(\big(|b^{(1)}_{\nu\mu}||\kappa_{1n,0}|+|b^{(2)}_{\nu\mu}||\kappa_{2n,0}|\\
&+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{nn,0}|\big)+\big(|b^{(1)}_{\nu\mu}||\kappa_{1n,1}|+|b^{(2)}_{\nu\mu}||\kappa_{2n,1}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{nn,1}|\big)\\
&+\big(|b^{(1)}_{\nu\mu}||\kappa_{1n,2}|+|b^{(2)}_{\nu\mu}||\kappa_{2n,2}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{nn,2}|\big)+\ldots+\big(|b^{(0)}_{\nu\mu}|+|b^{(1)}_{\nu\mu}|\\
&\times|\kappa_{1n,n}|+|b^{(2)}_{\nu\mu}||\kappa_{2n,n}|+\ldots+|b^{(n)}_{\nu\mu}||\kappa_{nn,n}|\big)\Big)l_\mu\Big]|\epsilon^{(n)}_\mu(t-\phi(\mathbf{t}))|\\
&+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\Big[\Big(\big(|d^{(1)}_{\nu\mu}||\kappa_{1n,0}|+|d^{(2)}_{\nu\mu}||\kappa_{2n,0}|+\ldots+|d^{(n)}_{\nu\mu}||\kappa_{nn,0}|\big)+\big(|d^{(1)}_{\nu\mu}||\kappa_{1n,1}|
\end{aligned}
$$

$$+ |d_{\nu\mu}^{(2)}||\kappa_{2n,1}| + \ldots + |d_{\nu\mu}^{(n)}||\kappa_{nn,1}|\big) + \big(|d_{\nu\mu}^{(1)}||\kappa_{1n,2}| + |d_{\nu\mu}^{(2)}||\kappa_{2n,2}| + \ldots$$
$$+ |d_{\nu\mu}^{(n)}||\kappa_{nn,2}|\big) + \ldots + \big(|d_{\nu\mu}^{(0)}| + |d_{\nu\mu}^{(1)}||\kappa_{1n,n}| + |d_{\nu\mu}^{(2)}||\kappa_{2n,n}| + \ldots + |d_{\nu\mu}^{(n)}|$$
$$\times |\kappa_{nn,n}|\big)\Big)l_\mu\Big] \int_{\mathbf{t}-\psi(\mathbf{t})}^{\mathbf{t}} |\epsilon_\mu^{(n)}(r)|\,dr - \sum_{\mu=1}^{N} m_{4\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_1} + |\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_1}$$
$$+ |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_1} + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_1}\Big) - \sum_{\mu=1}^{N} m_{5\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_2} + |\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_2}$$
$$+ |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_2} + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_2}\Big) \tag{4.10}$$

From Lemma 4.2.1, we obtain

$$\begin{aligned}
\dot{V}(\mathbf{t}) \leq &- eV(\mathbf{t}) - \sum_{\mu=1}^{N} m_{4\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_1} + |\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_1} + |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_1} + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_1}\Big) \\
&- \sum_{\mu=1}^{N} m_{5\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})|^{\lambda_2} + |\epsilon_\mu^{(1)}(\mathbf{t})|^{\lambda_2} + |\epsilon_\mu^{(2)}(\mathbf{t})|^{\lambda_2} + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|^{\lambda_2}\Big) \\
\leq &- eV(\mathbf{t}) - \sum_{\mu=1}^{N} m_{4\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})| + |\epsilon_\mu^{(1)}(\mathbf{t})| + |\epsilon_\mu^{(2)}(\mathbf{t})| + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|\Big)^{\lambda_1} \\
&- \sum_{\mu=1}^{N} (n+1)^{1-\lambda_2} m_{5\mu}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})| + |\epsilon_\mu^{(1)}(\mathbf{t})| + |\epsilon_\mu^{(2)}(\mathbf{t})| + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|\Big)^{\lambda_2} \\
\leq &- eV(\mathbf{t}) - \min_\mu(m_{4\mu})\Big(|\epsilon_\mu^{(0)}(\mathbf{t})| + |\epsilon_\mu^{(1)}(\mathbf{t})| + |\epsilon_\mu^{(2)}(\mathbf{t})| + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|\Big)^{\lambda_1} \\
&- \min_\mu(m_{5\mu})(N(n+1))^{1-\lambda_2}\Big(|\epsilon_\mu^{(0)}(\mathbf{t})| + |\epsilon_\mu^{(1)}(\mathbf{t})| + |\epsilon_\mu^{(2)}(\mathbf{t})| + \ldots + |\epsilon_\mu^{(n)}(\mathbf{t})|\Big)^{\lambda_2} \\
= &- \min_\mu(m_{4\mu})V^{\lambda_1}(\mathbf{t}) - \min_\mu(m_{5\mu})(N(n+1))^{1-\lambda_2}V^{\lambda_2}(\mathbf{t}) - eV(\mathbf{t}).
\end{aligned}$$

Hence, using Lemma 4.2.1 and the controllers given in (4.8), the FTS between the master-response systems (4.1) and (4.5) can be achieved. Furthermore, the settling time can be evaluated as $\mathcal{T}_{\text{set}}^{(1)} = \frac{1}{e(1-\lambda_1)} \ln\big(1 + \frac{e}{\min_\mu(m_{4\mu})}\big) + \frac{1}{e(\lambda_2-1)} \ln\big(1 + \frac{e}{\min_\mu(m_{5\mu})(N(n+1))^{1-\lambda_2}}\big)$. □

Corollary 4.3.1. The system (4.7) achieves FTS* if the controllers (4.8) meet same criteria as in Theorem 4.3.1, under Assumption 4.2.1 with settling time

$$\mathcal{T}_{\text{set}}^{(2)} = \frac{1}{\min_\mu(m_{4\mu})(1-\lambda_1)} + \frac{1}{\min_\mu(m_{5\mu})(N(n+1))^{1-\lambda_2}(\lambda_2-1)}. \tag{4.11}$$

Proof. Choose the Lyapunov functional as

$$V(\mathbf{t}) = \sum_{\rho=0}^{n} V^{(\rho)}(\mathbf{t}).$$

The derivative of $V(\mathbf{t})$ along a system's trajectories as computed in equation (4.10), and with the use of Lemma 4.2.2 and Lemma 4.2.3, we get

$$\dot{V}(\mathbf{t}) \leq -\min_{\mu}(m_{4\mu})V^{\lambda_1}(\mathbf{t}) - \min_{\mu}(m_{5\mu})(N(n+1))^{1-\lambda_2}V^{\lambda_2}(\mathbf{t}),$$

and expression for the corresponding settling time is given in (4.11). □

4.4 Numerical results

In this section, the effectiveness of the derived findings is explained through following numerical examples.

Example 4.4.1. Let $n = 1$, then equation (4.1) becomes CVNNs with mixed time-varying delays and multiplication is defined in Table 1.2. For $N = 2$, the parameters of equation (4.1) are given by $C = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $A = \begin{bmatrix} 0.6+0.2\gamma_1 & 1.1+2\gamma_1 \\ 0.9-0.7\gamma_1 & 1.2+0.8\gamma_1 \end{bmatrix}$, $B = \begin{bmatrix} 0.2-0.3\gamma_1 & 0.1-0.3\gamma_1 \\ 0.8+0.2\gamma_1 & 1.3-0.7\gamma_1 \end{bmatrix}$, $D = \begin{bmatrix} 0.5+0.1\gamma_1 & 1+0.5\gamma_1 \\ 0.8-0.3\gamma_1 & 0.6+0.3\gamma_1 \end{bmatrix}$, $R = (1.2 + 0.5\gamma_1, 0.2+0.6\gamma_1)^T$, $\phi(\mathbf{t}) = \psi(\mathbf{t}) = sin^2(\mathbf{t})$. The activation function is given by

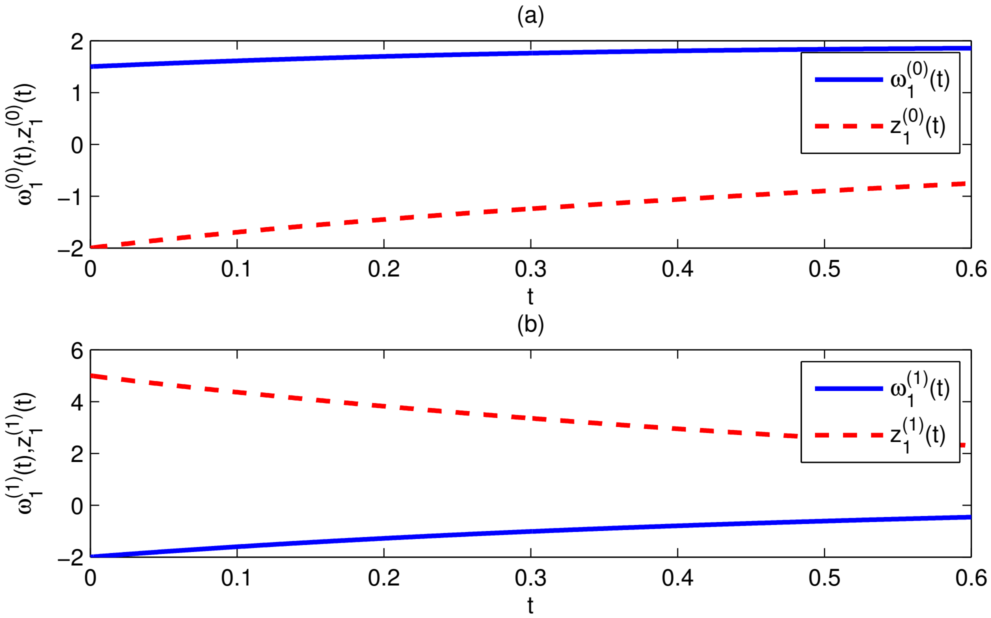

FIGURE 4.1: Plots of trajectories of $\omega_1^{(0)}(\mathbf{t})$ and $z_1^{(0)}(\mathbf{t})$, $\omega_1^{(1)}(\mathbf{t})$ and $z_1^{(1)}(\mathbf{t})$, for the master-response systems (4.1) and (4.5) without controllers.

$g_\mu(\omega_\mu^{(\rho)}) = tanh(\omega_\mu^{(\rho)})$ and we obtain $l_\mu = 1$ for $\mu = 1, 2$ and $\rho = 0, 1$ with the initial conditions $\omega_1(0) = 1.5 - 2\gamma_1$, $\omega_2(0) = 2 - 1\gamma_1$, $z_1(0) = -2 + 5\gamma_1$, $z_2(0) = 0.5 + 2.5\gamma_1$. Now, choosing $m_{11} = 1$, $m_{12} = 2$, $m_{21}^{(\rho)} = 2$, $m_{22}^{(\rho)} = 3$, $m_{31}^{(\rho)} = 2$, $m_{32}^{(\rho)} = 3$, $m_{4\mu} = 20$, $m_{5\mu} = 10$ for $\mu = 1, 2$ and $\rho = 0, 1$, the controller functions become

$$
\begin{aligned}
L_1^{(\rho)}(\mathbf{t}) = &-\epsilon_1^{(\rho)}(\mathbf{t}) - sgn(\epsilon_1^{(\rho)}(\mathbf{t}))[2|\epsilon_1^{(\rho)}(\mathbf{t} - sin^2(\mathbf{t}))| + 2\int_{\mathbf{t}-sin^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_1^{(\rho)}(s)|ds \\
&+ 20|\epsilon_1^{(\rho)}(\mathbf{t})|^{0.2} + 10|\epsilon_1^{(\rho)}(\mathbf{t})|^{1.5}], \\
L_2^{(\rho)}(\mathbf{t}) = &-2\epsilon_2^{(\rho)}(\mathbf{t}) - sgn(\epsilon_2^{(\rho)}(\mathbf{t}))[3|\epsilon_2^{(\rho)}(\mathbf{t} - sin^2(\mathbf{t}))| + 3\int_{\mathbf{t}-sin^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_2^{(\rho)}(s)|ds \\
&+ 20|\epsilon_2^{(\rho)}(\mathbf{t})|^{0.2} + 10|\epsilon_2^{(\rho)}(\mathbf{t})|^{1.5}].
\end{aligned}
$$

In this case, Figure 4.1 and Figure 4.2 depict the systems' non-synchronized and synchronized trajectories without and with controllers, respectively. The settling

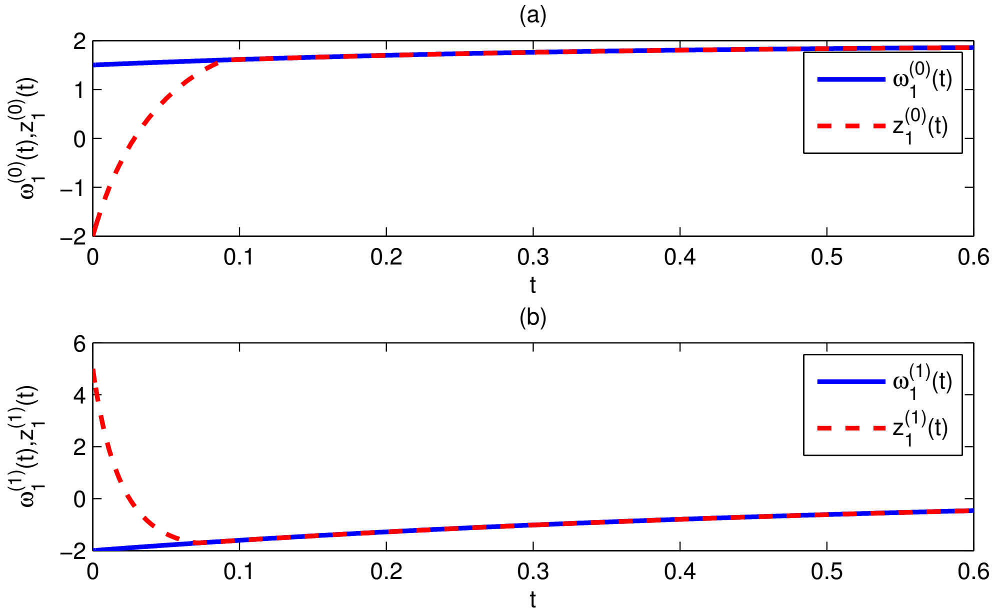

FIGURE 4.2: Plots of trajectories of $\omega_1^{(0)}(\mathbf{t})$ and $z_1^{(0)}(\mathbf{t})$, $\omega_1^{(1)}(\mathbf{t})$ and $z_1^{(1)}(\mathbf{t})$, for the master-response systems (4.1) and (4.5) with controllers.

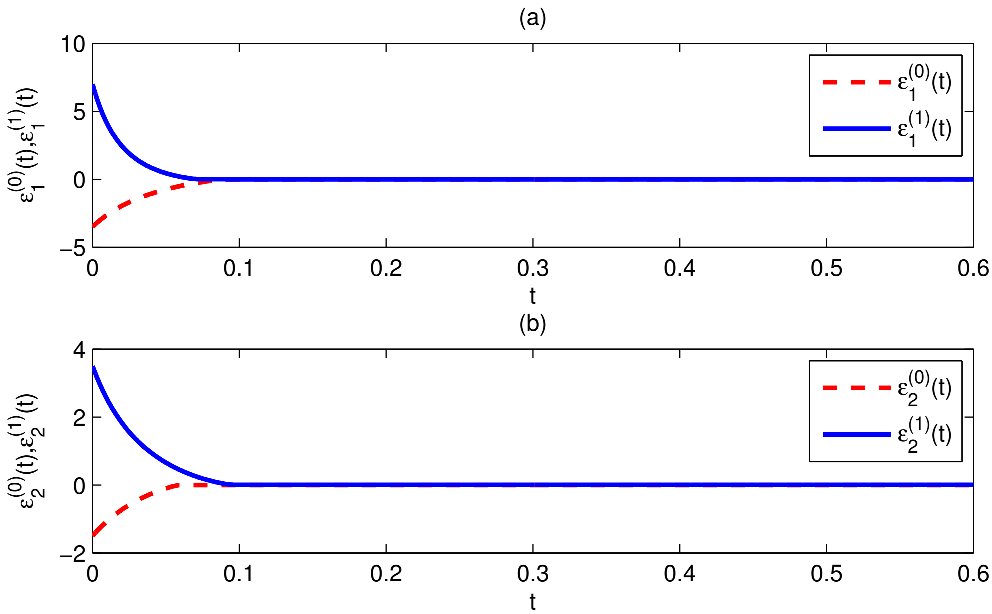

FIGURE 4.3: Plots of synchronization errors of CVNNs.

time is evaluated as $\mathcal{T}_{set}^{(1)} = 0.1634$ and $\mathcal{T}_{set}^{(2)} = 0.4625$. As a result, the settling time produced by Lemma 4.2.1 yields a more accurate result than that derived by Lemma 4.2.2. The synchronization of the error system for CVNNs is shown through Figure

4.3.

Example 4.4.2. Let $n = 3$, then equation (4.1) becomes QVNNs with mixed time-varying delays and multiplication is defined in Table 1.5. For $N = 2$, the parameters of equation (4.1) are given by $C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$,

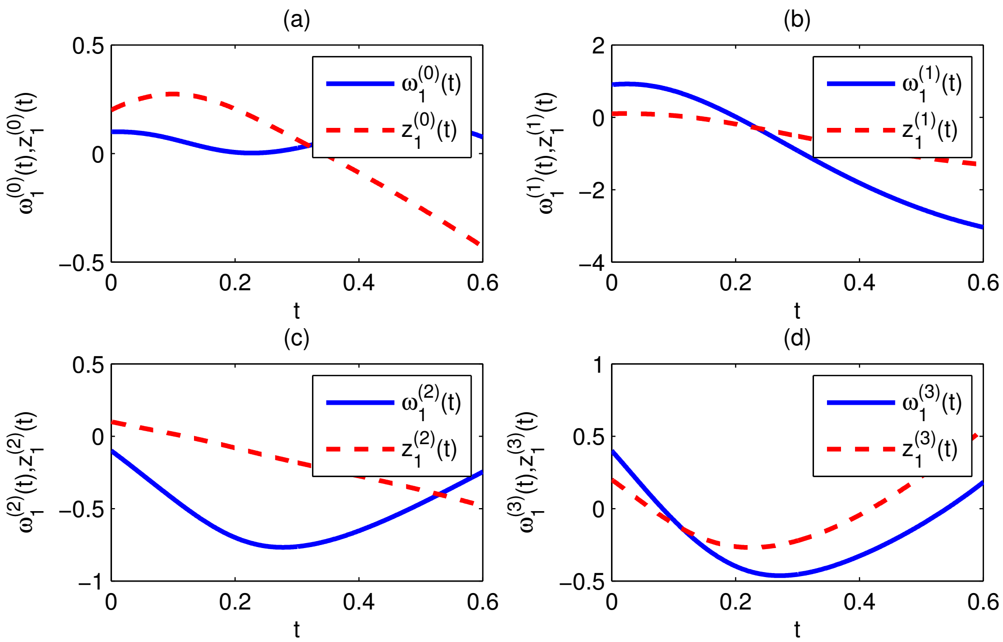

FIGURE 4.4: Plots of trajectories of $\omega_1^{(0)}(\mathbf{t})$ and $z_1^{(0)}(\mathbf{t})$, $\omega_1^{(1)}(\mathbf{t})$ and $z_1^{(1)}(\mathbf{t})$, $\omega_1^{(2)}(\mathbf{t})$ and $z_1^{(2)}(\mathbf{t})$, $\omega_1^{(3)}(\mathbf{t})$ and $z_1^{(3)}(\mathbf{t})$ for the master-response systems (4.1) and (4.5) without controllers.

$$A = \begin{bmatrix} 1.5 + 0.2\gamma_1 + 2\gamma_2 - 1.5\gamma_3 & -2.5 - 2\gamma_1 + 3\gamma_2 - 1.5\gamma_3 \\ -0.5 + 2\gamma_1 + 1.5\gamma_2 + 2\gamma_3 & 1.5 - 0.5\gamma_1 + 1.5\gamma_2 - 2\gamma_3 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 + 2.3\gamma_1 + 2.5\gamma_2 - 1\gamma_3 & 0.5 + 1.5\gamma_1 - 2\gamma_2 - 1.5\gamma_3 \\ 2 - 1.5\gamma_1 + 2\gamma_2 - 1.5\gamma_3 & 3 - 1\gamma_1 + 2.5\gamma_2 - 1\gamma_3 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.4 + 1.5\gamma_1 + 2\gamma_2 + 0.3\gamma_3 & -2 - 0.2\gamma_1 + 2.5\gamma_2 - 1.5\gamma_3 \\ -0.5 - 0.3\gamma_1 + 3.1\gamma_2 - 0.5\gamma_3 & 1.5 + 0.5\gamma_1 - 1.5\gamma_2 + 2\gamma_3 \end{bmatrix},$$
$R = (1.3 - 1.4\gamma_1 + 1.2\gamma_2 + 1.5\gamma_3, 1.2 + 1.4\gamma_1 + 1.5\gamma_2 - 1.3\gamma_3)^T$, $\phi(\mathbf{t}) = \psi(\mathbf{t}) = sin^2(\mathbf{t})$.

The activation function is given by $g_\mu(\omega_\mu^{(\rho)}) = tanh(\omega_\mu^{(\rho)})$ and we obtain $l_\mu = 1$ for

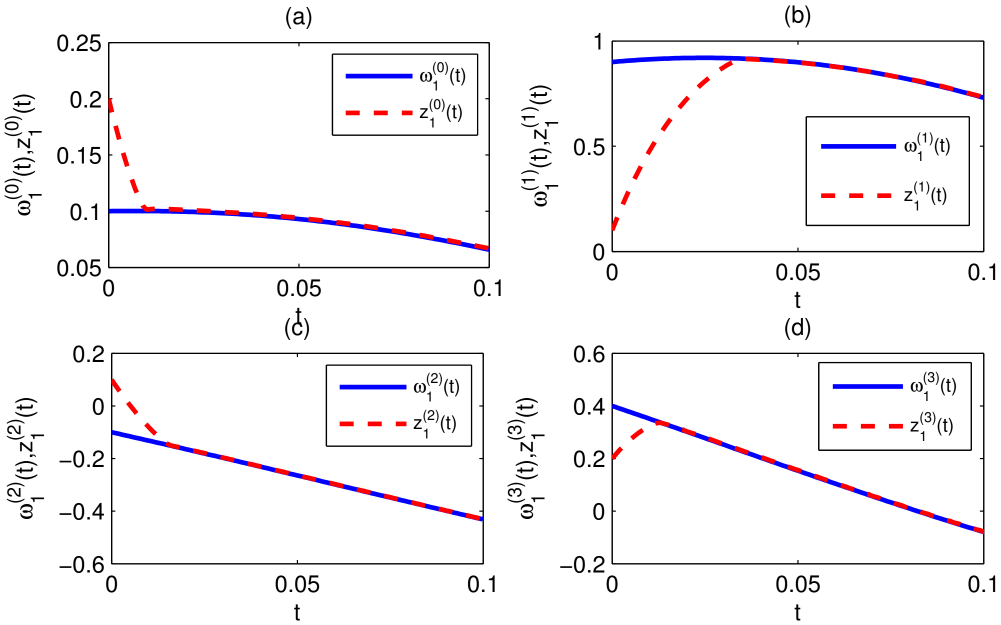

FIGURE 4.5: Plots of trajectories of $\omega_1^{(0)}(\mathbf{t})$ and $z_1^{(0)}(\mathbf{t})$, $\omega_1^{(1)}(\mathbf{t})$ and $z_1^{(1)}(\mathbf{t})$, $\omega_1^{(2)}(\mathbf{t})$ and $z_1^{(2)}(\mathbf{t})$, $\omega_1^{(3)}(\mathbf{t})$ and $z_1^{(3)}(\mathbf{t})$ for the master-response systems (4.1) and (4.5) with controllers.

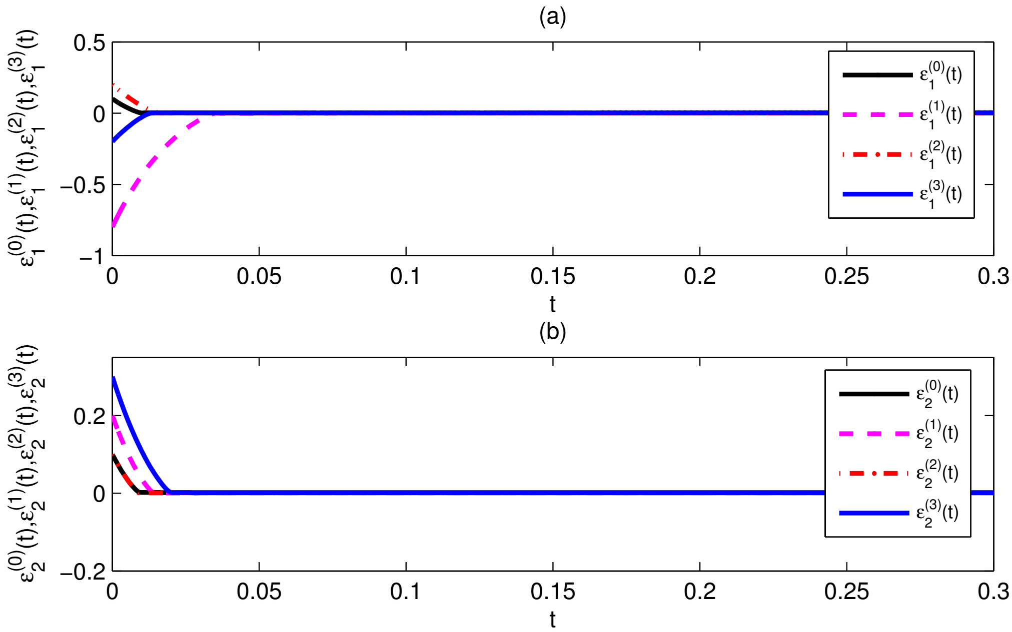

FIGURE 4.6: Plots of synchronization errors of QVNNs.

$\mu = 1,2$ and $\rho = 0,1,2,3$ with the initial conditions $\omega_1(0) = 50+60\gamma_1+70\gamma_2+80\gamma_3$, $\omega_2(0) = 10+20\gamma_1+30\gamma_2+40\gamma_3$, $z_1(0) = 80+70\gamma_1+60\gamma_2+50\gamma_3$, $z_2(0) = 40+30\gamma_1+20\gamma_2+10\gamma_3$.

Now, choosing $m_{11} = 10$, $m_{12} = 12$, $m_{21}^{(\rho)} = 14$, $m_{22}^{(\rho)} = 13.5$, $m_{31}^{(\rho)} = 9$, $m_{32}^{(\rho)} = 11$, $m_{4\mu} = 20$, $m_{5\mu} = 10$ for $\mu = 1,2$ and $\rho = 0,1,2,3$, the controller functions are obtained as

$$\begin{aligned}
L_1^{(\rho)}(\mathbf{t}) =& -10\epsilon_1^{(\rho)}(\mathbf{t}) - sgn(\epsilon_1^{(\rho)}(\mathbf{t}))[14|\epsilon_1^{(\rho)}(\mathbf{t}-sin^2(\mathbf{t}))| + 9\int_{\mathbf{t}-sin^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_1^{(\rho)}(s)|ds \\
&+ 20|\epsilon_1^{(\rho)}(\mathbf{t})|^{0.2} + 10|\epsilon_1^{(\rho)}(\mathbf{t})|^{1.5}], \\
L_2^{(\rho)}(\mathbf{t}) =& -12\epsilon_2^{(\rho)}(\mathbf{t}) - sgn(\epsilon_2^{(\rho)}(\mathbf{t}))[13.5|\epsilon_2^{(\rho)}(\mathbf{t}-sin^2(\mathbf{t}))| + 11\int_{\mathbf{t}-sin^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_2^{(\rho)}(s)|ds \\
&+ 20|\epsilon_2^{(\rho)}(\mathbf{t})|^{0.2} + 10|\epsilon_2^{(\rho)}(\mathbf{t})|^{1.5}].
\end{aligned}$$

In this case, Figure 4.4 and Figure 4.5 show the systems' non-synchronized and synchronized trajectories without and with controllers, respectively. The settling time is evaluated as $\mathcal{T}_{set}^{(1)} = 0.5893$ and $\mathcal{T}_{set}^{(2)} = 0.6281$. Also the synchronization of the error system for QVNNs is shown through Figure 4.6.

Example 4.4.3. Let $n = 7$, then equation (4.1) becomes OVNNs with mixed time-varying delays and multiplication is defined in Table 1.6. For $N = 2$, the parameters of equation (4.1) are given by $C = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$, $R_1 = 0.2 - 1.3\gamma_1 + 1.2\gamma_2 + 1.5\gamma_3 + 1.3\gamma_4 - 1.5\gamma_5 + 1\gamma_6 + 1.8\gamma_7$, $R_2 = 1.3+1.2\gamma_1+1.5\gamma_2-1.2\gamma_3+1.6\gamma_4+1.5\gamma_5+1.7\gamma_6-1.3\gamma_7$, $\phi(\mathbf{t}) = \psi(\mathbf{t}) = cos^2(\mathbf{t})$, and $a_{11} = 1.5 + 2\gamma_1 + 1\gamma_2 + 1.4\gamma_3 + 2.5\gamma_4 + 0.5\gamma_5 + 2\gamma_6 + 1.3\gamma_7$

$a_{12} = -2 - 1.5\gamma_1 - 2.5\gamma_2 - 3\gamma_3 - 1\gamma_4 - 3.2\gamma_5 - 2.3\gamma_6 - 1.8\gamma_7$

$a_{21} = 1 + 1.3\gamma_1 + 2.2\gamma_2 + 3\gamma_3 + 0.5\gamma_4 + 1.8\gamma_5 + 2.5\gamma_6 + 3.2\gamma_7$

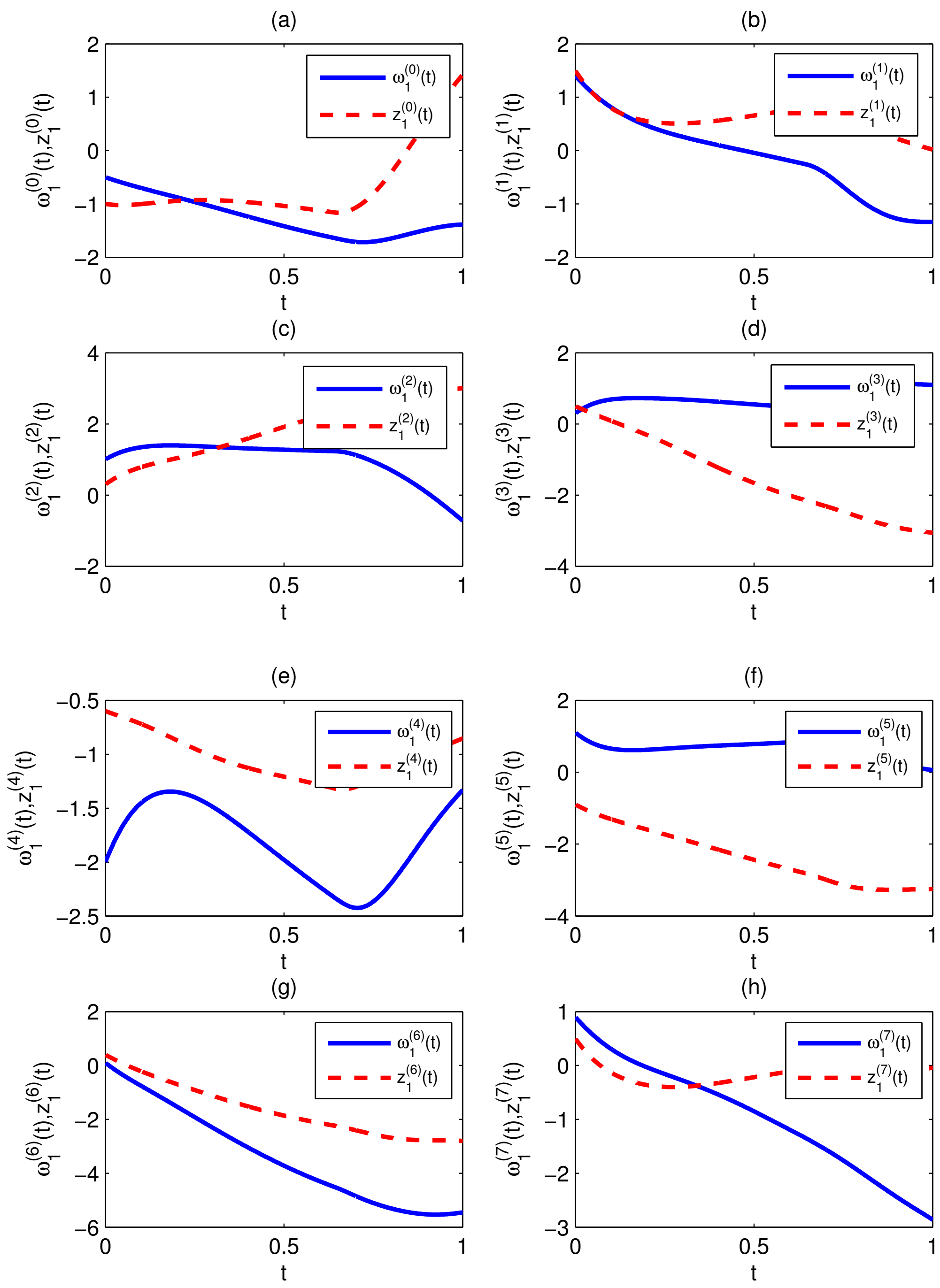

FIGURE 4.7: Plots of trajectories of $\omega_1^{(\rho)}(\mathbf{t})$ and $z_1^{(\rho)}(\mathbf{t})$ where $\rho = 0,1,2,3,4,5,6,7$, for the master-response systems (4.1) and (4.5) without controllers.

$a_{22} = 0.3 + 0.4\gamma_1 + 0.5\gamma_2 + 0.6\gamma_3 + 0.7\gamma_4 + 0.8\gamma_5 + 0.9\gamma_6 + 1.3\gamma_7$

$b_{11} = 2 - 1.5\gamma_1 + 1.3\gamma_2 + 3\gamma_3 + 1.6\gamma_4 - 0.5\gamma_5 + 2.1\gamma_6 + 1.4\gamma_7$

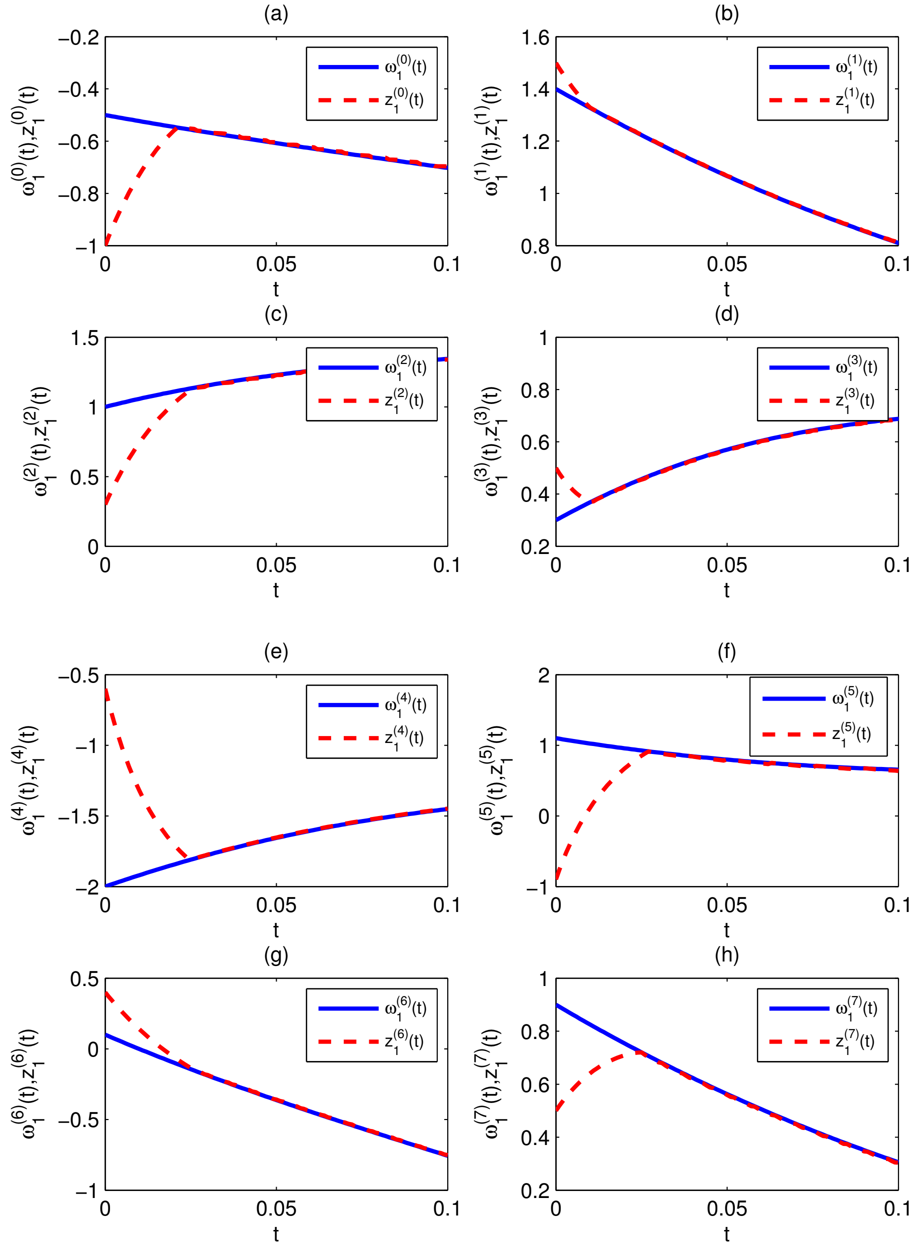

FIGURE 4.8: Plots of trajectories of $\omega_1^{(\rho)}(\mathbf{t})$ and $z_1^{(\rho)}(\mathbf{t})$ where $\rho = 0,1,2,3,4,5,6,7$, for the master-response systems (4.1) and (4.5) with controllers.

$$b_{12} = 3 + 3.2\gamma_1 - 2\gamma_2 + 1\gamma_3 + 1.8\gamma_4 + 2.2\gamma_5 - 1\gamma_6 - 1\gamma_7$$

$$b_{21} = 0.9 - 0.8\gamma_1 + 0.3\gamma_2 + 0.5\gamma_3 + 3.2\gamma_4 - 1\gamma_5 + 0.5\gamma_6 + 0.3\gamma_7$$

$$b_{22} = 0.1 - 1.2\gamma_1 + 2.2\gamma_2 + 0.9\gamma_3 + 0.3\gamma_4 - 0.5\gamma_5 + 1.9\gamma_6 + 1.4\gamma_7$$

$$d_{11} = 1 + 1\gamma_1 + 1.5\gamma_2 + 1.3\gamma_3 + 2\gamma_4 + 1.2\gamma_5 + 1.5\gamma_6 + 1.3\gamma_7$$

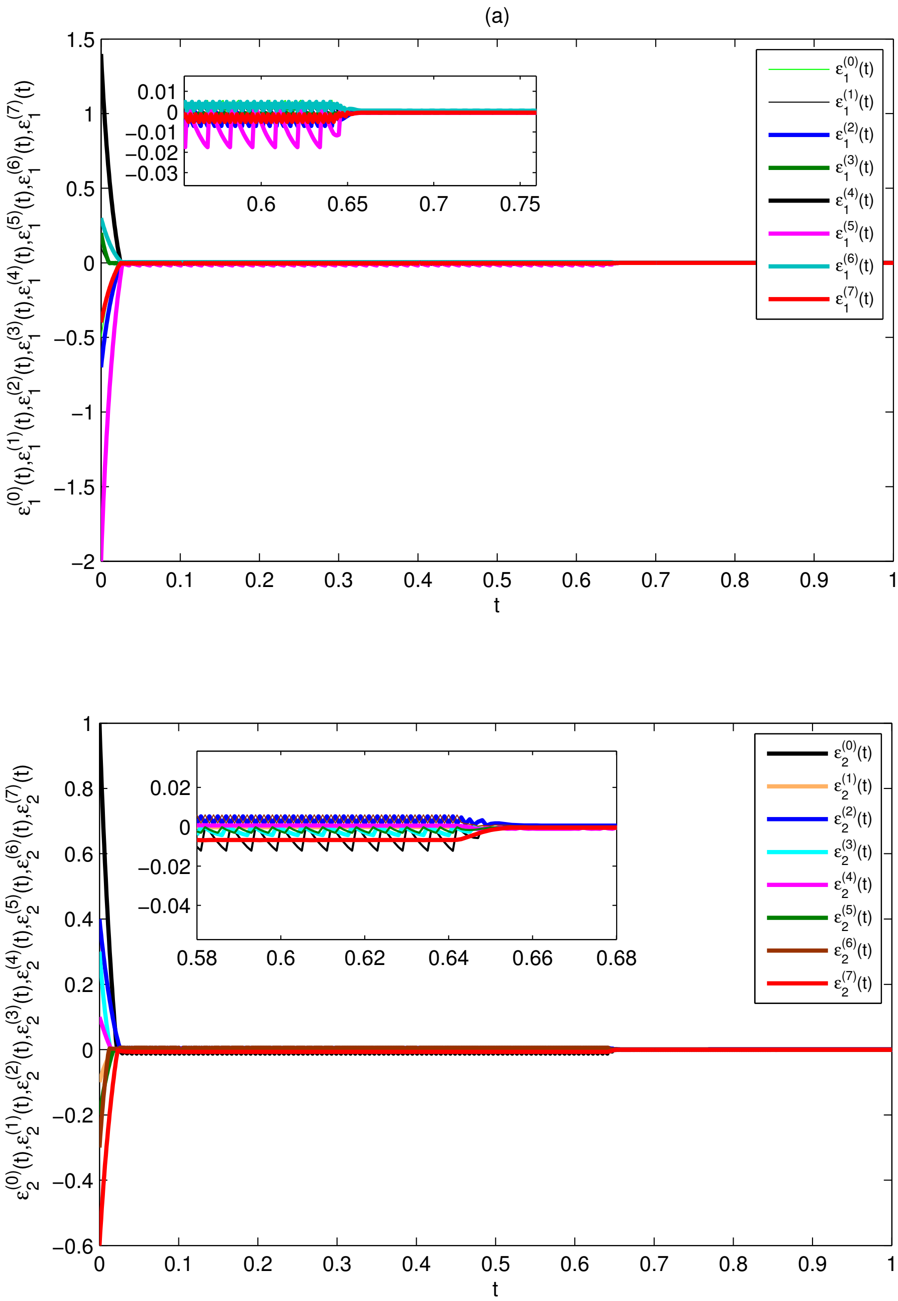

FIGURE 4.9: Plots of synchronization errors of OVNNs.

$d_{12} = -1 - 1\gamma_1 - 2.3\gamma_2 - 3.2\gamma_3 - 1\gamma_4 - 4.5\gamma_5 - 2.3\gamma_6 - 3.2\gamma_7$

$d_{21} = 1 + 1.7\gamma_1 + 1.4\gamma_2 + 1.6\gamma_3 + 1\gamma_4 + 1.7\gamma_5 + 1\gamma_6 + 0.6\gamma_7$

$d_{22} = 0.8 + 0.9\gamma_1 + 0.6\gamma_2 + 0.5\gamma_3 + 0.8\gamma_4 + 0.9\gamma_5 + 0.6\gamma_6 + 0.5\gamma_7.$

The activation function is given by $g_\mu(\omega_\mu^{(\rho)}) = tanh(\omega_\mu^{(\rho)})$ and we obtain $l_\mu = 1$ for

$\mu = 1, 2$ and $\rho = 0, 1, 2, 3, 4, 5, 6, 7$ with the initial conditions

$\omega_1(0) = -0.5 + 1.4\gamma_1 + 1.0\gamma_2 + 0.3\gamma_3 - 2.0\gamma_4 + 1.1\gamma_5 + 0.1\gamma_6 + 0.9\gamma_7,$

$\omega_2(0) = -0.5 + 0.6\gamma_1 - 0.5\gamma_2 - 0.1\gamma_3 + 0.3\gamma_4 + 0.8\gamma_5 + 0.5\gamma_6 + 0.1\gamma_7,$

$z_1(0) = -1 + 1.5\gamma_1 + 0.3\gamma_2 + 0.5\gamma_3 - 0.6\gamma_4 - 0.9\gamma_5 + 0.4\gamma_6 + 0.5\gamma_7,$

$z_2(0) = 0.5 + 0.5\gamma_1 - 0.1\gamma_2 + 0.2\gamma_3 + 0.4\gamma_4 + 0.6\gamma_5 + 0.2\gamma_6 - 0.5\gamma_7.$

Now, choose $m_{11} = 23$, $m_{12} = 19$, $m_{21}^{(\rho)} = 16$, $m_{22}^{(\rho)} = 20$, , $m_{31}^{(\rho)} = 16$, $m_{32}^{(\rho)} = 21$, $m_{4\mu} = 10$, $m_{5\mu} = 20$ for $\mu = 1, 2$ and $\rho = 0, 1, 2, 3, 4, 5, 6, 7$, the controller functions become

$$
\begin{aligned}
L_1^{(\rho)}(\mathbf{t}) =& -23\epsilon_1^{(\rho)}(\mathbf{t}) - sgn(\epsilon_1^{(\rho)}(\mathbf{t}))[16|\epsilon_1^{(\rho)}(t - cos^2(\mathbf{t}))| + 16\int_{t-cos^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_1^{(\rho)}(s)|ds \\
&+ 10|\epsilon_1^{(\rho)}(\mathbf{t})|^{0.2} + 20|\epsilon_1^{(\rho)}(\mathbf{t})|^{1.5}], \\
L_2^{(\rho)}(\mathbf{t}) =& -19\epsilon_2^{(\rho)}(\mathbf{t}) - sgn(\epsilon_2^{(\rho)}(\mathbf{t}))[20|\epsilon_2^{(\rho)}(t - cos^2(\mathbf{t}))| + 21\int_{t-cos^2(\mathbf{t})}^{\mathbf{t}} |\epsilon_2^{(\rho)}(s)|ds \\
&+ 10|\epsilon_2^{(\rho)}(\mathbf{t})|^{0.2} + 20|\epsilon_2^{(\rho)}(\mathbf{t})|^{1.5}].
\end{aligned}
$$

In this case, Figure 4.7 and Figure 4.8 depict the systems' non-synchronized and synchronized trajectories without and with controllers, respectively. The settling time is evaluated as $\mathcal{T}_{set}^{(1)} = 0.7123$ and $\mathcal{T}_{set}^{(2)} = 0.8625$. The synchronization of the error system of OVNNs shown through Figure 4.9.

Remark 4.4.1. The stability of the hypercomplex recurrent correlation NNs using synchronous and asynchronous update modes has been addressed in [102]. The first theorem in [102] demonstrates that if the activation function in the output layer is a B-function, the hypercomplex recurrent correlation NNs will always reach equilibrium through synchronous or asynchronous update modes. While in this chapter, the FTS and FNTS of HCNNs have been investigated with mixed time-varying delays. In FTS, the HCNN state variables reach a synchronized state within a set amount of time, called the settling time, that doesn't depend on initial conditions.

For HCNNs to reach FTS, the controller is designed to ensure convergence within a specific time frame. Examples of complex, quaternion and octonion-valued HCNNs are provided to demonstrate the theoretical findings. As a result, our proposed model and derived results are more general as compared to [102].

4.5 Conclusion

The present chapter focuses on the FTS of master-response systems with time-varying delays, marking the first exploration of FTS in HCNNs with such delays. Here, the state vector is represented using hypercomplex numbers. A set of sufficient conditions is derived to achieve the desired synchronization through the development of a novel controller and careful selection of the appropriate Lyapunov functional. Two distinct settling time formulations are obtained by employing two different lemmas, which demonstrate that the settling time produced by Lemma 1 yields a more accurate result than that has been derived by Lemma 2. To address the non-commutativity and non-associativity inherent in hypercomplex numbers, HCNNs are decomposed into $(n+1)$ equivalent RVNNs, which guide the investigation into HCNNs akin to RVNNs. Finally, numerical simulations corroborate the accuracy of the theoretical findings, extending to CVNNs, QVNNs, and OVNNs.

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