Chapter 1

Introduction

In this chapter, we provide the background theories, preliminaries, notations, and definitions that are used in this thesis. We also offer the rationale behind the research and a comprehensive summary of all the chapters included in this thesis.

1.1 Topological fixed point theory

Fixed points of a function are points that remain invariant under the application of a function. More precisely, for any two non-empty sets X and Y with $X \cap Y \neq \emptyset$, a point $x \in X$ is designated as a fixed point of the mapping $T : X \to Y$, if $T(x) = x$. Brouwer's fixed point theorem is a noteworthy consequence in the study of fixed points in topological spaces, and the theorem says that *"A continuous self-map T on a compact, convex set always possess at least one fixed point"*.

French mathematicians Henri Poincaré and Charles Émile Picard initially explored the theorem within the context of their research on differential equations. To establish outcomes such as the Poincaré-Bendixson theorem, topological techniques need to be employed. Jacques Hadamard [46] first demonstrated the situation of differentiable mappings of the closed n-dimensional ball in 1910, and the general case for continuous mappings was then proven by Brouwer [30] in 1911.

There are several formulations of the Brouwer fixed point theorem, each differing based on the particular context and degree of generality applied. The most fundamental version is sometimes articulated in the following way.

- Any continuous self mapping T on a closed disk C possesses at least one fixed point in C.

This statement can be generalized in Euclidean space as below:

- Any continuous self mapping T on a closed ball K of an arbitrary Euclidean space $\mathbb{R}^n$ has a fixed point in it.

A more general version of the above statement is as follows:

- Any continuous self mapping T on an arbitrary compact convex subset of $\mathbb{R}^n$ admits at least one fixed point.

In the year 1930, Schuder [90] expanded further the Brouwer fixed point theorem to encompass infinite dimensional spaces, thereby elevating the level of generality and broadening its domain of application.

Example 1.1. Let us define the set of all continuous functions defined on the interval $[0, 1]$, denoted by $\mathfrak{C}[0, 1]$. To quantify the distance between two functions within this collection, we employ the supremum norm defined as $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ for all $f \in \mathfrak{C}[0, 1]$. By using this norm, we endow $\mathfrak{C}[0, 1]$ with a structure known as Banach space.

Consider, the set $E = \{f \in \mathfrak{C}[0, 1] : \|f\|_\infty \leq 2\}$, a compact and convex subset of $\mathfrak{C}[0, 1]$. We now introduce a mapping $T : E \to E$ defined by

$$(Tf)(x) = \int_0^x tf(t)dt.$$

Then for all $f, g \in E$,

$$\begin{aligned}
\|Tf - Tg\|_\infty &= \sup_{x\in[0,1]} \left| \int_0^x [tf(t) - tg(t)]dt \right| \\
&\leq \sup_{x\in[0,1]} \int_0^x t|f(t) - g(t)|dt \\
&\leq \|f - g\|_\infty \sup_{x\in[0,1]} \int_0^x tdt \\
&= \frac{1}{2}\|f - g\|_\infty.
\end{aligned}$$

This serves as an evidence that T is a mapping on E that exhibits continuity. With the aid of Schauder's fixed point theorem, it can be inferred that T possesses a fixed point located within E. An instance of a fixed point for T could be $e^{\frac{x^2}{2}}$.

In 1935, Tychonoff [97] made significant advancements to Schauder's theorem regarding fixed points by extending its application to locally convex linear spaces, with particular emphasis on compact convex subsets. The core of this extension posits that every continuous transformation acting on such a subset possesses at least one stationary point. Subsequently, numerous research studies have been undertaken to explore the theory of fixed points within the context of topological vector spaces, for instance; Glicksberg [43], Fan [42], Bogachev and Smolyanov [27].

1.2 Metric fixed point theory

Metric fixed point theory focuses on fixed point results that depend on the presence of a distance metric in the given space. This means that the findings of this theory are closely tied to the metric properties of the space and the way distances between points are measured. In essence, the theory examines the concept of closeness between points in the space and its role in determining whether fixed points exist or not. Let X be a non-empty set and d be a metric on X. A mapping $T : X \to X$ that satisfies

$$d(Tx, Ty) \leq Md(x, y) \text{ for some fixed constant } M \text{ and } \; x, y \in X, \tag{1.1}$$

is called the Lipschitz function; the smallest such M is called the Lipschitz constant k of T. If $k < 1$, the map T is called contractive with contraction constant k. If $k = 1$, the map is said to be non-expansive. Note that a Lipschitz map is necessarily continuous. Let X be a non-empty set and $T : X \to X$ be a map. For any given $x \in X$, define $T^n x$ inductively by $T^0 x = x$ and $T^{n+1}x = T(T^n x)$; we call $T^n x$ the n-th iterate of x under T.

In 1922, Stefan Banach, a mathematician hailing from Poland, established the existence and uniqueness of fixed points of certain types of maps in metric space. This outcome made a significant impact because of its broad theoretical and practical implications.

Theorem 1.2. (Banach [9]) *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction i.e., there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } \; x, y \in X. \tag{1.2}$$

Then, T has a unique fixed point on a complete metric space (X, d) and the Picard sequence $\{T^n(x)\}$ of iterates generated from any element $x \in X$ converges to the unique fixed point.

Theorem 1.2 gives a sufficient condition for T in order to have a unique fixed point. One limitation of using the Banach contraction principle is that it requires the mapping to be continuous over its entire domain. This condition may not always be beneficial or feasible in some situations. To eliminate this obstacle, Caccioppoli [34] made an extension of Banach's theorem in 1930 as mentioned below by taking into account the characteristics of n-th iteration T^n of the mapping T.

1.2.1 Extension of the Banach contraction principle

Numerous generalizations of the Theorem 1.2 can be found in the literature. Here we point out some results.

Theorem 1.3. (Caccioppoli [34]) *Let (X, d) be a complete metric space and T be an operator on X such that for every positive integer n,*

$$d(T^n x, T^n y) \leq a_n d(x, y) \quad \text{for all} \quad x, y \in X,$$

where a_n is a positive constant that is independent of x and y, then T has a unique fixed point in X, provided $\sum_{n=1}^{\infty} a_n$ is convergent.

In 1968, Bryant [33] extended the Caccioppoli theorem by incorporating the contraction requirement of the map T, thereby, providing a more generalized version of the theorem.

Theorem 1.4. (Bryant [33]) *Let X be a complete metric space and let $T : X \to X$ be an operator on X. If T^n is a contraction, for some $n \geq 1$, then T has a unique fixed point.*

Example 1.5. Consider the map

$$Tx = \begin{cases} \frac{1}{2} + 2x, & \text{if} \quad x \in [0, \frac{1}{4}) \\ \frac{1}{2} & \text{if} \quad x \in (\frac{1}{4}, 1] \end{cases}$$

mapping $[0,1]$ onto itself. Then, T is not even continuous, but it possesses a unique fixed point $(x = \frac{1}{2})$.

While the theorems established by Caccioppoli and Bryant do not mandate the continuity of the mapping T, they rely on the characteristics of the n-th iteration. However, R. Kannan, an Indian mathematician proposed a novel contractive type condition in 1968 that eliminates the necessity to explore T's n-th iteration and doesn't require T to exhibit continuity behavior.

Theorem 1.6. (Kannan [55]) *Let (X,d) be a complete metric space and T be an operator on X such that*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \quad \text{for all} \quad x,y \in X,$$

for some $k < \frac{1}{2}$. Then, T has a unique fixed point in X.

Theorem 1.7. (Ćirić [2]) *Let (X,d) be a complete metric space and T be an operator on X such that*

$$d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for some $k < 1$ and every $x,y \in X$. Then, T has a unique fixed point in X.

Theorem 1.8. (Boyd-Wong [29]) *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying the following contractive condition:*

$$d(Tx,Ty) \leq \phi(d(x,y)) \quad \text{for all} \quad x,y \in X,$$

where $\phi : [0,\infty) \to [0,\infty)$ satisfies the following conditions:

(i) $\phi(s) < s$ for all $s > 0$;

(ii) $\limsup_{s \to e+} \phi(s) < e$ *for all* $e > 0$.

Then, T *has a unique fixed point in* X*.*

Theorem 1.9. (Proinov [76]) *Suppose* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping such that*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)) \quad \textit{for all} \;\; x, y \in X \qquad \textit{with } d(Tx, Ty) > 0, \tag{1.3}$$

where $\psi, \phi : (0, \infty) \to \mathbb{R}$ *satisfy the following three conditions:*

(a_1*)* ψ *is monotone increasing;*

(a_2*)* $\phi(s) < \psi(s)$ *for all* $s > 0$;

(a_3*)* $\limsup_{s \to e+} \phi(s) < \lim_{s \to e+} \psi(s)$ *for all* $e > 0$.

Then, T *has a unique fixed point in* X*.*

On the other hand, to study the generalized notion of distance which provides more flexibility and versatile application than traditional metric spaces, it was Bakhtin [8] who pioneered the inception of the b-metric space concept in 1989 and is formally defined by Czerwik [38] in 1993.

Subsequently, numerous mathematical frameworks were developed that built upon and extended the basic concept of a metric space, leading to the identification of several intriguing fixed point theorems by various researchers working within these new foundational structures. We cite here; Partial metric space [67], Cone metric space [79], *JS* metric space [51], and Non-triangular metric space [59].

1.2.2 Non-triangular metric space

In 2020, Khojasteh and Khandani [59] have introduced non-triangular metric spaces. Furthermore, the introduction of non-triangular metric spaces has demonstrated that the presence of the triangle inequality is not a fundamental requirement for the validity of various fixed point theorems. Several useful properties in the squeal are established in non-triangular metric spaces.

Definition 1.10. [59] Let $d : X \times X \to \mathbb{R}^+$ be a mapping on a non-empty set X. d is said to be a non-triangular metric on X, if it satisfies the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) for each $x, y \in X$, and $\{x_l\} \subset X$ such that $\lim_{l\to\infty} d(x_l, x) = 0$, and $\lim_{l\to\infty} d(x_l, y) = 0$, then $x = y$.

Then, (X, d) is said to be non-triangular metric space.

Definition 1.11. [59] Let (X, d) be a non-triangular metric space then the following conclusions hold:

(i) $\{x_l\}$ is d-convergent to x if $\lim_{l\to\infty} d(x_l, x) = 0$;

(ii) $\{x_l\}$ is a d-Cauchy sequence if $\lim_{l\to\infty} \sup\{d(x_l, x_m) : m \geq l\} = 0$;

(iii) if every d-Cauchy sequence in X is d-convergent to some element in X then (X, d) is called d-complete.

Remark 1.12. If (X, d) is a non-triangular metric space and $x, y \in X$, then $d(x, y) = 0$ implies $x = y$.

Example 1.13. [40] Let $X = [0, \infty)$, define

$$d(x,y) = \begin{cases} \frac{x+y}{x+y+1}, & \text{if } 0 \neq x \neq y \neq 0, \\ \frac{x}{2}, & \text{if } y = 0, \\ \frac{y}{2}, & \text{if } x = 0, \\ 0, & \text{if } x = y. \end{cases}$$

First of all, it is not a metric as it does not satisfy the triangle inequality: take $x = 0.3$, $y = 0$, and $z = 0.4$, then $d(x,z) \not< d(x,y) + d(y,z)$.

On the other hand, $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Now, consider a point $x \in X$ and $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, x) = 0$. Now, either the real sequence $\{d(x_n, x)\}$ is eventually constant (to 0), or it is not. For the eventually constant, it is obvious that $\{x_n\}$ itself is eventually convergent to x. However, when $\{d(x_n, x)\}$ is not eventually constant sequence in X, we have the following cases:-

if $x = 0$, then $d(x_n, x) = \frac{x_n}{2} \to 0$. In addition, if $x \neq 0$, then $d(x_n, x) = \frac{x_n+x}{x_n+x+1} \to 0$ iff $x_n + x \to 0$. However, since $x > 0$, $x_n + x \not\to 0$. That is, $\lim_{n\to\infty} d(x_n, x) = 0$ is possible iff either $\{x_n\}$ is eventually convergent to x or $x = 0$ and $x_n \to 0$ in the usual sense. Now, let $x, y \in X$ and $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $d(x_n, y) = 0$.

If $\{x_n\}$ is an eventually constant sequence (in either x or y), then it is easy to say that $x = y$. Therefore, let us consider that $\{x_n\}$ is not eventually constant to x. Then, we know that the only possibility is that $x = 0$ and $x_n \to 0$ in the usual sense. If $y \neq 0$, then we would have $d(x_n, y) = \frac{x_n+y}{x_n+y+1} \to \frac{y}{y+1} \neq 0$, which contradicts our hypothesis. Thereby, $x = y = 0$ must hold. In both cases, we get $x = y$, and hence, (X, d) is a non-triangular metric space.

Theorem 1.14. [57] *Suppose (X, d) be a non-triangular metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq k\ d(x, y) \quad \text{for all} \quad x, y \in X \quad \text{and} \quad \text{for each} \quad k \in [0, 1).$$

If there exists $x_0 \in X$ such that $\{d(T_i(x_0), T_j(x_0)) : i, j \geq 1\}$ is bounded. Then, T has a fixed point.

For related results and extensions, see, [40, 77, 89, 95].

1.3 Non-expansive mapping in the Banach space

In the context of metric space, when the Lipschitz constant of the contractive map is exactly 1, then it is called a non-expansive map. The study of non-expansive mappings has predominantly concentrated on the realm of linear spaces, with comparatively less importance on the scope of metric space. When we turn around our attention for the precise definition of the non-expansive map in the Banach space as follows: Let $(X, \|.\|)$ be a normed linear space and a map is called a non-expansive map in the normed linear space if it satisfies the following condition:

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} \quad x, y \in X.$$

Non-expansive mappings are generally found in the field of nonlinear analysis:

(i) Browder [32] : the relationship between non-expansive mappings and monotone operators;

(ii) Kirk [60]: the significance of the geometric characteristics of the norm for the existence of fixed points for non-expansive mappings;

(iii) Minty [72]: if $A : H \to H$ is maximally monotone, then its resolvent $J_A = (I + A)^{-1}$ is single-valued and firmly non-expansive;

(iv) most of the problems in nonlinear analysis and optimization can be embedded in the framework of metrical fixed point theory.

Four important classes of non-expansive map

(i) T is non-expansive: $\|Tx - Ty\| \leq \|x - y\| \quad \text{for all} \quad x, y \in X.$

(ii) T is quasi non-expansive: if $Fix(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\| \quad \text{for all} \quad x \in X.$

(iii) T is k-strictly pseudo-contractive of the Browder-Petryshyn type if there exists $k \in (0, 1)$ and $y \in Fix(T)$ such that $\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2 \quad \text{for all} \quad x \in X.$

(iv) T is demi-contractive or k demi-contractive if $Fix(T) \neq \emptyset$ and there exists $k < 1$ such that $\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2 \quad \text{for all} \quad x \in X.$

Definition 1.15. [12] A self-map T on a normed linear space $(X, \|.\|)$ is called an enriched non-expansive mapping (b-enriched non-expansive mapping) if there exists $b \in [0, \infty)$ such that

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\| \quad \text{for all} \quad x, y \in X. \tag{1.4}$$

Remark 1.16. A non-expansive mapping T is considered a 0-enriched mapping because it satisfies (1.4) with $b = 0$. However, it is important to note that the reverse statement is not necessarily valid. Consider the set X as the real numbers with the standard norm, and we define

$$T : [\frac{1}{2}, 2] \to [\frac{1}{2}, 2] \quad \text{by} \quad T(x) = \frac{1}{x}.$$

In this case, T is not a non-expansive (NE) mapping, but rather an enriched non-expansive (ENE) mapping.

It should be noted that the Picard iteration ($x_{n+1} = Tx_n, n \geq 0$) does not converge in general for both the non-expansive map and the enriched non-expansive map, but the Krasnoselskii-Mann iteration ($x_{n+1} = (1-\lambda)x_n + Tx_n, n \geq 0$), converges for any $\lambda \in (0, 1)$. As for example let us take $Tx = 2 - x$, $x \in [0, 1]$ for the case of non-expansive map and let us take $Tx = \frac{1}{x}$, $x \in [\frac{1}{2}, 2]$ for enriched non-expansive map.

1.3.1 The technique of enriching non-expansive mapping

The concept of the enriching process was first proposed in Krasnoselskii's article [61] in 1955. Browder and Petryshyn [31] formally introduced this idea in 1966 as asymptotic regularity when studying fixed point results for non-expansive mappings. Let $(X, \|.\|)$ be a Banach space and E be closed and convex subset of X and $T : E \to E$ be an operator and we denote $T_\lambda = (1-\lambda)I + \lambda T$ for $\lambda \in (0, 1)$ where I is the identity operator.
Then,

(i) If T is non-expansive then T_λ is non-expansive;

(ii) $Fix(T) = Fix(T_\lambda)$;

(iii) $\left\|T_\lambda^{n+1}x - T_\lambda^n x\right\| \to 0$ as $n \to \infty$ (T_λ is asymptotically regular, although T is in general not asymptotically regular).

From the point of view of fixed point theory T_λ is richer than T ($T_\lambda^n x \to x^* \in Fix(T)$, while $T^n x$ does not converge, in general). A mapping $T : X \to X$ is said to be C-enriched or enriched with respect to C if there exists $\lambda \in (0,1]$ such that $T_\lambda \in C$. Using C^e, we denote the set of all enriched mappings with respect to C.

There are several classes of enriching contractive classes such as Banach contraction [16], Kannan [21], Chatterjea [17], Ćirić-Reich-Rus [19, 22].

Definition 1.17. [20] A class C is considered a saturated class of mappings when $C = C^e$; otherwise, it is classified as an unsaturated class of mappings.

It should be noted that a class of mappings C is considered unsaturated if and only if there is a strict inclusion $C \subset C^e$.

For a specific class of mappings C, the enriched mappings class is always saturated, meaning C^e is consistently a saturated class where $(C^e)^e = C^e$. Consequently, it is not possible to further enrich an already enriched class of mappings. In 2021, Berinde and Păcurar [20] have proved that in the context of Hilbert space, the class of strictly pseudo-contractive maps coincides with the class of enriched non-expansive maps. Therefore, the class of strictly pseudo-contractive maps is a saturated class of mapping in the Hilbert space.

1.4 Non-expansive mapping in the quasi-Banach space

Recently, there has been a huge growth in the research in the area of quasi-Banach spaces. In fact, it has become a very flourishing field of research in recent years due to the fact that it became extremely natural to see if existing Banach space properties and results could be transposed onto quasi-Banach spaces in a less narrower context and have a more direct correlation. We mention the following very recent contributions from the rich literature dedicated to investigating quasi-Banach spaces as important settings in most of the areas in which they play a crucial part:

(i) Zada and Mashal [100]: study of the stability of solutions to ODEs and PDEs involving impulses;

(ii) Torres et al. [96]: harmonic analysis on the Euclidean space;

(iii) Sullivan [93]: inverse problems for PDEs and Bayesian inference;

(iv) Kwun et al. [63], Maligranda [65]: structure and geometry of normed linear spaces;

(v) Grafakos and Mastylo [44]: bilinear operator interpolation;

(vi) Albiac and Ansorena [4], Kalton [54]: functional analysis on the infinite-dimensional spaces which are not locally convex.

In 1938, Hyers [49] introduced the concept of quasi-norm under the name pseudo-norm; later, in 1943, Bourgin [28] actually proposed quasi-norm as follows:

Definition 1.18. Let X be a real vector space and a quasi-norm on it is a map $\|.\| : X \to [0, \infty)$, which satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$;

(iii) $\|x + y\| \leq C[\|x\| + \|y\|]$ for all $x, y \in X$, where $C \geq 1$.

It is obvious that if $C = 1$, then the quasi-norm converts to the usual norm, but the converse is not true. Vulpe et al. [98] have introduced the notation to a quasi-norm is that of b-metric or quasi-metric but was initially well-known mostly because of Czerwik [38, 39] and Bakhtin [7].

Definition 1.19. Let X be a non-empty set. Then a map $d : X \times X \to \mathbb{R}$ is said to be quasi-metric if it satisfies the following conditions:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) there exists a constant $K \geq 1$ such that $d(x, z) \leq K[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

The pair (X, d) is called quasi-metric space.

It should be noted that the quasi-norm and quasi-metric are closely related, as quasi-norm $\|.\|$ on a vector space X induces a quasi-metric d by

$$d(x, y) = \|x - y\| \quad \text{for all} \quad x, y \in X.$$

According to Pietsch [75], in 1935, Tychonoff [97] has provided the initial illustration of a quasi-Banach space, e.g., the space $l_{\frac{1}{2}}$,

$$l_{\frac{1}{2}} = \{x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{i=1}^{\infty} \sqrt{|x_i|} < \infty\},$$

and the quasi-norm is determined by

$$\|x\| = \Big(\sum_{i=1}^{\infty} \sqrt{|x_i|}\Big)^2.$$

It is evident that the quasi-triangle inequality with $C = 2$ is satisfied by the aforementioned quasi-norm $\|.\|$ on $l_{\frac{1}{2}}$. Taking $p \in (0,1)$ instead of $p = \frac{1}{2}$ in the previous consideration, it leads to more general example.

Any locally bounded topology on X can be induced by a quasi-norm and vice versa; see, for example, Maligranda [65].

Suppose $\|.\|$ is a quasi-norm and $p \in (0,1]$ and it is said to be a p-norm if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p \quad \text{for all} \;\; x, y \in X.$$

If there exists an equivalent p-norm $\|.\|_*$ on X and a constant C_1 with

$$\|x_1 + \cdots + x_n\|_* \leq C_1 \Big(\|x_1\|_*^p + \cdots + \|x_n\|_*^p \Big)^{\frac{1}{p}} \quad \text{for all} \;\; x_1, \ldots, x_n \in X,$$

then, X is said to be a p-normable space, $p \in (0,1]$. If $p = 1$, then the space is simply called normable space and it is obvious that any p-normable space is clearly a quasi-normed space. However, for some $p > 0$, quasi-norm is a p-norm. Indeed, a p-norm is equivalent to a quasi-norm where p satisfies $C = 2^{\frac{1}{p}-1}$. More precisely, the norm is

$$\|x\|_1 = \inf \Big(\sum_k \|x_k\|^p \Big)^{\frac{1}{p}},$$

and here the infimum is taken over all finite sequences $\{x_k\} \subset X$ with $\sum_k x_k = x$, this determines a p-norm such that, for any $x \in X$, there exists

$$\|x\|_1 \leq \|x\| \leq 4^{\frac{1}{p}} \|x\|_1.$$

X is said to be a quasi-Banach space if and only if p-norm $\|.\|$ is a quasi-norm on X and it defines a complete metrizable topology. For complementary and related results, we refer [52, 53].

1.5 Generation of fractals

We introduce here fractals as another application of contraction. In [48], Hutchinson conducted a study on fractals as a self-similar set. In the thesis, we have also adopted this view of fractals, specifically referring to Hutchinson's definition. Many theories have been developed based on the idea of fractals as self-similar sets, meaning that they possess a geometric shape that exhibits intricate structures even at extremely small scales. In [48], Hutchinson utilized the concept of an *iterated function system* to define the self-similarity of a set in his paper. In what follows, we will provide an overview of Hutchinson's theory on fractals.

Definition 1.20. [10] An *iterated function system* (IFS) is a finite set of contraction mappings $F_j : X \to X$ with contractivity factor r_j for $j \in \mathbb{N}_N$ (First N natural numbers) on the complete metric space (X, d). It is denoted by $\{X; F_j, j \in \mathbb{N}_N\}$.

Definition 1.21. [10] Let $\{F_j : j \in \mathbb{N}_N\}$ be the family of finite set maps on (X, d). Let $C(X)$ be the set of all compact subsets of X, then the Hutchinson mapping

$F : C(X) \to C(X)$ is interpreted as

$$F(B) = \bigcup_{j \in \mathbb{N}_N} \tilde{F}_j(B),$$

where $\tilde{F}_j(B) = \{F_j(b) : b \in B\}$.

Definition 1.22. [10] Let $\{X; F_j, j \in \mathbb{N}_N\}$ be an *iterated function system.* A set $P \in C(X)$ is said to be an attractor or deterministic fractal of the IFS if $F(P) = P$ and for all $Q \in C(X)$, $\lim_{n\to\infty} F^n(Q) = P$.

Let $F : X^m \to X$ be a continuous map on the complete metric space (X, d). Now, we define the mapping $\tilde{F} : (C(X))^m \to C(X)$ by

$$\tilde{F}(A_1, A_2, \ldots, A_m) = \{F(x_1, x_2, \ldots, x_m) : x_i \in A_i, \quad \text{and for all} \quad i \in \{1, 2, \ldots, m\}\}.$$

A tuple $(A_1, A_2, \ldots, A_m) \in (C(X))^m$ is said to be m-tuple fractal if

$$\begin{cases} \tilde{F}(A_1, A_2, \ldots, A_m) = A_2, \\ \tilde{F}(A_2, A_3, \ldots, A_1) = A_3, \\ \vdots \\ \tilde{F}(A_m, A_1, \ldots, A_{m-1}) = A_1. \end{cases}$$

When $A_1 = A_2 = \cdots = A_m = A$, then it becomes

$$\tilde{F}(A, A, A, \ldots, A) = A.$$

Definition 1.23. Let $F : X^m \to X$ be a mapping. A set $A \in C(X)$ is said to be strong m-tuple fractal if $\tilde{F}(A, A, \ldots, A) = A$.

1.6 Motivation and Objective of the Thesis

A significant amount of research is available on the fixed point theory. Additionally, the research works available focuses on the development of fixed point theory for different kinds of contraction in various spaces. Despite the significant interest and practical applications, there have been limited investigations into Ćirić quasi contraction, Ćirić-Reich-Rus contraction, Kannan contraction, and Proinov contraction.

Recently Mihiţ et al. [71] have introduced a new type of Ćirić operators to establish common fixed point results for a pair of operators in the complete metric space. They have proposed a related problem involving a pair of operators in such a space. This motivated us to address the problem raised in [71] by utilizing a pair of Ćirić quasi-contraction operators. We provide sufficient conditions for the common fixed point problem, ensuring well-posedness, Ulam–Hyers stability, and Ostrowski property. Additionally, we demonstrate an application involving a system composed of an alternating point problem and a fixed point problem.

In [20], Berinde and Păcurar have introduced the idea of contractive mappings that are saturated. They also have proposed a problem regarding the saturated/unsaturated class of Ćirić quasi-contractive type mappings in the Banach space. This also motivated us to develop this problem by constructing the enriched Ćirić quasi-contraction class in the Banach space. Further in the present work, we attempt to establish the existence and approximation of enriched Ćirić quasi-contraction in the Banach space and convex metric space.

Furthermore, we define enriched Ćirić-Reich-Rus contraction and Kanna contraction in the quasi-Banach space and obtain their approximating fixed point results and unifying error estimations. Additionally, we investigate the approximation of fixed points for k-strictly pseudo-contractive mappings by employing the Krasnoselskii-Mann iteration and its modified version. Our numerical experiments reveal that, when it comes to approximating fixed points of certain pseudo-contractive mappings, the Krasnoselskii-Mann iteration proves to be more effective than its modified counterpart. Furthermore, through the application of enrichment techniques, we determined that pseudo-contractive map as an unsaturated class of mappings in the Banach space.

On the other hand, to study generalized notation of distance which provides more flexibility and versatile application than traditional metric spaces. In addition, for the case of non-triangular metric space there is no inherent necessity for the triangle inequality for several fixed point results to be true. We generalize Proinov contraction in the non-triangular metric space from the metric space. Further, as an application, we find the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in non-triangular metric space using Proinov contraction.

In [73], Adrian and Gabriela have introduced novel methods for generating fractals in a complete metric space by utilizing fixed points from Meir-Keeler operators. They have also suggested a challenge involving the creation of multidimensional fractals. To tackle this issue, we create multidimensional fractals by employing strong m-tuple fixed points.

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