Chapter 5

Exponential and adaptive chaotic lag synchronization of inertial Cohen-Grossberg neural networks with discrete and unbounded distributed delays

5.1 Introduction

The Cohen-Grossberg neural networks (CGNNs), introduced by M.A. Cohen and S. Grossberg in 1983 [82], encompass the traditional Hopfield neural networks. These neural networks have been effectively used in neuroscience, population ecology and evolutionary theory [83, 84, 85, 86, 87]. Therefore, the researchers are keen to look into the dynamical properties of CGNNs. In most of the previous works concerned with the synchronization and stabilization of inertial neural networks (INNs), sufficient conditions are obtained by using the variable substitution method. This approach involves transforming the original INNs into two first-order systems, which can lead to an increase in system dimensions and complicate theoretical analysis. However, in the study by Li et al. [88], a novel Lyapunov functional is constructed to directly investigate the asymptotic stability and synchronization of INNs. Consequently, this non-reduced-order methodology has been employed to explore the stability and synchronization of various INNs [89, 90]. Nonetheless, these findings primarily concentrate on aspects like stability and synchronization. In contrast, the present chapter looks into the lag synchronization of inertial CGNNs. In reality, it is unrealistic that the neural system gets completely synchronized without time lag due to finite signal transmission speed. The transmission of signals from the transmitter to the receiver side often takes a definite time. For example, in the communication systems, the voice heard on the receiver side at time $t + \delta$ corresponds to the voice on the transmitter side at time t. In recent years, the study of lag synchronization has been of great importance, and several results have been published in this field by different researchers [91, 92, 93, 94, 95]. However, the above-mentioned results on lag synchronization are mainly based on first-order NNs. Also, to the best of the knowledge the lag synchronization of CGNNs has not been studied in the literature.

5.2 Preliminaries

Consider a class of CGINNs with unbounded distributed delay as

$$\begin{aligned}\ddot{s}_i(t) =& -\beta_i\dot{s}_i(t) - \alpha_i(s_i(t))\Big[h_i(s_i(t)) - \sum_{j=1}^{n} a_{ij}f_j(s_j(t)) - \sum_{j=1}^{n} b_{ij}f_j(s_j(t - \tau_j(t)) \\ &- \sum_{j=1}^{n} c_{ij}\int_0^\infty K_j(\theta)f_j(s_j(t-\theta))d\theta\Big], \end{aligned} \tag{5.1}$$

for $i \in I = \{1, 2, 3, ..., n\}$, where $s_i(t)$ denotes the state variable of the $i-th$ neuron at time t, $\ddot{s}_i(t)$ is called an inertial term of (5.1), $\beta_i > 0$ is the damping coefficient, $\alpha_i(.)$ denotes the amplification function, $h_i(.)$ is the behaved function, a_{ij}, b_{ij} and c_{ij} are the connection weights from jth neuron to i-th neuron; $f_j(.)$ denotes the activation function of the j-th neuron at time t; $\tau_j(t)$ is the discrete time delay of the j-th neuron at time t, the delay kernel $K_j(.)$: $[0,\infty) \to [0,\infty)$ is continuous function which satisfies $\int_0^\infty K_j(\theta)d\theta = 1$ and $\int_0^\infty K_j(\theta)e^{2\lambda t}d\theta < +\infty$ for some positive value of λ.

The response system corresponding to the master system (5.1) is given by

$$\begin{aligned}\ddot{w}_i(t) =& -\beta_i\dot{w}_i(t) - \alpha_i(w_i(t))\Big[h_i(w_i(t)) - \sum_{j=1}^{n} a_{ij}f_j(w_j(t)) - \sum_{j=1}^{n} b_{ij}f_j(w_j(t - \tau_j(t)) \\ &- \sum_{j=1}^{n} c_{ij}\int_0^\infty K_j(\theta)f_j(w_j(t-\theta))d\theta\Big] + u_i(t), \end{aligned} \tag{5.2}$$

where $u_i(t)$ is the feedback controller defined by

$$u_i(t) = -\delta_i(w_i(t) - s_i(t-\tau_d)), \quad i \in I. \tag{5.3}$$

Let $e_i(t) = w_i(t) - s_i(t-\tau_d)$, be the lag synchronization error where $\tau_d > 0$, for $i \in I$. The error system corresponding to the master-slave systems is obtained by subtracting (5.2) from (5.1) as

$$\begin{aligned}\ddot{e}_i(t) =& -\delta_i e_i(t) - \beta_i \dot{e}_i(t) - [\alpha_i(w_i(t))h_i(w_i(t)) - \alpha_i(s_i((t-\tau_d)))h_i(s_i((t-\tau_d)))] \\ & + \alpha_i(w_i(t))\sum_{j=1}^{n} a_{ij} f_j(e_j(t)) + \alpha_i(w_i(t))\sum_{j=1}^{n} b_{ij} f_j(e_j(t-\tau_j(t))) \\ & + \alpha_i(w_i(t))\sum_{j=1}^{n} c_{ij}\int_0^\infty K_j(\theta) f_j(e_j(t-\theta))d\theta + \Big(\alpha_i(w_i(t)) - \alpha_i(s_i(t-\tau_d))\Big) \\ & \times \Big(\sum_{j=1}^{n} a_{ij} f_j(s_j(t-\tau_d)) + \sum_{j=1}^{n} b_{ij} f_j(s_j(t-\tau_j(t)(t)-\tau_d)) \\ & + \sum_{j=1}^{n} c_{ij}\int_0^\infty K_j(\theta) f_j(s_j(t-\theta-\tau_d))d\theta\Big), \end{aligned} \tag{5.4}$$

where $f_j(e_j(t)) = f_j(w_j(t)) - f_j(s_j(t-\tau_d))$, $f_j(e_j(t-\tau_j(t))) = f_j(w_j(t-\tau_j(t)) - f_j(s_j(t-\tau_j(t)-\tau_d)$ and $f_j(e_j(t-\theta)) = f_j(w_j(t-\theta) - f_j(s_j(t-\theta-\tau_d)$.

In this chapter, the following assumptions are made.

Assumption 8. $\alpha_i(s_i(t))$ is differentiable and satisfies $|\dot{\alpha}_i(s_i(t))| \le \overline{A}_i$ and $0 \le \underline{\alpha}_i \le \alpha_i(s_i(t)) \le \overline{\alpha}_i$.

Assumption 9. Let $g_i(s_i(t)) = \alpha_i(s_i(t))h_i(s_i(t))$, where $g_i(s_i(t))$ is a differentiable function such that there are constants $\underline{L}_i,\ \overline{L}_i > 0$ satisfying $0 < \underline{L}_i \le \dot{g}_i(s_i(t)) \le \overline{L}_i$.

Assumption 10. The time varying delays $\tau_j(t)$ are bounded, differentiable and satisfy $\tau_j(t) \le \tau$ and $\dot{\tau}_j(t) \le \tau_0 < 1$, where τ is a positive constant.

Assumption 11. The activation fuctions f_j satisfy Lipschitz condition and are bounded, i.e., there exist constants $M_j,\ \overline{f}_j > 0$ such that $|f_j(s_k) - f_j(s_i)| \le M_j|s_k - s_i|$, and $|f_j(s_j)| \le \overline{f}_j$, for $i, j, k \in I$..

Lemma 5.1. *[96] If the function $f(x)$ is uniformly continuous and $\lim_{x\to\infty}\int_0^t f(s)ds$ exists and is bounded, then $f(x)$ converges to zero whenever $x \to \infty$.*

5.3 Main results

Theorem 5.2. Under the Assumptions 8-11 the considered system is exponentially lag synchronized under the feedback controller (5.3) if

$$
\begin{aligned}
&-\delta_i-\underline{L}_i+\left|\frac{3}{2}+\frac{\delta_i}{2}-\frac{\beta_i}{2}\right|+\sum_{j=1}^{n}\Big(|a_{ij}|+|b_{ij}|+|c_{ij}|\Big)\Big(\frac{3}{2}\overline{A}_i\overline{f}_j+\overline{\alpha}_iM_j\Big)+\frac{\overline{L}_i}{2}\\
&+\sum_{j=1}^{n}\left(|a_{ji}|+\frac{|b_{ji}|}{(1-\tau_0)}+|c_{ji}|\right)M_i\overline{\alpha}_j<0,\text{and}\\
&2-\beta_i+\left|\frac{3}{2}+\frac{\delta_i}{2}-\frac{\beta_i}{2}\right|+\frac{\overline{L}_i}{2}+\sum_{j=1}^{n}\big(|a_{ij}|+|b_{ij}|+|c_{ij}|\big)\big(\frac{1}{2}\overline{A}_if_j+\frac{1}{2}\overline{\alpha}_iM_j\big)<0.
\end{aligned}
$$

Proof. Let us consider the following Lyapunav functional

$$
\begin{aligned}
V(t)=&\frac{1}{2}\sum_{i=1}^{n}\left[e_i^2(t)+(e_i(t)+\dot{e}_i(t))^2\right]e^{2\lambda(t-\tau_d)}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_iM_j}{(1-\tau_0)}\int_{t-\tau_j(t)}^{t}e_j^2(\theta)e^{2\lambda(\theta+\tau-\tau_d)}d\theta\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}M_j\overline{\alpha}_i|c_{ij}|\int_0^{\infty}\int_{t-\theta}^{t}K_j(\theta)e_j^2(r)e^{2\lambda(r+\theta-\tau_d)}drd\theta. \qquad (5.5)
\end{aligned}
$$

Calculating the time derivative of $V(t)$ along the solutions of equation (5.4), we obtain

$$
\begin{aligned}
\dot{V}(t)=&\sum_{i=1}^{n}\Big(e_i(t)\dot{e}_i(t)+(e_i(t)+\dot{e}_i(t))(\dot{e}_i(t)+\ddot{e}_i(t))\Big)e^{2\lambda(t-\tau_d)}\\
&+\sum_{i=1}^{n}\lambda\big(e_i^2(t)+(e_i(t)+\dot{e}_i(t))^2\big)e^{2\lambda(t-\tau_d)}\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_iM_j}{(1-\tau_0)}\Big(e_j^2(t)e^{2\lambda(t+\tau-\tau_d)}-e_j^2(t-\tau_j(t))e^{2\lambda(t-\tau_j(t)+\tau-\tau_d)}(1-\dot{\tau}_j(t))\Big)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|\overline{\alpha}_iM_j\Big(\int_0^{\infty}K_j(\theta)e_j^2(t)e^{2\lambda\theta}d\theta-\int_0^{\infty}K_j(\theta)e_j^2(t-\theta)d\theta\Big)e^{2\lambda(t-\tau_d)}
\end{aligned}
$$

$$\begin{aligned}\dot{V}(t) =& \sum_{i=1}^{n} \lambda \left(e_i^2(t) + (e_i(t)+\dot{e}_i(t))^2\right) e^{2\lambda(t-\tau_d)} + \sum_{i=1}^{n} \left(e_i(t)\dot{e}_i(t) + (e_i(t)+\dot{e}_i(t))\dot{e}_i(t)\right) e^{2\lambda(t-\tau_d)}\\
&+ \sum_{i=1}^{n}(e_i(t)+\dot{e}_i(t))e^{2\lambda(t-\tau_d)}\Big\{-\delta_i e_i(t) + (1-\beta_i)\dot{e}_i(t) - [\alpha_i(w_i(t))h_i(w_i(t))\\
&- \alpha_i(s_i((t-\tau_d)))h_i(s_i((t-\tau_d)))] + \alpha_i(w_i(t))\sum_{j=1}^{n} a_{ij}f_j(e_j(t))\\
&+ \alpha_i(w_i(t))\sum_{j=1}^{n} b_{ij}f_j(e_j(t-\tau_j(t))) + \alpha_i(w_i(t))\sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(e_j(t-\theta))d\theta\\
&+ \Big(\alpha_i(w_i(t)) - \alpha_i(s_i(t-\tau_d)\Big)\Big(\sum_{j=1}^{n} a_{ij}f_j(s_j(t-\tau_d)) + \sum_{j=1}^{n} b_{ij}f_j(s_j(t-\tau_j(t)-\tau_d))\\
&+ \sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(s_j(t-\theta-\tau_d))d\theta\Big)\Big\}\\
&+ \sum_{i}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}\Big(e_j^2(t)e^{2\lambda(t+\tau-\tau_d)} - e_j^2(t-\tau_j(t))e^{2\lambda(t-\tau_j(t)+\tau-\tau_d)}(1-\dot{\tau}_j(t))\Big)\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|\overline{\alpha}_i M_j\Big(\int_0^{\infty} K_j(\theta)e_j^2(t)e^{2\lambda\theta}d\theta - \int_0^{\infty} K_j(\theta)e_j^2(t-\theta)d\theta\Big)e^{2\lambda(t-\tau_d)}.
\end{aligned} \tag{5.6}$$

Under the Assumption (8), and using the differential mean value theorem, we get

$$\begin{aligned}\dot{V}(t) =& \sum_{i=1}^{n}\Big\{\Big(e_i(t)\dot{e}_i(t) + (e_i(t)+\dot{e}_i(t))\dot{e}_i(t)\Big)e^{2\lambda(t-\tau_d)} + \lambda\big(e_i^2(t) + (e_i(t)+\dot{e}_i(t))^2\big)e^{2\lambda(t-\tau_d)}\\
&+ (e_i(t)+\dot{e}_i(t))e^{2\lambda(t-\tau_d)}\Big[-\delta_i e_i(t) + (1-\beta_i)\dot{e}_i(t) - \dot{g}_i(w_i + \theta_1(w_i - s_i))e_i(t)\\
&+ \alpha_i(w_i(t))\sum_{j=1}^{n} a_{ij}f_j(e_j(t)) + \alpha_i(w_i(t))\sum_{j=1}^{n} b_{ij}f_j(e_j(t-\tau_j(t)))\\
&+ \alpha_i(w_i(t))\sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(e_j(t-\theta))d\theta + \dot{\alpha}_i\big(w_i + \theta_2(w_i - s_i)\big)\\
&\times e_i(t)\Big(\sum_{j=1}^{n} a_{ij}f_j(s_j(t-\tau_d)) + \sum_{j=1}^{n} b_{ij}f_j(s_j(t-\tau_j(t)-\tau_d))
\end{aligned}$$

$$+\sum_{j=1}^{n} c_{ij} \int_0^{\infty} K_j(\theta) f_j(e_j(t-\theta-\tau_d))d\theta\Big)\Big]$$
$$+\sum_{j=1}^{n} \frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}\Big(e_j^2(t)e^{2\lambda(t+\tau-\tau_d)} - e_j^2(t-\tau_j(t))e^{2\lambda(t-\tau_j(t)+\tau-\tau_d)}(1-\dot{\tau}_j(t))\Big)$$
$$+\sum_{j=1}^{n} |c_{ij}|\overline{\alpha}_i M_j\Big(\int_0^{\infty} K_j(\theta)e_j^2(t)e^{2\lambda\theta}d\theta - \int_0^{\infty} K_j(\theta)e_j^2(t-\theta)d\theta\Big)e^{2\lambda(t-\tau_d)}\Big\},$$

where $0 < \theta_1, \theta_2 < 1$.

Using Assumptions (8)-(11), we have

$$\begin{aligned}
\dot{V}(t) \leq & e^{2\lambda(t-\tau_d)} \sum_{i=1}^{n}\Big\{\Big(\Big(2\lambda - \delta_i - \underline{L}_i + \overline{A}_i\Big(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j + \sum_{j=1}^{n}|b_{ij}|\overline{f}_j \\
&+ \sum_{j=1}^{n}|c_{ij}|\overline{f}_j\int_0^{\infty}K_j(\theta)d\theta\Big)\Big)e_i^2(t) + (\lambda+2-\beta_i)\dot{e}_i^2(t) + (2\lambda+3+\delta_i-\beta_i)e_i(t)\dot{e}_i(t) \\
&+ \Big(\overline{A}_i\Big(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j + \sum_{j=1}^{n}|b_{ij}|\overline{f}_j + \sum_{j=1}^{n}|c_{ij}|\overline{f}_j\int_0^{\infty}K_j(\theta)d\theta\Big) + \overline{L}_i\Big)|e_i(t)||\dot{e}_i(t)| \\
&+ \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|M_j|e_i(t)||e_j(t)| + \overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j|e_i(t)||e_j(t-\tau_j(t))| \\
&+ \overline{\alpha_i}\sum_{j=1}^{n}|c_{ij}|M_j\int_0^{\infty}K_j(\theta)|e_i(t)||e_j(t-\theta)|d\theta + \overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|M_j|\dot{e}_i(t)||e_j(t)| \\
&+ \overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j|\dot{e}_i(t)|||e_j(t-\tau_j(t)) + \overline{\alpha}_i\sum_{j=1}^{n}|c_{ij}|M_j\int_0^{\infty}K_j(\theta)|\dot{e}_i(t)||e_j(t-\theta)|d\theta \\
&+ \sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}e_j^2(t)e^{2\lambda\tau} - \sum_{j=1}^{n}|b_{ij}|\overline{\alpha}_i M_j e_j^2(t-\tau_j(t)) \\
&+ \sum_{j=1}^{n}|c_{ij}|\overline{\alpha}_i M_j\Big(\int_0^{\infty}K_j(\theta)e_j^2(t)e^{2\lambda\theta}d\theta - \int_0^{\infty}K_j(\theta)e_j^2(t-\theta)d\theta\Big)\Big\}
\end{aligned}$$

Now by using mean inequality, i.e., $uv \leq \frac{1}{2}(u^2+v^2)$, we have

$$\begin{aligned}\dot{V}(t) \leq & e^{2\lambda(t-\tau_d)} \sum_{i=1}^{n} \Bigg\{ \Bigg[2\lambda - \delta_i - \underline{L}_i + \overline{A}_i \Bigg(\sum_{j=1}^{n} |a_{ij}|\overline{f}_j + \sum_{j=1}^{n} |b_{ij}|\overline{f}_j + \sum_{j=1}^{n} |c_{ij}|\overline{f}_j \int_0^\infty K_j(\theta)d\theta \Bigg) \Bigg] \\ & \times e_i^2(t) + (\lambda + 2 - \beta_i)\dot{e}_i^2(t) + \left| \lambda + \frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2} \right| \big(e_i^2(t) + \dot{e}_i^2(t) \big) + \Bigg[\frac{\overline{A}_i}{2} \Bigg(\sum_{j=1}^{n} |a_{ij}|\overline{f}_j \\ & + \sum_{j=1}^{n} |b_{ij}|\overline{f}_j + \sum_{j=1}^{n} |c_{ij}|\overline{f}_j \int_0^\infty K_j(\theta)d\theta \Bigg) + \frac{\overline{L}_i}{2} \Bigg] (e_i^2(t) + \dot{e}_i^2(t)) + \frac{\overline{\alpha}_i}{2} \sum_{j=1}^{n} |a_{ij}| M_j (e_i^2(t) \\ & + e_j^2(t)) + \frac{\overline{\alpha}_i}{2} \sum_{j=1}^{n} |b_{ij}| M_j (e_i^2(t) + e_j^2(t - \tau_j(t))) + \frac{\overline{\alpha_i}}{2} \sum_{j=1}^{n} |c_{ij}| M_j \int_0^\infty K_j(\theta)((e_i^2(t) \\ & + e_j^2(t-\theta))d\theta + \frac{\overline{\alpha}_i}{2} \sum_{j=1}^{n} |a_{ij}||M_j|(\dot{e}_i^2(t) + e_j^2(t)) + \frac{\overline{\alpha}_i}{2} \sum_{j=1}^{n} |b_{ij}||M_j|(\dot{e}_i^2(t) \\ & + e_j^2(t - \tau_j(t))) + \frac{\overline{\alpha}_i}{2} \sum_{j=1}^{n} |c_{ij}||M_j| \int_0^\infty (\dot{e}_i^2(t) + e_j^2(t-\theta)) + \sum_{j=1}^{n} \frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)} e_j^2(t) e^{2\lambda\tau} \\ & - \sum_{j=1}^{n} |b_{ij}|\overline{\alpha}_i M_j e_j^2(t - \tau_j(t)) + \sum_{j=1}^{n} |c_{ij}|\overline{\alpha}_i M_j \Bigg(\int_0^\infty K_j(\theta) e_j^2(t) e^{2\lambda\theta} d\theta \\ & - \int_0^\infty K_j(\theta) e_j^2(t-\theta)d\theta \Bigg) \Bigg\}\end{aligned}$$

$$\begin{aligned}\dot{V}(t) \leq & e^{2\lambda(t-\tau_d)} \sum_{i=1}^{n} \Bigg\{ \Bigg[2\lambda - \delta_i - \underline{L}_i + \left| \lambda + \frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2} \right| + \sum_{j=1}^{n} \Bigg(|a_{ij}| + |b_{ij}| \\ & + |c_{ij}| \int_o^\infty K_j(\theta)d\theta \Bigg) \Bigg(\frac{3}{2}\overline{A}_i\overline{f}_j + \overline{\alpha}_i M_j \Bigg) + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n} \Bigg(|a_{ji}| + \frac{|b_{ji}|}{(1-\tau_0)} e^{2\lambda\tau} \\ & + |c_{ji}| \int_0^\infty K_i(\theta) e^{2\lambda\theta} d\theta \Bigg) M_i \overline{\alpha}_j \Bigg] e_i^2(t) + \Bigg[\lambda + 2 - \beta_i + \left| \lambda + \frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2} \right| \\ & + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n} \big(|a_{ij}| + |b_{ij}| + |c_{ij}| \int_o^\infty K_j(\theta)d\theta \big) \big(\frac{1}{2}\overline{A}_i f_j + \frac{1}{2}\overline{\alpha}_i M_j \big) \Bigg] \dot{e}_i^2(t) \Bigg\}.\end{aligned} \tag{5.7}$$

As, $\int_0^\infty K_j(\theta)d\theta = 1$ and $\int_0^\infty K_j(\theta)e^{2\lambda\theta}d\theta < \infty$. Therefore, we can choose a small $\lambda > 0$ such that

$$2\lambda - \delta_i - \underline{L}_i + \left|\lambda + \frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \sum_{j=1}^{n}\bigg(|a_{ij}| + |b_{ij}| + |c_{ij}|\int_o^\infty K_j(\theta)d\theta\bigg)\bigg(\frac{3}{2}\overline{A}_i\overline{f}_j + \overline{\alpha}_i M_j\bigg)$$

$$+\frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\bigg(|a_{ji}| + \frac{|b_{ji}|}{(1-\tau_0)}e^{2\lambda\tau} + |c_{ji}|\int_0^\infty K_i(\theta)e^{2\lambda\theta}d\theta\bigg)M_i\overline{\alpha}_j \leq 0, \text{and}$$

$$\lambda + 2 - \beta_i + \left|\lambda + \frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\big(|a_{ij}| + |b_{ij}|$$

$$+|c_{ij}|\int_o^\infty K_j(\theta)d\theta\big)\big(\frac{1}{2}\overline{A}_i f_j + \frac{1}{2}\overline{\alpha}_i M_j\big) \leq 0, \quad \text{for } i = 1, 2, ...n.$$

Then from equation (5.7), $\dot{V}(t) \leq 0$, which implies that $V(t) \leq V(0)$, for all $t \geq 0$. Now, from the equation (5.4), we get

$$V(t) \geq \sum_{i=1}^{n} e_i^2(t)e^{2\lambda(t-\tau_d)}. \tag{5.8}$$

Now

$$\begin{aligned}
V(0) =& \frac{1}{2}\sum_{i=1}^{n}\big[e_i^2(0) + (e_i(0) + \dot{e}_i(0))^2\big]e^{-2\lambda\tau_d} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}\int_{-\tau_j(0)}^{0} e_j^2(\theta)e^{2\lambda(\theta+\tau-\tau_d)}d\theta \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} M_j\overline{\alpha}_i|c_{ij}|\int_0^\infty\int_{-\theta}^{0} K_j(\theta)e_j^2(r)e^{2\lambda(r+\theta-\tau_d)}drd\theta \\
\leq& \frac{1}{2}\sum_{i=1}^{n}\sup_{\theta\in(-\infty,\ 0]}\big[e_i^2(0) + (e_i(0) + \dot{e}_i(0))^2\big] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}\tau e^{2\lambda\tau}\sup_{\theta\in(-\infty,\ 0]} e_j^2(\theta) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} M_j\overline{\alpha}_i|c_{ij}|\int_0^\infty K_j(\theta)e^{2\lambda\theta}d\theta \sup_{\theta\in(-\infty,\ 0]} e_j^2(\theta) &(5.9) \\
\triangleq& \Xi^2\|e_{\tau_d}(\theta)\|^2, &(5.10)
\end{aligned}$$

It follows from (5.8) and (5.10) that $\sum_{i=1}^{n}(w_i(t) - s_i(t-\tau_d)) \leq \Xi\|e_{\tau_d}(\theta)\|e^{-\lambda(t-\tau_d)}, \quad t \geq \tau_d$.

This completes the proof. □

Corollary 5.3. *Under the Assumptions 8-9 and 11, the systems* (5.1) *and* (5.2) *with constant discrete delay* τ_j *and unbounded distributed delay are exponentially lag synchronized via control law* (5.3) *if*

$$-\delta_i - \underline{L}_i + \left|\frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \sum_{j=1}^{n}\Big(|a_{ij}| + |b_{ij}| + |c_{ij}|\Big)\Big(\frac{3}{2}\overline{A}_i\overline{f}_j + \overline{\alpha}_i M_j\Big) + \frac{\overline{L}_i}{2}$$
$$+ \sum_{j=1}^{n}\Big(|a_{ji}| + |b_{ji}| + |c_{ji}|\Big)M_i\overline{\alpha}_j < 0,$$
$$2 - \beta_i + \left|\frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\big(|a_{ij}| + |b_{ij}| + |c_{ij}|\big)\big(\frac{1}{2}\overline{A}_i f_j + \frac{1}{2}\overline{\alpha}_i M_j\big) < 0,$$

for $i \in I$.

Corollary 5.4. *It follows from the Assumptions 8-11 that the systems* (5.1) *and* (5.2) *are globally lag synchronized via control law* (5.3) *for small value of* λ *with discrete time delays* $\tau_j(t)$ *and without distributed delay if the following are satisfied:*

$$-\delta_i - \underline{L}_i + \left|\frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \sum_{j=1}^{n}\Big(|a_{ij}| + |b_{ij}|\Big)\Big(\frac{3}{2}\overline{A}_i\overline{f}_j + \overline{\alpha}_i M_j\Big)$$
$$+ \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\Big(|a_{ji}| + \frac{|b_{ji}|}{(1-\tau_0)}e^{2\lambda\tau}\Big)M_i\overline{\alpha}_j < 0,$$
$$2 - \beta_i + \left|\frac{3}{2} + \frac{\delta_i}{2} - \frac{\beta_i}{2}\right| + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\big(|a_{ij}| + |b_{ij}|\big)\big(\frac{1}{2}\overline{A}_i f_j + \frac{1}{2}\overline{\alpha}_i M_j\big) < 0,$$

for each $i \in I$.

Remark 5.3.1. When the control gains become excessively large in a feedback controller, it can lead to a high control cost, making the control strategy expensive to implement. To address this issue and design a more cost-effective controller from a practical standpoint, we will utilize the following adaptive controller in the subsequent sections, formulated and based on the given results in Theorem 5.5.

Theorem 5.5. *Under the Assumptions 8-11, the systems (5.1) and (5.2) acheive global lag synchronization under the adaptive control scheme given by*

$$\begin{cases} u_i(t) = -\eta_i(t)e_i(t) - \psi_i(t)\dot{e}_i(t), \\ \dot{\eta}_i(t) = \eta_i e_i(t)(e_i(t) + \dot{e}_i(t)), \qquad \textit{where} \quad \eta_i, \ \psi_i > 0 \ \textit{for} \ i \in I \\ \dot{\psi}_i(t) = \psi_i \dot{e}_i(t)(\dot{e}_i(t) + e_i(t)). \end{cases} \tag{5.11}$$

Proof. Consider the following Lyapunov functional as

$$\begin{aligned} V(t) =& \frac{1}{2}\sum_{i=1}^{n}\left[\rho_i e_i^2(t) + (e_i(t) + \dot{e}_i(t))^2\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}\int_{t-\tau_j(t)}^{t} e_j^2(\theta)d\theta + \sum_{i=1}^{n}\sum_{j=1}^{n} M_j \\ & \times \overline{\alpha}_i |c_{ij}| \int_0^{\infty}\int_{t-\theta}^{t} K_j(\theta)e_j^2(r)drd\theta + \frac{1}{2}\sum_{i=1}^{n}\left[\frac{1}{\eta_i}(\eta_i(t) - \overline{\eta}_i)^2 + \frac{1}{\psi_i}(\psi_i(t) - \overline{\psi}_i)^2\right], \end{aligned} \tag{5.12}$$

where $\rho_i, \overline{\eta}_i, \ \overline{\psi}_i$ are positive constants, which will be given later.

Now, calculating the time derivative of $V(t)$ along the solutions of equation (5.4), we obtain

$$\begin{aligned} \dot{V}(t) =& \sum_{i=1}^{n}\left[\rho_i e_i(t)\dot{e}_i(t) + (e_i(t) + \dot{e}_i(t))(\dot{e}_i(t) + \ddot{e}_i(t))\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}(e_j^2(t) \\ & - e_j^2(t-\tau_j(t))(1-\dot{\tau}_j(t))) + \sum_{i=1}^{n}\sum_{j=1}^{n} M_j\overline{\alpha}_i|c_{ij}|\int_0^{\infty} K_j(\theta)(e_j^2(t) - e_j^2(t-\theta))d\theta \\ & + \sum_{i=1}^{n}\left[\frac{1}{\eta_i}(\eta_i(t) - \overline{\eta}_i)\dot{\eta}_i + \frac{1}{\psi_i}(\psi_i(t) - \overline{\psi}_i)\dot{\psi}_i\right] \end{aligned}$$

or,

$$\dot{V}(t) = \sum_{i=1}^{n} \left[\rho_i e_i(t)\dot{e}_i(t) + e_i(t)\dot{e}_i(t) + \dot{e}_i^2(t)\right] + \sum_{i=1}^{n}(e_i(t) + \dot{e}_i(t))\Bigg\{-\eta_i(t)e_i(t)$$
$$- \psi_i(t)\dot{e}_i(t) - \beta_i\dot{e}_i(t) - [\alpha_i(w_i(t))h_i(w_i(t)) - \alpha_i(s_i((t-\tau_d)))h_i(s_i((t-\tau_d)))]$$
$$+ \alpha_i(w_i(t))\sum_{j=1}^{n} a_{ij}f_j(e_j(t)) + \alpha_i(w_i(t))\sum_{j=1}^{n} b_{ij}f_j(e_j(t-\tau_j(t)))$$
$$+ \alpha_i(w_i(t))\sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(e_j(t-\theta))d\theta + \Big(\alpha_i(w_i(t)) - \alpha_i(s_i(t-\tau_d)\Big)$$
$$\times \Bigg(\sum_{j=1}^{n} a_{ij}f_j(s_j(t-\tau_d)) + \sum_{j=1}^{n} b_{ij}f_j(s_j(t-\tau_j(t)-\tau_d))$$
$$+ \sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(s_j(t-\theta-\tau_d))d\theta\Bigg)\Bigg\} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}(e_j^2(t) -$$
$$e_j^2(t-\tau_j(t-\tau_d)(1-\dot{\tau}_j(t))) + \sum_{i=1}^{n}\sum_{j=1}^{n} M_j\overline{\alpha}_i|c_{ij}|\int_0^{\infty} K_j(\theta)(e_j^2(t) - e_j^2(t-\theta))d\theta$$
$$+ \sum_{i=1}^{n}\Big[(\eta_i(t) - \overline{\eta}_i)(e_i^2(t) + e_i(t)\dot{e}_i(t)) + (\psi_i(t) - \overline{\psi}_i)(\dot{e}_i^2(t) + e_i(t)\dot{e}_i(t))\Big] \tag{5.13}$$

By using the Assumption 8 and the differential mean value theorem, we get

$$\dot{V}(t) = \sum_{i=1}^{n} \left[\rho_i e_i(t)\dot{e}_i(t) + e_i(t)\dot{e}_i(t) + \dot{e}_i^2(t)\right] + \sum_{i=1}^{n}(e_i(t) + \dot{e}_i(t))\Bigg\{-\beta_i\dot{e}_i(t)$$
$$- \dot{g}_i(w_i + \theta_1(w_i - s_i))e_i(t) + \alpha_i(w_i(t))\sum_{j=1}^{n} a_{ij}f_j(e_j(t))$$
$$+ \alpha_i(w_i(t))\sum_{j=1}^{n} b_{ij}f_j(e_j(t-\tau_j(t))) + \alpha_i(w_i(t))\sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(e_j(t-\theta))d\theta$$
$$+ \dot{\alpha}_i\big(w_i + \theta_2(w_i - s_i)\big)e_i(t)\Bigg(\sum_{j=1}^{n} a_{ij}f_j(s_j(t-\tau_d)) + \sum_{j=1}^{n} b_{ij}f_j(s_j(t-\tau_j(t)-\tau_d))$$
$$+ \sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_j(\theta)f_j(s_j(t-\theta-\tau_d))d\theta\Bigg)\Bigg\} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|\overline{\alpha}_i M_j}{(1-\tau_0)}(e_j^2(t) -$$

$$e_j^2(t-\tau_j(t-\tau_d)(1-\dot{\tau}_j(t))) + \sum_{i=1}^{n}\sum_{j=1}^{n} M_j\overline{\alpha}_i|c_{ij}|\int_0^\infty K_j(\theta)(e_j^2(t)-e_j^2(t-\theta))d\theta$$
$$+\sum_{i=1}^{n}\Big[-\overline{\eta}_i e_i^2(t)-\overline{\eta}_i e_i(t)\dot{e}_i(t)-\overline{\psi}_i\dot{e}_i^2(t)-\overline{\psi}_i e_i(t)\dot{e}_i(t))\Big], \tag{5.14}$$

for $0<\theta_1,\theta_2<1$.

Using Assumptions (8)-(11), we have

$$\begin{aligned}\dot{V}(t) \leq &\sum_{i=1}^{n}\Big\{\Big(-\underline{L}_i+\overline{A}_i\Big(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j+\sum_{j=1}^{n}|b_{ij}|\overline{f}_j+\sum_{j=1}^{n}|c_{ij}|\overline{f}_j\int_0^\infty K_j(\theta)d\theta\Big)\Big)e_i^2(t)\\
&+(1-\beta_i)\dot{e}_i^2(t)+(\rho_i+1-\beta_i)e_i(t)\dot{e}_i(t)+\Big[\overline{A}_i\Big(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j+\sum_{j=1}^{n}|b_{ij}|\overline{f}_j\\
&+\sum_{j=1}^{n}|c_{ij}|\overline{f}_j\int_0^\infty K_j(\theta)d\theta\Big)+\overline{L}_i\Big]|e_i(t)||\dot{e}_i(t)|+\overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|M_j|e_i(t)||e_j(t)|\\
&+\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j|e_i(t)||e_j(t-\tau_j(t))|+\overline{\alpha_i}\sum_{j=1}^{n}|c_{ij}|M_j\int_0^\infty K_j(\theta)|e_i(t)||e_j(t-\theta)|d\theta\\
&+\overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|M_j|\dot{e}_i(t)||e_j(t)|+\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j|\dot{e}_i(t)|||e_j(t-\tau_j(t))|\\
&+\overline{\alpha}_i\sum_{j=1}^{n}|c_{ij}|M_j\int_0^\infty K_j(\theta)|\dot{e}_i(t)||e_j(t-\theta)|d\theta+\overline{\alpha}_i\sum_{j=1}^{n}\frac{|b_{ij}|M_j}{(1-\tau_0)}e_j^2(t)\\
&-\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j e_j^2(t-\tau_j(t))+\sum_{j=1}^{n}|c_{ij}|\overline{\alpha}_i M_j\Big(\int_0^\infty K_j(\theta)e_j^2(t)d\theta\\
&-\int_0^\infty K_j(\theta)e_j^2(t-\theta)d\theta\Big)\Big\}+\sum_{i=1}^{n}\Big[-\overline{\eta}_i e_i^2(t)-\overline{\eta}_i e_i(t)\dot{e}_i(t)-\overline{\psi}_i\dot{e}_i^2(t)\\
&-\overline{\psi}_i e_i(t)\dot{e}_i(t))\Big].\end{aligned}$$

Now by the use of mean inequality, we get

$$\begin{aligned}
\dot{V}(t) \leq \sum_{i=1}^{n}\Bigg\{ \Bigg(-\underline{L}_i - \overline{\eta}_i + \overline{A}_i\Bigg(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j + \sum_{j=1}^{n}|b_{ij}|\overline{f}_j + \sum_{j=1}^{n}|c_{ij}|\overline{f}_j \int_0^{\infty} K_j(\theta)d\theta\Bigg)\Bigg)e_i^2(t) \\
+ (1-\beta_i-\overline{\psi}_i)\dot{e}_i^2(t) + \big(\rho_i + 1 - \beta_i - \overline{\eta}_i - \overline{\psi}_i\big)\, e_i(t)\dot{e}_i(t) + \Bigg[\frac{\overline{A}_i}{2}\Bigg(\sum_{j=1}^{n}|a_{ij}|\overline{f}_j \\
+ \sum_{j=1}^{n}|b_{ij}|\overline{f}_j + \sum_{j=1}^{n}|c_{ij}|\overline{f}_j \int_0^{\infty} K_j(\theta)d\theta\Bigg) + \frac{\overline{L}_i}{2}\Bigg]\big(e_i^2(t) + \dot{e}_i^2(t)\big) + \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|a_{ij}| \\
\times M_j(e_i^2(t) + e_j^2(t)) + \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|b_{ij}|M_j(e_i^2(t) + e_j^2(t-\tau_j(t))) + \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|c_{ij}|M_j \\
\times \int_0^{\infty} K_j(\theta)((e_i^2(t) + e_j^2(t-\theta))d\theta + \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|a_{ij}||M_j|(\dot{e}_i^2(t) + e_j^2(t)) \\
+ \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|b_{ij}||M_j|(\dot{e}_i^2(t) + e_j^2(t-\tau_j(t))) + \frac{\overline{\alpha}_i}{2}\sum_{j=1}^{n}|c_{ij}||M_j|\int_0^{\infty}(\dot{e}_i^2(t) + e_j^2(t-\theta)) \\
+ \overline{\alpha}_i\sum_{j=1}^{n}\frac{|b_{ij}|M_j}{(1-\tau_0)}e_j^2(t) - \overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|M_j e_j^2(t-\tau_j(t)) \\
+ \sum_{j=1}^{n}|c_{ij}|\overline{\alpha}_i M_j\Bigg(\int_0^{\infty}K_j(\theta)e_j^2(t)d\theta - \int_0^{\infty}K_j(\theta)e_j^2(t-\theta)d\theta\Bigg)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\dot{V}(t) \leq \sum_{i=1}^{n}\Bigg\{\Bigg[-\underline{L}_i - \overline{\eta}_i + \sum_{j=1}^{n}\Bigg(|a_{ij}| + |b_{ij}| + |c_{ij}|\int_o^{\infty}K_j(\theta)d\theta\Bigg)\Bigg(\frac{3}{2}\overline{A}_i\overline{f}_j + \frac{1}{2}\overline{\alpha}_i M_j\Bigg) \\
+ \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\Bigg(|a_{ji}| + \frac{|b_{ji}|}{(1-\tau_0)} + |c_{ji}|\int_0^{\infty}K_i(\theta)d\theta\Bigg)M_i\overline{\alpha}_j\Bigg]e_i^2(t) + (\rho_i + 1 - \beta_i \\
- \overline{\eta}_i - \overline{\psi}_i)e_i(t)\dot{e}_i(t) + \Bigg[1 - \beta_i - \overline{\psi}_i + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n}\big(|a_{ij}| + |b_{ij}| \\
+ |c_{ij}|\int_o^{\infty}K_j(\theta)d\theta\big)\big(\frac{1}{2}\overline{A}_i\overline{f}_j + \frac{1}{2}\overline{\alpha}_i M_j\big)\Bigg]\dot{e}_i^2(t)\Bigg\}. \qquad (5.15)
\end{aligned}$$

For each $i \in I$, choose

$$\begin{aligned}
\overline{\eta}_i =& \sum_{j=1}^{n} \left(|a_{ij}| + |b_{ij}| + |c_{ij}| \int_o^\infty K_j(\theta)d\theta \right) \left(\frac{3}{2}\overline{A}_i \overline{f}_j + \frac{1}{2}\overline{\alpha}_i M_j \right) + \frac{\overline{L}_i}{2} \\
&+ \sum_{j=1}^{n} \left(|a_{ji}| + \frac{|b_{ji}|}{(1-\tau_0)} + |c_{ji}| \int_0^\infty K_i(\theta)d\theta \right) M_i \overline{\alpha}_j - \underline{L}_i + 1, && (5.16) \\
\overline{\psi}_i =& 1 - \beta_i + \frac{\overline{L}_i}{2} + \sum_{j=1}^{n} \big(|a_{ij}| + |b_{ij}| + |c_{ij}| \int_o^\infty K_j(\theta)d\theta \big) \big(\frac{1}{2}\overline{A}_i \overline{f}_j + \frac{1}{2}\overline{\alpha}_i M_j \big), && (5.17) \\
\rho_i =& \beta_i + \overline{\psi}_i + \overline{\eta}_i - 1. && (5.18)
\end{aligned}$$

From equation (5.15), it can be easily verified that $\rho_i > 0$. Therefore,

$$\dot{V}(t) \leq -\sum_{i=1}^{n} e_i^2(t), \tag{5.19}$$

which shows that

$$\lim_{x\to\infty} \int_0^t \sum_{i=1}^{n} e_i^2(s)ds \leq V(0) < +\infty. \tag{5.20}$$

Hence, $V(t) \leq V(0)$ for $t \geq 0$.

Then by the definition of $V(t)$, both $e_i(t)$ and $\dot{e}_i(t)$ are bounded for each $i \in I$ and $t \geq 0$. This means that the derivative of $\sum_{i=1}^n e_i^2(t)$ is bounded and hence $\sum_{i=1}^n e_i^2(t)$ is uniformly continuous.

By applying the lemma 5.1, we have

$$\lim_{t\to\infty} \sum_{i=1}^{n} e_i^2(t) = 0.$$

Hence, the master-slave systems (5.1) and (5.2) are globally asymptotically synchronized, which shows that proof of the theorem is completed. □

Corollary 5.3.1. Under the Assumption 8-9, and Assumption 11 system (5.1) and (5.2) with constant discrete $\tau_\jmath$ and unbounded distributed delay is exponentially lag synchronized via control law (5.11).

Corollary 5.3.2. Under the Assumption 8-11, system (5.1) and (5.2) with discrete delays $\tau_\jmath(t)$ without unbounded distributed delay is exponentially lag synchronized via control law (5.11).

Remark 5.3.2. Indeed, the Lyapunov functional method is a well-known approach for pursuing the synchronization of NNs. In many Lyapunov functionals, the state variables of the model in question are utilized. In the case of Eq. (5.1), treated as a second-order differential system for studying its lag synchronization, the authors have introduced a newly developed Lyapunov functional (5.5) and (5.12). This particular Lyapunov functional includes the state variables and their derivatives, which distinguishes it from the traditional Lyapunov functional. This novel approach likely allows for a more comprehensive analysis and potentially different insights into the synchronizations of the system.

Remark 5.3.3. Up to this point, research on synchronization and stability for ICGNNs using the non-reduced order approach has yielded a limited number of results, as cited in [97, 98, 99, 100]. However, it's noteworthy that these research authors have focused on scenarios involving either constant or bounded discrete delays. In practical terms, the functioning of a neuron indeed relies on its entire previous state history. Therefore, considering infinite delays in NNs would represent real-world behavior and dynamics more. Addressing these infinite delays in ICGNNs presents an interesting avenue for further research.

Remark 5.3.4. If we opt for the behavior function as $h_\imath(\upsilon_\imath(t)) = h_\imath \upsilon_\imath(t)$ and set the amplification factor as $\alpha_\imath(\upsilon_\imath(t)) = 1$, then the system described in Eq. (5.1) transforms into Hopfield-type Neural Networks (NNs). In this context, the outcomes related to lag synchronization, synchronization and stability that have been presented in works [101, 102, 103, 46, 88] become specific instances of the more general results obtained for a broader class of NNs. This demonstrates the versatility and applicability of the proposed framework to a wide range of neural network models beyond just the Hopfield-type.

Remark 5.3.5. If you select the lag factor $\tau_d = 0$, and there is no presence of an unbounded distributed delay term, then the results presented in the work would indeed encompass the synchronization of ICGNNs, as mentioned in the article [99]. In such a scenario, it's reasonable to conclude that the results that have derived are more general and applicable to a broader range of NNs model, thereby extending the scope and relevance of your findings.

Remark 5.3.6. It's notable that, up until now, there have been no results regarding the adaptive lag synchronization of ICGNNs using the non-reduced order approach. To address this research gap, this article has successfully introduced an adaptive control scheme as presented in Equation (5.11). This scheme has been designed to explore the global lag synchronization of ICGNNs, and this result has been substantiated in Theorem 5.5. This contribution fills an important void in the field and provides valuable insights into the lag synchronization of ICGNNs with an adaptive control approach.

5.4 Examples

Example 5.1. *Let us consider the two-dimensional ICGNNs with discrete and unbounded distributed delays*

$$\ddot{\upsilon}_\imath(t) = -\beta_\imath\dot{\upsilon}_\imath(t) - \alpha_\imath(\upsilon_\imath(t))\Bigg[h_\imath(\upsilon_\imath(t)) - \sum_{j=1}^{n} a_{\imath\jmath}f_\jmath(\upsilon_\jmath(t)) - \sum_{j=1}^{n} b_{\imath\jmath}f_\jmath(\upsilon_\jmath(t - \tau_j(t)) - \sum_{j=1}^{n} c_{\imath\jmath}\int_0^\infty K_\jmath(\theta)f_\jmath(\upsilon_\jmath(t-\theta))d\theta\Bigg], \quad i = 1, 2, \tag{5.21}$$

and the corresponding response system is

$$\ddot{w}_\imath(t) = -\beta_\imath\dot{w}_\imath(t) - \alpha_\imath(w_\imath(t))\Bigg[h_\imath(w_\imath(t)) - \sum_{j=1}^{n} a_{\imath\jmath}f_\jmath(w_\jmath(t)) - \sum_{j=1}^{n} b_{\imath\jmath}f_\jmath(w_\jmath(t - \tau_j(t)) - \sum_{j=1}^{n} c_{\imath\jmath}\int_0^\infty K_\jmath(\theta)f_\jmath(w_\jmath(t-\theta))d\theta\Bigg] + u_\imath(t), \quad i = 1, 2, \tag{5.22}$$

where $\beta_1 = 8, \beta_2 = 10.4, \alpha_\imath(\upsilon_\imath(t)) = 1 + \frac{1}{(1+\upsilon_i^2(t))}, h_\imath(\upsilon_\imath(t)) = \frac{1}{2}\upsilon_\imath(t), f_\imath(\upsilon_\imath(t)) = \frac{1}{4}\tanh(\upsilon_\imath(t)), \tau_\jmath(t) = \frac{e^t}{1+e^t},$

$K_\jmath(\theta) = \frac{\sin(2\theta)}{1+\theta^2}$ *for* $j = 1, 2.$ $a_{11} = 0.5, a_{12} = 0.7, a_{21} = -0.4, a_{22} = -0.2, b_{11} = 0.4, b_{12} = 0.6, b_{21} = -0.3, b_{22} = -0.2, c_{11} = -0.2, c_{12} = -0.4, c_{21} = 0.3, c_{22} = -0.6$

Also control scheme is defined as, $u_1(t) = -10e_\imath(t),\ u_2(t) = -12e_2(t)$

Clearly, $1 \leq \alpha_\imath(\upsilon_\imath(t)) \leq 2, |\dot{\alpha}_\imath(\upsilon_\imath(t))| \leq 0.65,\ g_\imath(\upsilon_\imath(t)) = \alpha_\imath(\upsilon_\imath(t))h_\imath(\upsilon_\imath(t)) = \frac{1}{2}\left(\upsilon_\imath(t) + \frac{\upsilon_\imath(t)}{1+\upsilon_i^2(t)}\right), 0.5 \leq \dot{g}_\imath(\upsilon_\imath) \leq 1$ *Select* $\underline{\alpha}_\imath = 1, \overline{\alpha}_\imath = 2, \overline{A}_\imath = 0.65, \underline{L}_\imath = 0.5, \overline{L}_\imath = 1, \overline{f}_\imath = \frac{1}{4}, M_\imath = \frac{1}{4}, \tau = 1, \tau_0 = \frac{1}{4} < 0$ *for* $i = 1, 2.$

By some simple calculations, we get

$$-\delta_1 - \underline{L}_1 + \left|\frac{3}{2} + \frac{\delta_1}{2} - \frac{\beta_1}{2}\right| + \sum_{j=1}^{2}\Big(|a_{1j}| + |b_{1j}| + |c_{1j}|\Big)\Big(\frac{3}{2}\overline{A}_1\overline{f}_\jmath + \overline{\alpha}_1 M_\jmath\Big) + \frac{\overline{L}_1}{2} + \sum_{j=1}^{2}\left(|a_{j1}| + \frac{|b_{j1}|}{(1-\tau_0)} + |c_{j1}|\right)M_1\overline{\alpha}_\jmath = -5.251 < 0,$$

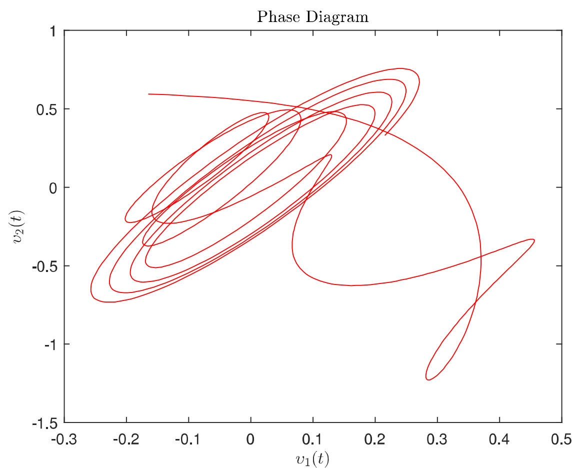

FIGURE 5.1: Phase diagram of the system (5.21)

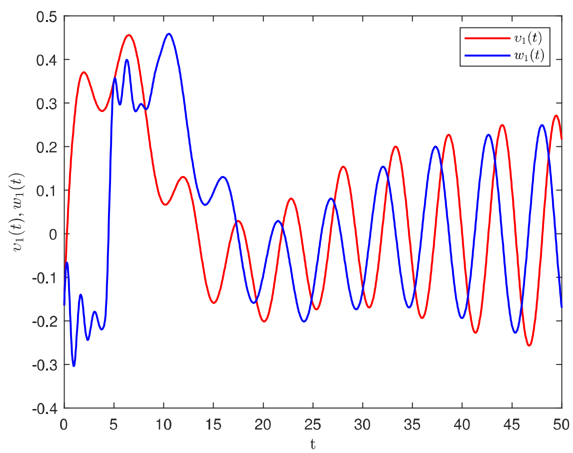

FIGURE 5.2: Lag synchronization curve of $v_1(t)$ and $w_1(t)$ for $\tau_d = 4$

$$-\delta_2 - \underline{L}_2 + \left|\frac{3}{2} + \frac{\delta_2}{2} - \frac{\beta_2}{2}\right| + \sum_{j=1}^{2}\Big(|a_{2j}| + |b_{2j}| + |c_{2j}|\Big)\left(\frac{3}{2}\overline{A}_2\overline{f}_j + \overline{\alpha}_2 M_j\right) + \frac{\overline{L}_2}{2}$$
$$+ \sum_{j=1}^{2}\left(|a_{j2}| + \frac{|b_{j2}|}{(1-\tau_0)} + |c_{j2}|\right) M_2\overline{\alpha}_j = -7.427 < 0,$$

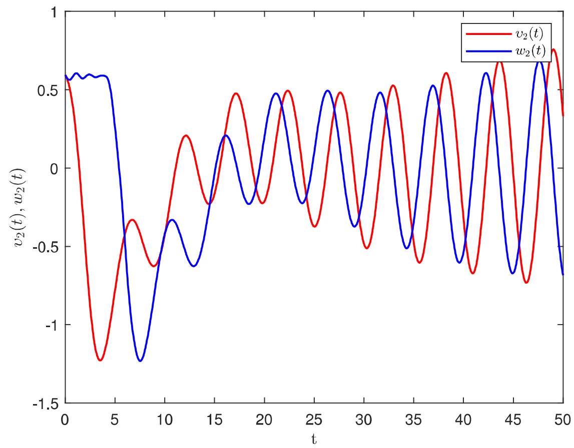

FIGURE 5.3: Lag synchronization curve of $v_2(t)$ and $w_2(t)$ for $\tau_d = 4$.

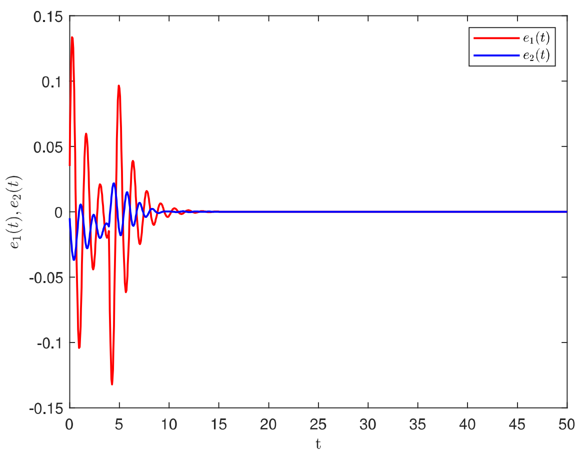

FIGURE 5.4: Time response of $e_1(t), e_2(t)$ via feedback controller (5.3).

$$2 - \beta_1 + \left|\frac{3}{2} + \frac{\delta_1}{2} - \frac{\beta_1}{2}\right| + \frac{\overline{L}_1}{2} + \sum_{j=1}^{2}\Big(|a_{1j}| + |b_{1j}| + |c_{1j}|\Big)\left(\frac{3}{2}\overline{A}_1\overline{f}_j + \frac{1}{2}\overline{\alpha}_1 M_j\right) = -2.5726 < 0,$$

$$2 - \beta_2 + \left|\frac{3}{2} + \frac{\delta_2}{2} - \frac{\beta_2}{2}\right| + \frac{\overline{L}_2}{2} + \sum_{j=1}^{2}\Big(|a_{2j}| + |b_{2j}| + |c_{2j}|\Big)\left(\frac{3}{2}\overline{A}_2\overline{f}_j + \frac{1}{2}\overline{\alpha}_2 M_j\right) = -4.713 < 0$$

The phase behavior of the considered system (5.21)*is depicted in Fig. 5.1, with initial values as* $x_1 = -0.2, x_2 = 0.6, y_1 = 0.7,$*and* $, y_2 = -0.1$

Firstly, let's consider the lag synchronization between the systems (5.21) *and* (5.22),

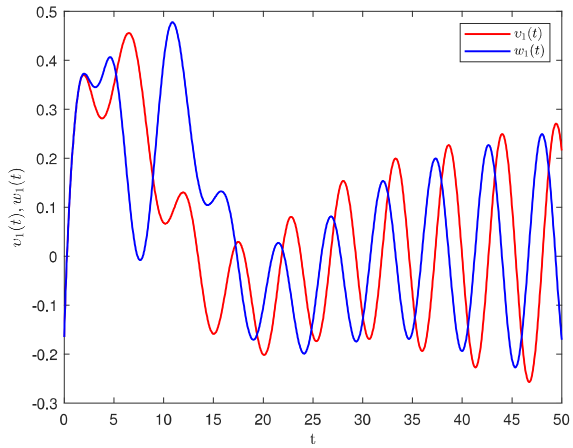

FIGURE 5.5: Lag synchronization curve of $v_1(t)$ and $w_2(t)$ for $\tau_d = 4$.

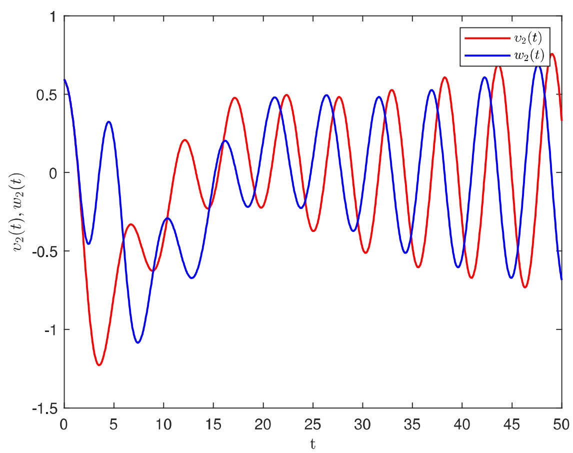

FIGURE 5.6: Lag synchronization curve of $v_1(t)$ and $w_2(t)$ for $\tau_d = 4$.

which are depicted through Fig. 5.2 and 5.3 for lag $\tau_d = 4$. *The error curve via the feedback control 5.3 is depicted in Fig. 5.4. Next, verify the adaptive lag synchronization between systems* (5.21) *and* (5.22). *Select* $\eta_\imath$, $\psi_\imath = 1$ *for* $\imath = 1, 2$ *in the adaptive control scheme* (5.11), *and The error curve via the adaptive control 5.11 is depicted in Fig. 5.7.*

Remark 5.6. A comparison of Figs. 5.4, and 5.7 shows that the synchronization speed between systems (5.21) and (5.22) is notably faster in Fig. 5.4. This observation

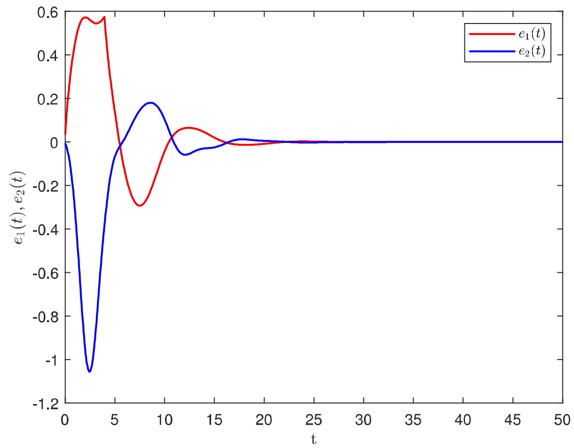

FIGURE 5.7: Time response of $e_1(t), e_2(t)$ via adaptive controller (5.11).

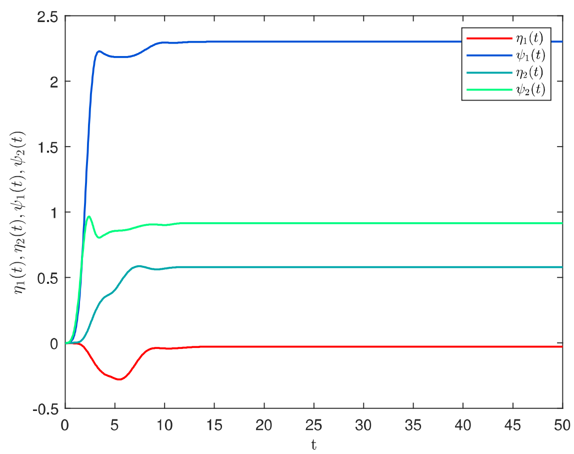

FIGURE 5.8: Time response for the control parameters $\eta_1(t), \eta_2(t), \psi_1(t)$, and $\psi_2(t)$.

validates the notion that exponential synchronization exhibits a faster convergence rate when compared to asymptotic synchronization.

5.5 Conclusion

In this chapter, two kinds of lag synchronization for inertial CGNNs have been investigated. Different from the traditional order reduction approach for INNs, two different kinds of Lyapunov functional are constructed, and two different kinds of controllers have been designed on the considered inertial CGNNs to ensure lag synchronization. In the first theorem, the feedback controller is used whereas an adaptive controller is designed in second. The two sufficient criteria are feasible, efficiently verified and maintain the originality of the proposed systems as compared to the variable substitution approach for INNs. In the near future, the drive can be made to investigate the global dissipativity and finite time stability for ICGNNs with hybrid unbounded and non-differentiable delays.

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