Chapter 5

Lagrange stability criteria for hypercomplex neural networks with time varying delays

5.1 Introduction

This chapter explores the Lagrange stability (LS) of HCNNs with time-varying delays. To address the challenges posed by the non-commutativity and non-associativity of HCNNs, they are decomposed into equivalent RVNNs. Subsequently, LS and the globally attractive exponential set are derived using the Lyapunov theory. From a dynamical systems standpoint, a monostable system with a single equilibrium point that asymptotically attracts all trajectories is globally stable in the Lyapunov sense. However, in scenarios involving multistable dynamics with multiple equilibrium points, some of which may be unstable, the conclusions regarding Lyapunov stability become inapplicable. Multistable dynamics become necessary to accommodate the complex neural computations required in many situations where monostable NNs exhibit computational limitations. Thus, a comprehensive understanding of stability is essential to examine multistable systems under such conditions. It is important to note that LS pertains to the system's overall stability without necessitating knowledge of the equilibrium points. This distinguishes it significantly from Lyapunov stability, which typically requires the existence and uniqueness of equilibrium points. This distinction arises due to attractive global sets and the boundedness of solutions, both of which are associated with LS [103]. Beyond the global attractive set, the system may lack equilibrium points, exhibit periodic states, nearly periodic states, or chaos attractors, as demonstrated in [104] [105]. In essence, identifying a globally attractive set enables the evaluation of a wide range of periodic states and chaotic attractors, facilitating system dynamics analysis. The results are inherently more general because CVNNs, QVNNs, and OVNNs are specific instances of HCNNs. Finally, three numerical examples of CVNNs, QVNNs, and OVNNs are presented to validate the results.

5.2 Some preliminaries and problem formulation

The HCNN model and its matrix form is represented as

$$\begin{aligned} \dot{s}_\jmath(t) =& - c_\jmath s_\jmath(t) + \sum_{\ell=1}^{N} a_{\jmath\ell} h_\ell(s_\ell(t)) + \sum_{\ell=1}^{N} b_{\jmath\ell} g_\ell(s_\ell(t-\sigma(t))) + G_\jmath(t), \\ \text{i.e.,}\quad \dot{s}(t) =& - Cs(t) + Ah(s(t)) + Bg(s(t-\sigma(t))) + G(t), \end{aligned} \tag{5.1}$$

where $\jmath = 1, 2, \ldots, N$. The state vector of NNs with N neurons denoted by $s = [s_1, s_2, \ldots, s_N]^T \in \mathbb{H}^N$, $c_\jmath \in \mathbb{R}$ with $c_\jmath > 0$ is the self-feedback connection weight matrix, the connection weight matrices are denoted by $A = (a_{\jmath\ell})_{N\times N} \in \mathbb{H}^{N\times N}$ and $B = (b_{\jmath\ell})_{N\times N} \in \mathbb{H}^{N\times N}$, $h(.) = [h_1(.), h_2(.), \ldots, h_N(.)]^T : \mathbb{H}^N \longrightarrow \mathbb{H}^N$ and $g(.) = [g_1(.), g_2(.), \ldots, g_N(.)]^T : \mathbb{H}^N \longrightarrow \mathbb{H}^N$ are the hypercomplex valued activation functions without and with time varying delay, the external input vector is given by $G_\jmath \in \mathbb{H}$, $\sigma(t)$ is the time varying delay.

Assumption 5.2.1. The activation functions $h(.)$ *and* $g(.)$ are defined as

$$h_\ell(s_\ell(t)) = \sum_{i=0}^{n} h_\ell^{(i)}(s_\ell^{(i)}(t)), g_\ell(s_\ell(t)) = \sum_{i=0}^{n} g_\ell^{(i)}(s_\ell^{(i)}(t)),$$

where $i = 0, 1, 2, \ldots, n$, $\ell = 1, 2, \ldots, N$.

Assumption 5.2.2. For $x_1, x_2 \in \mathbb{R}$, there exist constants $\hat{k}_\ell, \hat{f}_\ell \in \mathbb{R}$ s.t. the activation functions $h_\ell^{(i)}(.)$ and $g_\ell^{(i)}(.)$ satisfy the following Lipschitz condition as

$$|h_\ell^{(i)}(x_1) - h_\ell^{(i)}(x_2)| \leq \hat{k}_\ell |x_1 - x_2|,$$
$$|g_\ell^{(i)}(x_1) - g_\ell^{(i)}(x_2)| \leq \hat{f}_\ell |x_1 - x_2|,$$

where $i = 0, 1, 2, \ldots, n$ and $\ell = 1, 2, \ldots, N$.

Assumption 5.2.3. The external input vector $G_\jmath^{(i)}(t)$ is defined as

$$G_\jmath(t) = \sum_{i=0}^{n} G_\jmath^{(i)}(t)\psi_i,$$

where $G_\jmath^{(i)}(t)$ is bounded, i.e., there exists $d_\jmath > 0$ s.t. $|G_\jmath^{(i)}| \leq d_\jmath^{(i)}$, where $i = 0, 1, 2, \ldots, n$.

Assumption 5.2.4. $\sigma(t) \geq 0$ is a differential function with $0 \leq \dot{\sigma}(t) \leq \nu < 1, \ \forall\ t$, where ν is constant. This assumption is obviously ensured if $\sigma(t)$ is constant.

Remark 5.2.1. The key elements those affect the dynamic properties of the given NNs are the activation functions. The existence and uniqueness of the system (5.1) can be assured based on Assumption 5.2.2 because the activation functions are continuous [98]. Additionally, the assumption of the Lipschitz condition is supported by the following activation functions, such as the piecewise linear function, the logistic sigmoid function, and the hyperbolic tangent function.

Now, by using (1.65) and Assumption 5.2.1, the system (5.1) can be seprated into $n+1$ real-valued system of equations with following expressions.

$$\begin{aligned}
\dot{s}_{\jmath}^{(0)}(t) =& - c_{\jmath} s_{\jmath}^{(0)}(t) + \sum_{\ell=1}^{N} \Big(a_{\jmath\ell}^{(0)} h_{\ell}^{(0)}(s_{\ell}^{(0)}(t)) + \sum_{\gamma,\delta=1}^{n} a_{\jmath\ell}^{(\gamma)} h_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t)) \epsilon_{\gamma\delta,0} \Big) \\
& + \sum_{\ell=1}^{N} \Big(b_{\jmath\ell}^{(0)} g_{\ell}^{(0)}(s_{\ell}^{(0)}(t-\sigma(t))) + \sum_{\gamma,\delta=1}^{n} b_{\jmath\ell}^{(\gamma)} g_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t-\sigma(t))) \epsilon_{\gamma\delta,0} \Big) + G_{\jmath}^{(0)}(t), \\
\dot{s}_{\jmath}^{(1)}(t) =& - c_{\jmath} s_{\jmath}^{(1)}(t) + \sum_{\ell=1}^{N} \Big(a_{\jmath\ell}^{(0)} h_{\ell}^{(1)}(s_{\ell}^{(1)}(t)) + \sum_{\ell=1}^{N} a_{\jmath\ell}^{(1)} h_{\ell}^{(0)}(s_{\ell}^{(0)}(t)) + \sum_{\gamma,\delta=1}^{n} a_{\jmath\ell}^{(\gamma)} \\
& \times h_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t)) \epsilon_{\gamma\delta,1} \Big) + \sum_{\ell=1}^{N} \Big(b_{\jmath\ell}^{(0)} g_{\ell}^{(1)}(s_{\ell}^{(1)}(t-\sigma(t))) + b_{\jmath\ell}^{(1)} g_{\ell}^{(0)}(s_{\ell}^{(0)}(t-\sigma(t))) \\
& + \sum_{\gamma,\delta=1}^{n} b_{\jmath\ell}^{(\gamma)} g_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t-\sigma(t))) \epsilon_{\gamma\delta,1} \Big) + G_{\jmath}^{(1)}(t), \\
\dot{s}_{\jmath}^{(2)}(t) =& - c_{\jmath} s_{\jmath}^{(2)}(t) + \sum_{\ell=1}^{N} \Big(a_{\jmath\ell}^{(0)} h_{\ell}^{(2)}(s_{\ell}^{(2)}(t)) + \sum_{\ell=1}^{N} a_{\jmath\ell}^{(2)} h_{\ell}^{(0)}(s_{\ell}^{(0)}(t)) + \sum_{\gamma,\delta=1}^{n} a_{\jmath\ell}^{(\gamma)} \\
& \times h_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t)) \epsilon_{\gamma\delta,2} \Big) + \sum_{\ell=1}^{N} \Big(b_{\jmath\ell}^{(0)} g_{\ell}^{(2)}(s_{\ell}^{(2)}(t-\sigma(t))) + b_{\jmath\ell}^{(2)} g_{\ell}^{(0)}(s_{\ell}^{(0)}(t-\sigma(t))) \\
& + \sum_{\gamma,\delta=1}^{n} b_{\jmath\ell}^{(\gamma)} g_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t-\sigma(t))) \epsilon_{\gamma\delta,2} \Big) + G_{\jmath}^{(2)}(t), \\
& \vdots \\
\dot{s}_{\jmath}^{(n)}(t) =& - c_{\jmath} s_{\jmath}^{(n)}(t) + \sum_{\ell=1}^{N} \Big(a_{\jmath\ell}^{(0)} h_{\ell}^{(n)}(s_{\ell}^{(n)}(t)) + \sum_{\ell=1}^{N} a_{\jmath\ell}^{(n)} h_{\ell}^{(0)}(s_{\ell}^{(0)}(t)) + \sum_{\gamma,\delta=1}^{n} a_{\jmath\ell}^{(\gamma)} \\
& \times h_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t)) \epsilon_{\gamma\delta,n} \Big) + \sum_{\ell=1}^{N} \Big(b_{\jmath\ell}^{(0)} g_{\ell}^{(n)}(s_{\ell}^{(n)}(t-\sigma(t))) + b_{\jmath\ell}^{(n)} g_{\ell}^{(0)}(s_{\ell}^{(0)}(t-\sigma(t)))
\end{aligned}$$

$$+\sum_{\gamma,\delta=1}^{n} b_{\jmath\ell}^{(\gamma)} g_{\ell}^{(\delta)}(s_{\ell}^{(\delta)}(t-\sigma(t)))\epsilon_{\gamma\delta,n}\Big) + G_{\jmath}^{(n)}(t),\ \jmath = 1, 2, \ldots, N. \quad (5.2)$$

5.2.1 Definition and lemmas

In this section, some key definitions and lemmas, which will be required to derive the main results of the article, are discussed.

Definition 5.2.1. [106] Network (5.1) is said to be uniformly stable in Lagrange sense if for any $W > 0$, $\exists$ constant $M = M(W) > 0$ s.t. $\|s(t)\| < M$ for all $g \in \mathbb{H}_W = \{g \in \mathbb{H} : \|g\| \leq W\}$.

Definition 5.2.2. [106] If there exist constants $d > 0, z > 0$ and for any $W > 0$ there exists $M = M(W) > 0$ s.t. $\|s(t)\| < d + Me^{-zt}$, $\forall\ t \geq 0$, $s_0 \in \mathbf{H}_W$, then the Network (5.1) is called globally Lagrange exponentially stable (LES). The network (5.1) is also said to be globally uniform exponentially convergent with rate z in ball $K(d) = \{s(t) \in \mathbf{H}^n : \|s(t)\| \leq d\}$.

Definition 5.2.3. Suppose $L(z)$ is a real-valued function defined on the Interval I containing a point z_0, then the dini-derivative $D^+L(x_0)$ at x_0 is defined as

$$D^+L(z_0) = \limsup_{z \longrightarrow z_0} \frac{L(z) - L(z_0)}{z - z_0}.$$

As z approaches z_0, the limit superior (lim sup) captures the upper bound of the slopes of secant lines between $L(z)$ and $L(z_0)$.

Lemma 5.2.1. [107] Let us suppose that $L(t)$ is the the continuous function on $[0, \infty)$, and there exist κ_1, $\tilde{\kappa}_2 > 0$ such that

$$D^+L(t) \leq -\kappa_1 L(t) + \tilde{\kappa}_2,$$

then

$$L(t) - \frac{\tilde{\kappa}_2}{\kappa_1} \leq \Big(L(0) - \frac{\tilde{\kappa}_2}{\kappa_1}\Big) e^{-\kappa_1 t},$$

which implies that if $L(t) \geq \frac{\tilde{\kappa}_2}{\kappa_1}$, then $L(t)$ exponentially approaches to $\frac{\tilde{\kappa}_2}{\kappa_1}$ as t increases.

5.3 Main results

Theorem 5.3.1. If the Assumptions 5.2.2-5.2.4 hold, and there exist constants $\beta_\jmath^{(i)}, \omega_\jmath^{(i)} > 0, i = 0, 1, \dots, n$ such that

$$\begin{aligned}
\kappa_\jmath^{(0)} =& c_\jmath - \frac{\omega_\jmath^{(0)}}{\beta_\jmath^{(0)}(1-\nu)} - \sum_{\ell=1}^{N} \big(|a_{\ell\jmath}^{(0)}| + \frac{\beta_\jmath^{(1)}}{\beta_\jmath^{(0)}}|a_{\ell\jmath}^{(1)}| + \frac{\beta_\jmath^{(2)}}{\beta_\jmath^{(0)}}|a_{\ell\jmath}^{(2)}| + \dots + \frac{\beta_\jmath^{(n)}}{\beta_\jmath^{(0)}}|a_{\ell\jmath}^{(n)}|\big)\hat{k}_\jmath > 0 \\
\kappa_\jmath^{(1)} =& c_\jmath - \frac{\omega_\jmath^{(1)}}{\beta_\jmath^{(1)}(1-\nu)} - \sum_{\ell=1}^{N} \Big(\frac{\beta_\jmath^{(0)}}{\beta_\jmath^{(1)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,0}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,0}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big) \\
& + \big(|a_{\ell\jmath}^{(0)}| + |a_{\ell\jmath}^{(1)}||\epsilon_{11,1}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,1}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big) + \frac{\beta_\jmath^{(2)}}{\beta_\jmath^{(1)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,2}| \\
& + |a_{\ell\jmath}^{(2)}||\epsilon_{21,2}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big) + \dots + \frac{\beta_\jmath^{(n)}}{\beta_\jmath^{(1)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,n}| \\
& + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_\jmath > 0 \\
& \vdots \\
\kappa_\jmath^{(n)} =& -c_\jmath - \frac{\omega_\jmath^{(n)}}{\beta_\jmath^{(n)}(1-\nu)} - \sum_{\ell=1}^{N} \Big(\frac{\beta_\jmath^{(0)}}{\beta_\jmath^{(n)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,0}| + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,0}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big) \\
& + \frac{\beta_\jmath^{(1)}}{\beta_\jmath^{(n)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,1}| + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,1}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big) + \frac{\beta_\jmath^{(2)}}{\beta_\jmath^{(n)}}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,2}| \\
& + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,2}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big) + \dots + \big(|a_{\ell\jmath}^{(0)}| + |a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}| \\
& + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}| + \dots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\hat{k}_\jmath > 0. \qquad (5.3)
\end{aligned}$$

Then, HCNNs (5.2) is said to be LES. Additionally, the solution of the system (5.2) exponentially converges to the ball

$$\mathcal{R}_1 = \Big\{ s(t) \in \mathbb{H}^n : \|s(t)\| = \sum_{\jmath=1}^{N} |s_\jmath^{(i)}(t)| \leq \frac{\tilde{\kappa}_2}{\beta\kappa_1} \Big\}, \quad i = 0, 1, \ldots, n, \tag{5.4}$$

where

$$\begin{aligned}
\beta &= \min_{1\leq \jmath \leq N} \{\beta_\jmath^{(0)}, \beta_\jmath^{(1)}, \ldots, \beta_\jmath^{(n)}\},\\
\kappa_1 &= \min_{1\leq \jmath \leq N} \{\kappa_\jmath^{(0)}, \kappa_\jmath^{(1)}, \ldots, \kappa_\jmath^{(n)}\},\\
\tilde{\kappa}_2 &= \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_\jmath^{(0)}|a_{\ell\jmath}^{(0)}| + \beta_\jmath^{(1)}|a_{\ell\jmath}^{(1)}| + \beta_\jmath^{(2)}|a_{\ell\jmath}^{(2)}| + \ldots + \beta_\jmath^{(n)}|a_{\ell\jmath}^{(n)}|\Big)|h_\jmath^{(0)}(0)|\\
&\quad + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_\jmath^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,0}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,0}| + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big) + \beta_\jmath^{(1)}\big(|a_{\ell\jmath}^{(0)}|\\
&\quad + |a_{\ell\jmath}^{(1)}||\epsilon_{11,1}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,1}| + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big) + \beta_\jmath^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,2}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,2}|\\
&\quad + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big) + \ldots + \beta_\jmath^{(n)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}| + |a_{\ell\jmath}^{(2)}||\epsilon_{21,n}| + \ldots\\
&\quad + |a_{\ell\jmath}^{(2)}||\epsilon_{n1,n}|\big)\Big)|h_\jmath^{(1)}(0)|\\
&\qquad\qquad\qquad\qquad\vdots\\
&\quad + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_\jmath^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,0}| + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,0}| + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big) + \beta_\jmath^{(1)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|\\
&\quad + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,1}| + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big) + \beta_\jmath^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,2}| + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,2}| + \ldots + |a_{\ell\jmath}^{(n)}|\\
&\quad \times |\epsilon_{nn,2}|\big) + \ldots + \beta_\jmath^{(n)}\big(|a_{\ell\jmath}^{(0)}| + |a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}| + |a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}| + \ldots + |a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\\
&\quad \times |h_\jmath^{(n)}(0)| + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_\jmath^{(0)}|b_{\ell\jmath}^{(0)}| + \beta_\jmath^{(1)}|b_{\ell\jmath}^{(1)}| + \beta_\jmath^{(2)}|b_{\ell\jmath}^{(2)}| + \ldots + \beta_\jmath^{(n)}|b_{\ell\jmath}^{(n)}|\Big)|g_\jmath^{(0)}(0)|\\
&\quad + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N} \Big(|\beta_\jmath^{(0)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,0}| + |b_{\ell\jmath}^{(2)}||\epsilon_{21,0}| + \ldots + |b_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big) + \beta_\jmath^{(1)}\big(|b_{\ell\jmath}^{(0)}|\\
&\quad + |b_{\ell\jmath}^{(1)}||\epsilon_{11,1} + |b_{\ell\jmath}^{(2)}||\epsilon_{21,1}| + \ldots + |b_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big) + \beta_\jmath^{(2)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,2}| + |b_{\ell\jmath}^{(2)}||\epsilon_{21,2}| + \ldots
\end{aligned}$$

$$+ |b_{\ell j}^{(n)}||\epsilon_{n1,2}|\big) + \ldots + \beta_j^{(n)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,n}| + |b_{\ell j}^{(2)}||\epsilon_{21,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,n}|\big)\Big)|g_j^{(1)}(0)|$$

$$\vdots$$

$$+ \sum_{j=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_j^{(0)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,0}| + |b_{\ell j}^{(2)}||\epsilon_{2n,0}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,0}|\big) + \beta_j^{(1)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,1}|$$

$$+ |b_{\ell j}^{(2)}||\epsilon_{2n,1}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,1}|\big) + \beta_j^{(2)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,2}| + |b_{\ell j}^{(2)}||\epsilon_{2n,2}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,2}|\big)$$

$$+ \ldots + \beta_j^{(n)}\big(|b_{\ell j}^{(0)}| + |b_{\ell j}^{(1)}||\epsilon_{1n,n}| + |b_{\ell j}^{(2)}||\epsilon_{2n,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,n}|\big)\Big)|g_j^{(n)}(0)|$$

$$+ \sum_{j=1}^{N}\big(\beta_j^{(0)}d_j^{(0)} + \beta_j^{(1)}d_j^{(1)} + \ldots + \beta_j^{(n)}d_j^{(n)}\big). \tag{5.5}$$

Proof. Set the Lyapunov functional as

$$L(t) = \sum_{i=0}^{n} L^{(i)}(t),$$
$$L^{(i)}(t) = \sum_{j=1}^{N}\beta_j^{(i)}|s_j^{(i)}(t)| + \sum_{j=1}^{N}\frac{\omega_j^{(i)}}{1-\nu}\int_{t-\sigma(t)}^{t}|s_j^{(i)}(\theta)|\,d\theta,\ i = 0, 1, \ldots, n, \tag{5.6}$$

where

$$\omega_j^{(0)} = \sum_{\ell=1}^{N}\big(\beta_j^{(0)}|b_{\ell j}^{(0)}| + \beta_j^{(1)}|b_{\ell j}^{(1)}| + \beta_j^{(2)}|b_{\ell j}^{(2)}| + \ldots + \beta_j^{(n)}|b_{\ell j}^{(n)}|\big)\hat{f}_j$$

$$\omega_j^{(1)} = \sum_{\ell=1}^{N}\Big(\beta_j^{(0)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,0}| + |b_{\ell j}^{(2)}||\epsilon_{21,0}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,0}|\big) + \beta_j^{(1)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,1}|$$

$$+ |b_{\ell j}^{(2)}||\epsilon_{21,1}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,1}|\big) + \beta_j^{(2)}\big(|b_{\ell j}^{(0)}| + |b_{\ell j}^{(1)}||\epsilon_{11,2}| + |b_{\ell j}^{(2)}||\epsilon_{21,2}| + \ldots$$

$$+ \beta_j^{(n)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,n}| + |b_{\ell j}^{(2)}||\epsilon_{21,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_j$$

$$\vdots$$

$$\omega_j^{(n)} = \sum_{\ell=1}^{N}\Big(\beta_j^{(0)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,0}| + |b_{\ell j}^{(2)}||\epsilon_{2n,0}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,0}|\big) + \beta_j^{(1)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,1}|$$

$$+ |b_{\ell j}^{(2)}||\epsilon_{2n,1}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,1}|\big) + \beta_j^{(2)}\big(|b_{\ell j}^{(1)}||\epsilon_{1n,2}| + |b_{\ell j}^{(2)}||\epsilon_{2n,2}| + \ldots + |b_{\ell j}^{(n)}|$$

$$\times |\epsilon_{nn,2}|\Big) + \ldots + \beta_j^{(n)}\Big(|b_{\ell j}^{(0)}| + |b_{\ell j}^{(1)}||\epsilon_{1n,n}| + |b_{\ell j}^{(2)}||\epsilon_{2n,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,n}|\Big)\Big)\hat{k}_j$$

Then, along the trajectories of the proposed system, the dini derivative of $L^{(0)}(t)$ is equal to

$$D^+L^{(0)}(t) = \sum_{j=1}^{N} \beta_j^{(0)} sign(s_j^{(0)}(t))\dot{s}_j^{(0)}(t) + \sum_{j=1}^{N} \frac{\omega_j^{(0)}}{1-\nu}|s_j^{(0)}(t)| \\ - \sum_{j=1}^{N} \frac{(1-\dot{\sigma}(t))}{1-\nu}\omega_j^{(0)}|s_j^{(0)}(t-\sigma(t))|.$$

In light of Assumption 5.2.4, $0 \le \dot{\sigma}(t) \le \nu < 1$, one can obtain $-\Big[\frac{(1-\dot{\sigma}(t))}{1-\nu}\Big] \le -1$. Then using equation (5.2), we can get

$$D^+L^{(0)}(t) \le \sum_{j=1}^{N} \beta_j^{(0)} sign(s_j^{(0)}(t))\Big[- c_j s_j^{(0)}(t) + \sum_{\ell=1}^{N}\Big(a_{j\ell}^{(0)} h_\ell^{(0)}(s_\ell^{(0)}(t)) \\ + \sum_{\gamma,\delta=1}^{n} a_{j\ell}^{(\gamma)} h_\ell^{(\delta)}(s_\ell^{(\delta)}(t))\epsilon_{\gamma\delta,0}\Big) + \sum_{\ell=1}^{N}\Big(b_{j\ell}^{(0)} g_\ell^{(0)}(s_\ell^{(0)}(t-\sigma(t))) \\ + \sum_{\gamma,\delta=1}^{n} b_{j\ell}^{(\gamma)} g_\ell^{(\delta)}(s_\ell^{(\delta)}(t-\sigma(t)))\epsilon_{\gamma\delta,0}\Big) + G_j^{(0)}(t)\Big] + \sum_{j=1}^{N}\frac{\omega_j^{(0)}}{1-\nu}|s_j^{(0)}(t)| \\ - \sum_{j=1}^{N} \omega_j^{(0)}|s_j^{(0)}(t-\sigma(t))|. \tag{5.7}$$

The terms $h_\ell^{(i)}(0)$ and $g_\ell^{(i)}(0)$, $i = 0, 1, \ldots, n$ are added and subtracted together in (5.5) and for convenient $t - \sigma(t)$ can be written as $t_{\sigma(t)}$, so that

$$D^+L^{(0)}(t) \le -\sum_{j=1}^{N} \beta_j^{(0)} c_j |s_j^{(0)}(t)| + \sum_{j=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_j^{(0)}|a_{j\ell}^{(0)}||h_\ell^{(0)}(s_\ell^{(0)}(t)) - h_\ell^{(0)}(0)| \\ + \sum_{\gamma,\delta=1}^{n} \beta_j^{(0)}|a_{j\ell}^{(\gamma)}||h_\ell^{(\delta)}(s_\ell^{(\delta)}(t)) - h_\ell^{(\delta)}(0)||\epsilon_{\gamma\delta,0}|\Big)$$

$$+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_\jmath^{(0)}|b_{\jmath\ell}^{(0)}||g_\ell^{(0)}(s_\ell^{(0)}(t_{\sigma(t)}))-g_\ell^{(0)}(0)|+\sum_{\gamma,\delta=1}^{n}\beta_\jmath^{(0)}|b_{\jmath\ell}^{(\gamma)}||g_\ell^{(\delta)}(s_\ell^{(\delta)}(t_{\sigma(t)}))$$
$$-g_\ell^{(0)}(0)||\epsilon_{\gamma\delta,0}|\Big)+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(|a_{\jmath\ell}^{(0)}||h_\ell^{(0)}(0)|+|b_{\jmath\ell}^{(0)}||g_\ell^{(0)}(0)|+\sum_{\gamma,\delta=1}^{n}\Big(|a_{\jmath\ell}^{(\gamma)}||h_\ell^{(\delta)}(0)|$$
$$+|b_{\jmath\ell}^{(\gamma)}||g_\ell^{(\delta)}(0)|\Big)|\epsilon_{\gamma\delta,0}|\Big)+\sum_{\jmath=1}^{N}\beta_\jmath^{(0)}|G_\jmath^{(0)}(t)|+\sum_{\jmath=1}^{N}\frac{\omega_\jmath^{(0)}}{1-\nu}|s_\jmath^{(0)}(t)|-\sum_{\jmath=1}^{N}\omega_\jmath^{(0)}|s_\jmath^{(0)}(t_{\sigma(t)})|. \tag{5.8}$$

In light of Assumptions 5.2.2 and 5.2.3, we can conclude that

$$D^+L^{(0)}(t)\le-\sum_{\jmath=1}^{N}\beta_\jmath^{(0)}c_\jmath|s_\jmath^{(0)}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(|a_{\jmath\ell}^{(0)}|\hat{k}_\ell|s_\jmath^{(0)}(t)|+\sum_{\gamma,\delta=1}^{n}|a_{\jmath\ell}^{(\gamma)}|\hat{k}_\ell|s_\ell^{(\delta)}(t)|$$
$$\times|\epsilon_{\gamma\delta,0}|\Big)+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(|b_{\jmath\ell}^{(0)}|\hat{f}_\ell|s_\ell^{(0)}(t_{\sigma(t)})|+\sum_{\gamma,\delta=1}^{n}|b_{\jmath\ell}^{(\gamma)}|\hat{f}_\ell|s_\ell^{(\delta)}(t_{\sigma(t)})||\epsilon_{\gamma\delta,0}|\Big)$$
$$+\sum_{\jmath=1}^{N}\beta_\jmath^{(0)}d_\jmath^{(0)}+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(|a_{\jmath\ell}^{(0)}||h_\ell^{(0)}(0)|+|b_{\jmath\ell}^{(0)}||g_\ell^{(0)}(0)|+\sum_{\gamma,\delta=1}^{n}\Big(|a_{\jmath\ell}^{(\gamma)}||$$
$$\times h_\ell^{(\delta)}(0)|+|b_{\jmath\ell}^{(\gamma)}||g_\ell^{(\delta)}(0)|\Big)|\epsilon_{\gamma\delta,0}|\Big)+\sum_{\jmath=1}^{N}\frac{\omega_\jmath^{(0)}}{1-\nu}|s_\jmath^{(0)}(t)|-\sum_{\jmath=1}^{N}\omega_\jmath^{(0)}|s_\jmath^{(0)}(t_{\sigma(t)})|$$
$$\le-\sum_{\jmath=1}^{N}\big(\beta_\jmath^{(0)}c_\jmath-\sum_{\ell=1}^{N}\beta_\jmath^{(0)}|a_{\ell\jmath}^{(0)}|\hat{k}_\jmath\big)|s_\jmath^{(0)}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}|b_{\ell\jmath}^{(0)}|\hat{f}_\ell|s_\ell^{(0)}(t_{\sigma(t)})|$$
$$+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(\sum_{\gamma,\delta=1}^{n}|a_{\ell\jmath}^{(\gamma)}|\hat{k}_\jmath|s_\jmath^{(\delta)}(t)|+|b_{\ell\jmath}^{(\gamma)}|\hat{f}_\jmath|s_\jmath^{(\delta)}(t_{\sigma(t)})|\Big)|\epsilon_{\gamma\delta,0}|+\sum_{\jmath=1}^{N}\beta_\jmath^{(0)}$$
$$\times d_\jmath^{(0)}+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big(|a_{\ell\jmath}^{(0)}||h_\jmath^{(0)}(0)|+|b_{\ell\jmath}^{(0)}||g_\jmath^{(0)}(0)|+\sum_{\gamma,\delta=1}^{n}\big(|a_{\ell\jmath}^{(\gamma)}||h_\jmath^{(\delta)}(0)|$$
$$+|b_{\ell\jmath}^{(\gamma)}||g_\jmath^{(\delta)}(0)|\big)|\epsilon_{\gamma\delta,0}|\Big)+\sum_{\jmath=1}^{N}\frac{\omega_\jmath^{(0)}}{1-\nu}|s_\jmath^{(0)}(t)|-\sum_{\jmath=1}^{N}\omega_\jmath^{(0)}|s_\jmath^{(0)}(t_{\sigma(t)})|$$
$$\le-\sum_{\jmath=1}^{N}\big(\beta_\jmath^{(0)}c_\jmath-\sum_{\ell=1}^{N}\beta_\jmath^{(0)}|a_{\ell\jmath}^{(0)}|\hat{k}_\jmath\big)|s_\jmath^{(0)}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}|b_{\ell\jmath}^{(0)}|\hat{f}_\ell|s_\ell^{(0)}(t_{\sigma(t)})|$$
$$+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(0)}\Big[\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,0}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,0}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big)|s_\jmath^{(1)}(t)|$$

$$
\begin{aligned}
&+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,0}|+|a^{(2)}_{\ell\jmath}||\epsilon_{22,0}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{n2,0}|\big)|s^{(2)}_{\jmath}(t)|+\ldots+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{1n,0}| \\
&+|a^{(2)}_{\ell\jmath}||\epsilon_{2n,0}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|s^{(n)}_{\jmath}(t)|\Big]\hat{k}_{\jmath}+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(0)}_{\jmath}\Big[\big(|b^{(1)}_{\ell\jmath}||\epsilon_{11,0}|+|b^{(2)}_{\ell\jmath}||\epsilon_{21,0}| \\
&+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{n1,0}|\big)|s^{(1)}_{\jmath}(t_{\sigma(t)})|+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{12,0}|+|b^{(2)}_{\ell\jmath}||\epsilon_{22,0}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{n2,0}|\big) \\
&\times|s^{(2)}_{\jmath}(t_{\sigma(t)})|+\ldots+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{1n,0}|+|b^{(2)}_{\ell\jmath}||\epsilon_{2n,0}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|s^{(n)}_{\jmath}(t_{\sigma(t)})|\Big]\hat{f}_{\jmath} \\
&+\sum_{\jmath=1}^{N}\beta^{(0)}_{\jmath}d^{(0)}_{\jmath}+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(0)}_{\jmath}\Big(|a^{(0)}_{\ell\jmath}||h^{(0)}_{\jmath}(0)|+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{11,0}|+|a^{(2)}_{\ell\jmath}||\epsilon_{21,0}|+\ldots+|a^{(n)}_{\ell\jmath}| \\
&\times|\epsilon_{n1,0}|\big)|h^{(1)}_{\jmath}(0)|+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,0}|+|a^{(2)}_{\ell\jmath}||\epsilon_{22,0}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{n2,0}|\big)|h^{(2)}_{\jmath}(0)|+\ldots \\
&+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{1n,0}|+|a^{(2)}_{\ell\jmath}||\epsilon_{2n,0}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|h^{(n)}_{\jmath}(0)|+|b^{(0)}_{\ell\jmath}||g^{(0)}_{\jmath}(0)|+\big(|b^{(1)}_{\ell\jmath}| \\
&\times|\epsilon_{11,0}|+|b^{(2)}_{\ell\jmath}||\epsilon_{21,0}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{n1,0}|\big)|g^{(1)}_{\jmath}(0)|+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{12,0}|+|b^{(2)}_{\ell\jmath}||\epsilon_{22,0}|+\ldots \\
&+|b^{(n)}_{\ell\jmath}||\epsilon_{n2,0}|\big)|g^{(2)}_{\jmath}(0)|+\ldots+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{1n,0}|+|a^{(2)}_{\ell\jmath}||\epsilon_{2n,0}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|g^{(n)}_{\jmath}(0)|\Big) \\
&+\sum_{\jmath=1}^{N}\frac{\omega^{(0)}_{\jmath}}{1-\nu}|s^{(0)}_{\jmath}(t)|-\sum_{\jmath=1}^{N}\omega^{(0)}_{\jmath}|s^{(0)}_{\jmath}(t_{\sigma(t)})|. \qquad (5.9)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D^{+}L^{(1)}(t)\leq&-\sum_{\jmath=1}^{N}\big(\beta^{(1)}_{\jmath}c_{\jmath}-\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}|a^{(0)}_{\ell\jmath}|\hat{k}_{\jmath}\big)|s^{(1)}_{\jmath}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}|a^{(1)}_{\ell\jmath}|\hat{k}_{\ell}|s^{(0)}_{\ell}(t)| \\
&+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}|b^{(0)}_{\ell\jmath}|\hat{f}_{\ell}|s^{(1)}_{\ell}(t_{\sigma(t)})|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}|b^{(1)}_{\ell\jmath}|\hat{f}_{\ell}|s^{(0)}_{\ell}(t_{\sigma(t)})| \\
&+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}\Big[\big(|a^{(1)}_{\ell\jmath}||\epsilon_{11,1}|+|a^{(2)}_{\ell\jmath}||\epsilon_{21,1}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{n1,1}|\big)|s^{(1)}_{\jmath}(t)| \\
&+\big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,1}|+|a^{(2)}_{\ell\jmath}||\epsilon_{22,1}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{n2,1}|\big)|s^{(2)}_{\jmath}(t)|+\ldots+\big(|a^{(1)}_{\ell\jmath}| \\
&\times|\epsilon_{1n,1}|+|a^{(2)}_{\ell\jmath}||\epsilon_{2n,1}|+\ldots+|a^{(n)}_{\ell\jmath}||\epsilon_{nn,1}|\big)|s^{(n)}_{\jmath}(t)|\Big]\hat{k}_{\jmath}+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath} \\
&\times\Big[\big(|b^{(1)}_{\ell\jmath}||\epsilon_{11,1}|+|b^{(2)}_{\ell\jmath}||\epsilon_{21,1}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{n1,1}|\big)|s^{(1)}_{\jmath}(t_{\sigma(t)})|+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{12,1}| \\
&+|b^{(2)}_{\ell\jmath}||\epsilon_{22,1}|+\ldots+|b^{(n)}_{\ell\jmath}||\epsilon_{n2,1}|\big)|s^{(2)}_{\jmath}(t_{\sigma(t)})|+\ldots+\big(|b^{(1)}_{\ell\jmath}||\epsilon_{1n,1}|+|b^{(2)}_{\ell\jmath}|
\end{aligned}
$$

$$\times |\epsilon_{2n,1}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|s^{(n)}_{\jmath}(t_{\sigma(t)})|\Big]\hat{f}_{\jmath} + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(1)}_{\jmath}\Big(|a^{(0)}_{\ell\jmath}||h^{(1)}_{\jmath}(0)|$$
$$+ |a^{(1)}_{\ell\jmath}||h^{(0)}_{\jmath}(0)| + \big(|a^{(1)}_{\ell\jmath}||\epsilon_{11,1}| + |a^{(2)}_{\ell\jmath}||\epsilon_{21,1}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{n1,1}|\big)|h^{(1)}_{\jmath}(0)|$$
$$+ \big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,1}| + |a^{(2)}_{\ell\jmath}||\epsilon_{22,1}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{n2,1}|\big)|h^{(2)}_{\jmath}(0)| + \ldots + \big(|a^{(1)}_{\ell\jmath}|$$
$$\times |\epsilon_{1n,1}| + |a^{(2)}_{\ell\jmath}||\epsilon_{2n,1}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{nn,1}|\big)|h^{(n)}_{\jmath}(0)| + |b^{(0)}_{\ell\jmath}||g^{(1)}_{\jmath}(0)|$$
$$+ |b^{(1)}_{\ell\jmath}||g^{(0)}_{\jmath}(0)| + \big(|b^{(1)}_{\ell\jmath}||\epsilon_{11,1}| + |b^{(2)}_{\ell\jmath}||\epsilon_{21,1}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{n1,1}|\big)|g^{(1)}_{\jmath}(0)|$$
$$+ \big(|b^{(1)}_{\ell\jmath}||\epsilon_{12,1}| + |b^{(2)}_{\ell\jmath}||\epsilon_{22,1}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{n2,1}|\big)|g^{(2)}_{\jmath}(0)| + \ldots + \big(|b^{(1)}_{\ell\jmath}|$$
$$\times |\epsilon_{1n,1}| + |a^{(2)}_{\ell\jmath}||\epsilon_{2n,1}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{nn,1}|\big)|g^{(n)}_{\jmath}(0)|\Big) + \sum_{\jmath=1}^{N}\beta^{(1)}_{\jmath}d^{(1)}_{\jmath}$$
$$+ \sum_{\jmath=1}^{N}\frac{\omega^{(1)}_{\jmath}}{1-\nu}|s^{(1)}_{\jmath}(t)| - \sum_{\jmath=1}^{N}\omega^{(1)}_{\jmath}|s^{(1)}_{\jmath}(t_{\sigma(t)})|. \tag{5.10}$$

$$D^{+}L^{(2)}(t) \leq -\sum_{\jmath=1}^{N}\big(\beta^{(2)}_{\jmath}c_{\jmath} - \sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}|a^{(0)}_{\ell\jmath}|\hat{k}_{\jmath}\big)|s^{(2)}_{\jmath}(t)| + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}|a^{(2)}_{\ell\jmath}|\hat{k}_{\ell}|s^{(0)}_{\ell}(t)|$$
$$+ \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}|b^{(0)}_{\ell\jmath}|\hat{f}_{\ell}|s^{(2)}_{\ell}(t_{\sigma(t)})| + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}|b^{(2)}_{\ell\jmath}|\hat{f}_{\ell}|s^{(0)}_{\ell}(t_{\sigma(t)})|$$
$$+ \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}\Big[\big(|a^{(1)}_{\ell\jmath}||\epsilon_{11,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{21,2}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{n1,2}|\big)|s^{(1)}_{\jmath}(t)|$$
$$+ \big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{22,2}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{n2,2}|\big)|s^{(2)}_{\jmath}(t)| + \ldots + \big(|a^{(1)}_{\ell\jmath}|$$
$$\times |\epsilon_{1n,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{2n,2}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{nn,2}|\big)|s^{(n)}_{\jmath}(t)|\Big]\hat{k}_{\jmath} + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}$$
$$\times \Big[\big(|b^{(1)}_{\ell\jmath}||\epsilon_{11,2}| + |b^{(2)}_{\ell\jmath}||\epsilon_{21,2}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{n1,2}|\big)|s^{(1)}_{\jmath}(t_{\sigma(t)})| + \big(|b^{(1)}_{\ell\jmath}|$$
$$\times |\epsilon_{12,2}| + |b^{(2)}_{\ell\jmath}||\epsilon_{22,2}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{n2,2}|\big)|s^{(2)}_{\jmath}(t_{\sigma(t)})| + \ldots + \big(|b^{(1)}_{\ell\jmath}||\epsilon_{1n,2}|$$
$$+ |b^{(2)}_{\ell\jmath}||\epsilon_{2n,2}| + \ldots + |b^{(n)}_{\ell\jmath}||\epsilon_{nn,0}|\big)|s^{(n)}_{\jmath}(t_{\sigma(t)})|\Big]\hat{f}_{\jmath} + \sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta^{(2)}_{\jmath}\Big(|a^{(0)}_{\ell\jmath}|$$
$$\times |h^{(2)}_{\jmath}(0)| + |a^{(2)}_{\ell\jmath}||h^{(0)}_{\jmath}(0)| + \big(|a^{(1)}_{\ell\jmath}||\epsilon_{11,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{21,2}| + \ldots + |a^{(n)}_{\ell\jmath}|$$
$$\times |\epsilon_{n1,2}|\big)|h^{(1)}_{\jmath}(0)| + \big(|a^{(1)}_{\ell\jmath}||\epsilon_{12,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{22,2}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{n2,2}|\big)$$
$$\times |h^{(2)}_{\jmath}(0)| + \ldots + \big(|a^{(1)}_{\ell\jmath}||\epsilon_{1n,2}| + |a^{(2)}_{\ell\jmath}||\epsilon_{2n,2}| + \ldots + |a^{(n)}_{\ell\jmath}||\epsilon_{nn,2}|\big)$$
$$\times |h^{(n)}_{\jmath}(0)| + |b^{(0)}_{\ell\jmath}||g^{(2)}_{\jmath}(0)| + |b^{(2)}_{\ell\jmath}||g^{(0)}_{\jmath}(0)| + \big(|b^{(1)}_{\ell\jmath}||\epsilon_{11,2}| + |b^{(2)}_{\ell\jmath}||\epsilon_{21,2}|$$

$$
\begin{aligned}
&+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big)|g_\jmath^{(1)}(0)|+\big(|b_{\ell\jmath}^{(1)}||\epsilon_{12,2}|+|b_{\ell\jmath}^{(2)}||\epsilon_{22,2}|+\ldots+|b_{\ell\jmath}^{(n)}|\\
&\times|\epsilon_{n2,2}|\big)|g_\jmath^{(2)}(0)|+\ldots+\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,2}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,2}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)|g_\jmath^{(n)}(0)|\Big)\\
&+\sum_{\jmath=1}^{N}\beta_\jmath^{(2)}d_\jmath^{(2)}+\sum_{\jmath=1}^{N}\frac{\omega_\jmath^{(2)}}{1-\nu}|s_\jmath^{(2)}(t)|-\sum_{\jmath=1}^{N}\omega_\jmath^{(2)}|s_\jmath^{(2)}(t_{\sigma(t)})|.
\end{aligned}
\tag{5.11}
$$

and finally,

$$
\begin{aligned}
D^+L^{(n)}(t)\le&-\sum_{\jmath=1}^{N}\big(\beta_\jmath^{(n)}c_\jmath-\sum_{\ell=1}^{N}\beta_\jmath^{(n)}|a_{\ell\jmath}^{(0)}|\hat{k}_\jmath\big)|s_\jmath^{(n)}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}|a_{\ell\jmath}^{(n)}|\hat{k}_\ell|s_\ell^{(0)}(t)|\\
&+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}|b_{\ell\jmath}^{(0)}|\hat{f}_\ell|s_\ell^{(n)}(t_{\sigma(t)})|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}|b_{\ell\jmath}^{(n)}|\hat{f}_\ell|s_\ell^{(0)}(t_{\sigma(t)})|\\
&+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}\Big[\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)|s_\jmath^{(1)}(t)|\\
&+\big(|a_{\ell\jmath}^{(1)}||\epsilon_{12,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{22,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n2,n}|\big)|s_\jmath^{(2)}(t)|+\ldots+\big(|a_{\ell\jmath}^{(1)}|\\
&\times|\epsilon_{1n,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)|s_\jmath^{(n)}(t)|\Big]\hat{k}_\jmath+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}\\
&\times\Big[\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)|s_\jmath^{(1)}(t_{\sigma(t)})|+\big(|b_{\ell\jmath}^{(1)}|\\
&\times|\epsilon_{12,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{22,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n2,n}|\big)|s_\jmath^{(2)}(t_{\sigma(t)})|+\ldots+\big(|b_{\ell\jmath}^{(1)}|\\
&\times|\epsilon_{1n,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big)|s_\jmath^{(n)}(t_{\sigma(t)})|\Big]\hat{f}_\jmath+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\beta_\jmath^{(n)}\\
&\times\Big(|a_{\ell\jmath}^{(0)}||h_\jmath^{(n)}(0)|+|a_{\ell\jmath}^{(n)}||h_\jmath^{(0)}(0)|+\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots\\
&+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)|h_\jmath^{(1)}(0)|+\big(|a_{\ell\jmath}^{(1)}||\epsilon_{12,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{22,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n2,n}|\big)\\
&\times|h_\jmath^{(2)}(0)|+\ldots+\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\\
&\times|h_\jmath^{(n)}(0)|+|b_{\ell\jmath}^{(0)}||g_\jmath^{(n)}(0)|+|b_{\ell\jmath}^{(n)}||g_\jmath^{(0)}(0)|+\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,n}|\\
&+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)|g_\jmath^{(1)}(0)|+\big(|b_{\ell\jmath}^{(1)}||\epsilon_{12,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{22,n}|+\ldots+|b_{\ell\jmath}^{(n)}|
\end{aligned}
$$

$$\times |\epsilon_{n2,n}|\big)|g_j^{(2)}(0)| + \ldots + \big(|b_{\ell j}^{(1)}||\epsilon_{1n,n}| + |a_{\ell j}^{(2)}||\epsilon_{2n,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{nn,n}|\big)|g_j^{(n)}(0)|\Big)$$
$$+ \sum_{j=1}^{N} \beta_j^{(n)} d_j^{(n)} + \sum_{j=1}^{N} \frac{\omega_j^{(n)}}{1-\nu}|s_j^{(n)}(t)| - \sum_{j=1}^{N} \omega_j^{(n)}|s_j^{(n)}(t_{\sigma(t)})|. \tag{5.12}$$

By combining the equations (5.10)-(5.12) and changing the order, we obtain

$$\begin{aligned}
D^+L(t) =& \sum_{i=0}^{n} D^+L^{(i)}(t) \\
\leq& -\sum_{j=1}^{N} \Big(\beta_j^{(0)} c_j - \sum_{\ell=1}^{N} \big(\beta_j^{(0)}|a_{\ell j}^{(0)}| + \beta_j^{(1)}|a_{\ell j}^{(1)}| + \beta_j^{(2)}|a_{\ell j}^{(2)}| + \ldots + \beta_j^{(n)}|a_{\ell j}^{(n)}|\big)\hat{k}_j\Big) \\
& \times |s_j^{(0)}(t)| + \sum_{j=1}^{N}\sum_{\ell=1}^{N} \big(\beta_j^{(0)}|b_{\ell j}^{(0)}| + \beta_j^{(1)}|b_{\ell j}^{(1)}| + \beta_j^{(2)}|b_{\ell j}^{(2)}| + \ldots + \beta_j^{(n)}|b_{\ell j}^{(n)}|\big)\hat{f}_j\Big) \\
& \times |s_j^{(0)}(t_{\sigma(t)})| + \sum_{j=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_j^{(0)}|a_{\ell j}^{(0)}| + \beta_j^{(1)}|a_{\ell j}^{(1)}| + \beta_j^{(2)}|a_{\ell j}^{(2)}| + \ldots + \beta_j^{(n)}|a_{\ell j}^{(n)}|\Big) \\
& \times |h_j^{(0)}(0)| + \sum_{j=1}^{N}\sum_{\ell=1}^{N} \Big(\beta_j^{(0)}|b_{\ell j}^{(0)}| + \beta_j^{(1)}|b_{\ell j}^{(1)}| + \beta_j^{(2)}|b_{\ell j}^{(2)}| + \ldots + \beta_j^{(n)}|b_{\ell j}^{(n)}|\Big) \\
& \times |g_j^{(0)}(0)| + \sum_{j=1}^{N} \beta_j^{(0)} d_j^{(0)} + \sum_{j=1}^{N} \frac{\omega_j^{(0)}}{1-\nu}|s_j^{(0)}(t)| - \sum_{j=1}^{N} \omega_j^{(0)}|s_j^{(0)}(t_{\sigma(t)})| \\
& - \sum_{j=1}^{N} \Big(\beta_j^{(1)} c_j - \sum_{\ell=1}^{N} \Big[\Big(\beta_j^{(0)}\big(|a_{\ell j}^{(1)}||\epsilon_{11,0}| + |a_{\ell j}^{(2)}||\epsilon_{21,0}| + \ldots + |a_{\ell j}^{(n)}||\epsilon_{n1,0}|\big) \\
& + \beta_j^{(1)}\big(|a_{\ell j}^{(0)}| + |a_{\ell j}^{(1)}||\epsilon_{11,1}| + |a_{\ell j}^{(2)}||\epsilon_{21,1}| + \ldots + |a_{\ell j}^{(n)}||\epsilon_{n1,1}|\big) + \beta_j^{(2)}\big(|a_{\ell j}^{(1)}| \\
& \times |\epsilon_{11,2}| + |a_{\ell j}^{(2)}||\epsilon_{21,2}| + \ldots + |a_{\ell j}^{(n)}||\epsilon_{n1,2}|\big) + \ldots + \beta_j^{(n)}\big(|a_{\ell j}^{(1)}||\epsilon_{11,n}| + |a_{\ell j}^{(2)}| \\
& \times |\epsilon_{21,n}| + \ldots + |a_{\ell j}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_j\Big]|s_j^{(1)}(t)| + \sum_{j=1}^{N}\sum_{\ell=1}^{N} \Big[\Big(\beta_j^{(0)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,0}| \\
& + |b_{\ell j}^{(2)}||\epsilon_{21,0}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,0}|\big) + \beta_j^{(1)}\big(|b_{\ell j}^{(0)}| + |b_{\ell j}^{(1)}||\epsilon_{11,1}| + |b_{\ell j}^{(2)}||\epsilon_{21,1}| \\
& + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,1}|\big) + \beta_j^{(2)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,2}| + |b_{\ell j}^{(2)}||\epsilon_{21,2}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,2}|\big) \\
& + \ldots + \beta_j^{(n)}\big(|b_{\ell j}^{(1)}||\epsilon_{11,n}| + |b_{\ell j}^{(2)}||\epsilon_{21,n}| + \ldots + |b_{\ell j}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_j\Big]|s_j^{(1)}(t_{\sigma(t)})|
\end{aligned}$$

$$
\begin{aligned}
&+\ldots+\beta_{\jmath}^{(n)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_{\jmath}\Big]|s_{\jmath}^{(1)}(t_{\sigma(t)})|\\
&+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_{\jmath}^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,0}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,0}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big)+\beta_{\jmath}^{(1)}\big(|a_{\ell\jmath}^{(0)}|\\
&+|a_{\ell\jmath}^{(1)}||\epsilon_{11,1}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,1}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big)+\beta_{\jmath}^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,2}|+|a_{\ell\jmath}^{(2)}|\\
&\times|\epsilon_{21,2}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big)+\ldots+\beta_{\jmath}^{(n)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots\\
&+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)\Big)|h_{\jmath}^{(1)}(0)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big(|\beta_{\jmath}^{(0)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,0}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,0}|+\ldots\\
&+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big)+\beta_{\jmath}^{(1)}\big(|b_{\ell\jmath}^{(0)}|+|b_{\ell\jmath}^{(1)}||\epsilon_{11,1}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,1}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big)\\
&+\beta_{\jmath}^{(2)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,2}|+|b_{\ell\jmath}^{(2)}||\epsilon_{21,2}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big)+\ldots+\beta_{\jmath}^{(n)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{11,n}|\\
&+|b_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)\Big)|g_{\jmath}^{(1)}(0)|+\sum_{\jmath=1}^{N}\beta_{\jmath}^{(1)}d_{\jmath}^{(1)}+\sum_{\jmath=1}^{N}\frac{\omega_{\jmath}^{(1)}}{1-\nu}|s_{\jmath}^{(1)}(t)|\\
&-\sum_{\jmath=1}^{N}\omega_{\jmath}^{(1)}|s_{\jmath}^{(1)}(t_{\sigma(t)})|\\
&\qquad\qquad\qquad\qquad\vdots\\
&-\sum_{\jmath=1}^{N}\Big(\beta_{\jmath}^{(n)}c_{\jmath}-\sum_{\ell=1}^{N}\Big[\Big(\beta_{\jmath}^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,0}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,0}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big)\\
&+\beta_{\jmath}^{(1)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,1}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big)+\beta_{\jmath}^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,2}|+|a_{\ell\jmath}^{(2)}|\\
&\times|\epsilon_{2n,2}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)+\ldots+\beta_{\jmath}^{(n)}\big(|a_{\ell\jmath}^{(0)}|+|a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|\\
&+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\hat{k}_{\jmath}\Big]|s_{\jmath}^{(n)}(t)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big[\Big(\beta_{\jmath}^{(0)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,0}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,0}|\\
&+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big)+\beta_{\jmath}^{(1)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,1}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big)+\beta_{\jmath}^{(2)}\\
&\times\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,2}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,2}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)+\ldots+\beta_{\jmath}^{(n)}\big(|b_{\ell\jmath}^{(0)}|+|b_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|\\
&+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\hat{k}_{\jmath}\Big]|s_{\jmath}^{(n)}(t_{\sigma(t)})|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_{\jmath}^{(0)}\big(|a_{\ell\jmath}^{(1)}|\\
&\times|\epsilon_{1n,0}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,0}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big)+\beta_{\jmath}^{(1)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,1}|\\
&+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big)+\beta_{\jmath}^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,2}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,2}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\ldots+\beta_\jmath^{(n)}\big(|a_{\ell\jmath}^{(0)}|+|a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\\
&\times|h_\jmath^{(n)}(0)|+\sum_{\jmath=1}^{N}\sum_{\ell=1}^{N}\Big(\beta_\jmath^{(0)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,0}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,0}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,0}|\big)\\
&+\beta_\jmath^{(1)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,1}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big)+\beta_\jmath^{(2)}\big(|b_{\ell\jmath}^{(1)}||\epsilon_{1n,2}|\\
&+|b_{\ell\jmath}^{(2)}||\epsilon_{2n,2}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)+\ldots+\beta_\jmath^{(n)}\big(|b_{\ell\jmath}^{(0)}|+|b_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|+|b_{\ell\jmath}^{(2)}|\\
&\times|\epsilon_{2n,n}|+\ldots+|b_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)|g_\jmath^{(n)}(0)|+\sum_{\jmath=1}^{N}\beta_\jmath^{(n)}d_\jmath^{(n)}+\sum_{\jmath=1}^{N}\frac{\omega_\jmath^{(n)}}{1-\nu}|s_\jmath^{(n)}(t)|\\
&-\sum_{\jmath=1}^{N}\omega_\jmath^{(n)}|s_\jmath^{(n)}(t_{\sigma(t)})|\\
\leq&-\sum_{\jmath=1}^{N}\Big(\beta_\jmath^{(0)}c_\jmath-\frac{\omega_\jmath^{(0)}}{1-\nu}-\sum_{\ell=1}^{N}\big(\beta_\jmath^{(0)}|a_{\ell\jmath}^{(0)}|+\beta_\jmath^{(1)}|a_{\ell\jmath}^{(1)}|+\beta_\jmath^{(2)}|a_{\ell\jmath}^{(2)}|+\ldots+\beta_\jmath^{(n)}\\
&\times|a_{\ell\jmath}^{(n)}|\big)\hat{k}_\jmath\Big)|s_\jmath^{(0)}(t)|-\sum_{\jmath=1}^{N}\Big(\beta_\jmath^{(1)}c_\jmath-\frac{\omega_\jmath^{(1)}}{1-\nu}-\sum_{\ell=1}^{N}\Big[\Big(\beta_\jmath^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,0}|+|a_{\ell\jmath}^{(2)}|\\
&\times|\epsilon_{21,0}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,0}|\big)+\beta_\jmath^{(1)}\big(|a_{\ell\jmath}^{(0)}|+|a_{\ell\jmath}^{(1)}||\epsilon_{11,1}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,1}|+\ldots\\
&+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,1}|\big)+\beta_\jmath^{(2)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,2}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,2}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,2}|\big)+\ldots\\
&+\beta_\jmath^{(n)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{11,n}|+|a_{\ell\jmath}^{(2)}||\epsilon_{21,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{n1,n}|\big)\Big)\hat{k}_\jmath\Big]|s_\jmath^{(1)}(t)|\\
&\qquad\qquad\qquad\qquad\vdots\\
&-\sum_{\jmath=1}^{N}\Big(\beta_\jmath^{(n)}c_\jmath-\frac{\omega_\jmath^{(n)}}{1-\nu}-\sum_{\ell=1}^{N}\Big[\Big(\beta_\jmath^{(0)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,0}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,0}|+\ldots+|a_{\ell\jmath}^{(n)}|\\
&\times|\epsilon_{nn,0}|\big)+\beta_\jmath^{(1)}\big(|a_{\ell\jmath}^{(1)}||\epsilon_{1n,1}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,1}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,1}|\big)+\beta_\jmath^{(2)}\big(|a_{\ell\jmath}^{(1)}|\\
&\times|\epsilon_{1n,2}|+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,2}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,2}|\big)+\ldots+\beta_\jmath^{(n)}\big(|a_{\ell\jmath}^{(0)}|+|a_{\ell\jmath}^{(1)}||\epsilon_{1n,n}|\\
&+|a_{\ell\jmath}^{(2)}||\epsilon_{2n,n}|+\ldots+|a_{\ell\jmath}^{(n)}||\epsilon_{nn,n}|\big)\Big)\hat{k}_\jmath\Big]|s_\jmath^{(n)}(t)|+\tilde{\kappa}_2\\
\leq&-\sum_{\jmath=1}^{N}\beta_\jmath^{(0)}\kappa_\jmath^{(0)}|s_\jmath^{(0)}(t)|-\sum_{\jmath=1}^{N}\beta_\jmath^{(1)}\kappa_\jmath^{(1)}|s_\jmath^{(1)}(t)|-\ldots-\sum_{\jmath=1}^{N}\beta_\jmath^{(n)}\kappa_\jmath^{(n)}|s_\jmath^{(n)}(t)|+\tilde{\kappa}_2\\
D^+L(t)\leq&-\kappa_1\sum_{\jmath=1}^{N}\big(\beta_\jmath^{(0)}|s_\jmath^{(0)}(t)|+\beta_\jmath^{(1)}|s_\jmath^{(1)}(t)|+\ldots+\beta_\jmath^{(n)}|s_\jmath^{(n)}(t)|\big)+\tilde{\kappa}_2.
\end{aligned}
$$

Thus we obtain

$$D^+L(t) \leq -\kappa_1 L(t) + \tilde{\kappa}_2, \tag{5.13}$$

where $\kappa_1 = min_{1\leq j\leq N}\{\kappa_j^{(0)}, \kappa_j^{(1)}, \ldots, \kappa_j^{(n)}\}$.

By Lemma 5.2.1, when $L(t) \geq \frac{\tilde{\kappa}_2}{\kappa_1}$, we obtain

$$L(t) - \frac{\tilde{\kappa}_2}{\kappa_1} \leq \Big(L(0) - \frac{\tilde{\kappa}_2}{\kappa_1}\Big)e^{-\kappa_1 t}, \quad t \geq 0. \tag{5.14}$$

The state variable of the network is limited by

$$\|s(t)\| = \sum_{j=1}^{N}\Big(|s_j^{(0)}(t)| + |s_j^{(1)}(t)| + |s_j^{(2)}(t)| + \ldots + |s_j^{(n)}(t)|\Big) \leq \frac{\tilde{\kappa}_2}{\beta\kappa_1} + \frac{L(0)}{\beta}e^{-\kappa_1 t}. \tag{5.15}$$

By definition (5.2.2), the system (5.1) is globally LES. Also, the system (5.1) globally exponential convergent to the ball $\mathcal{R}_1$ with a rate κ_1. The proof is completed. □

Remark 5.3.1. Recently, Lyapunov stability has been the main focus of research on CVNN, QVNN, and OVNN ([108] [94] [109]). Though there are still not many results on multistability, some examples include the availability of LES for CVNN and QVNN but not for OVNN and HCNN. The Lemma 5.2.1 is reduced to Lyapunov exponential stability of HCNN if $\tilde{\kappa}_2$=0, and as a result, the network (5.1) is Lyapunov exponential stable, and we achieve exponential stability of (5.1) in the Lyapunov sense.

Therefore, the following outcome can be obtained.

Corollary 5.1. *Assume that Assumptions (5.2.2)-(5.2.4) hold and for* $i = 0, 1, \ldots, n$, $|h_j^{(i)}(\boldsymbol{0}))| = 0, |g_j^{(i)}(\boldsymbol{0}))| = 0$, $G_j^{(i)}(t) = 0$, *and there exist constants* $\beta_j^{(i)}, \omega_j^{(i)} > 0$ *such that all the conditions of* (5.3) *hold then we can deduce from Theorem 5.3.1 that*

$\tilde{\kappa}_2 = 0$. *Therefore, the HCNNs* (5.1) *is globally exponentially stable in Lyapunov sense.*

Remark 5.3.2. LS, in contrast to Lyapunov stability, describes the stability of the entire system rather than the stability of an equilibrium point. It is necessary to apply LS because the NN should have numerous equilibrium points. Here, we looked at the Lyapunov-Based Method as a potential means of reducing conservatism in stability analysis. Lyapunov-based techniques concentrate on generating Lyapunov functions that demonstrate a system's stability. By carefully selecting these functions, we can provide more conservative stability results than we can with algebraic techniques. Lyapunov functions may result in less conservative stability criteria and can better describe the dynamics of the system.

5.4 Numerical examples

In this section, the following numerical examples are used to discuss the effectiveness of the derived results.

Example 5.4.1. Suppose $n = 1$, then the equation (5.2) is transformed into CVNN with time-varying delay and multiplication is defined in Table 1.2. For $N = 2$, the parameters of equation (5.2) are given by

$$A = \begin{bmatrix} 0.3 + 0.5\psi_1 & 1.5 + 0.6\psi_1 \\ 0.7 - 0.3\psi_1 & 0.3 + 0.9\psi_1 \end{bmatrix} = \begin{bmatrix} 0.3 & 1.5 \\ 0.7 & 0.3 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.6 \\ -0.3 & 0.9 \end{bmatrix} \psi_1 = A^{(0)} + A^{(1)}\psi_1,$$

$$B = \begin{bmatrix} 1.2 + 0.3\psi_1 & 0.8 + 0.3\psi_1 \\ 0.9 - 0.5\psi_1 & 0.4 + 0.7\psi_1 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.8 \\ 0.9 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.3 & 0.3 \\ -0.5 & 0.7 \end{bmatrix} \psi_1 = B^{(0)} + B^{(1)}\psi_1,$$

$C = \begin{bmatrix} 7 & 0 \\ 0 & 10 \end{bmatrix}$, $G = (0.3cost + 0.2sint\psi_1, 0.2sint + 0.3cost\psi_1)^T$, $\sigma(t) = \frac{0.8}{1+e^{-t}}$ with $0 \leq \dot{\sigma}(t) \leq \nu = 0.2$, which fulfil the condition of Assumption 5.2.4.

The activation functions are given by $h_\jmath(s_\jmath^{(i)}) = g_\jmath(s_\jmath^{(i)}) = tanh(s_\jmath^{(i)})$, which give

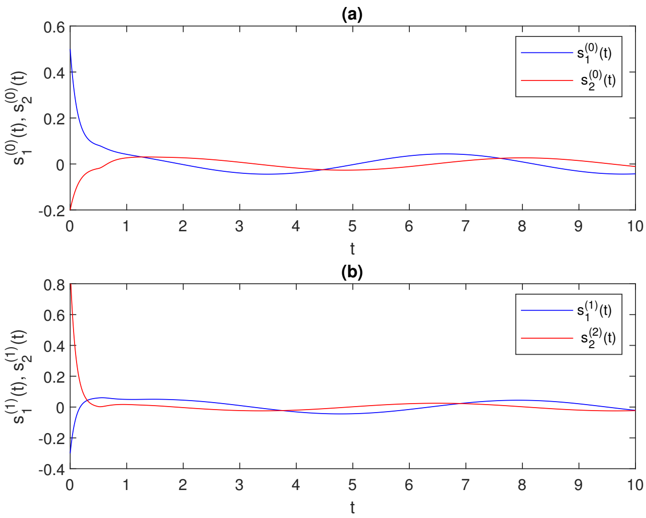

FIGURE 5.1: Plots of real component trajectories $s_1^{(0)}(t)$, $s_2^{(0)}(t)$, and imaginary component trajectories $s_1^{(1)}(t)$, $s_2^{(1)}(t)$ for the system (5.1).

$\hat{k}_\jmath = 1$, $\hat{f}_\jmath = 1$ for $\jmath = 1, 2$ and $i = 0, 1$ with the initial conditions $s_1(0) = 0.5 - 0.2\psi_1$, $s_2(0) = -0.3 + 0.9\psi_1$. In this case, Figure 5.1 depict the trajectories of the system 5.2. Now, choosing $\omega_\jmath^{(i)} = 1$, $\beta_\jmath^{(i)} = 1$, we get $\kappa_\jmath^{(0)} = 2.825 > 0$, $\kappa_\jmath^{(1)} = 1.95 > 0$, $\tilde{\kappa}_2 = 1$ and $\mathcal{R} = \frac{\tilde{\kappa}_2}{\beta \kappa_1} = 0.51$, which meet all the conditions of Theorem 5.3.1 for $\jmath = 1, 2$ and $i = 0, 1$. Also, the graph of the error system with Lagrange and Lyapunov stabilities is shown through Figure 5.2.

Example 5.4.2. Suppose $n = 3$, the equation (5.2) becomes QVNN with time-varying delay and multiplication is defined in Table 1.5. For $N = 2$, the parameters of the equation (5.1) are given by

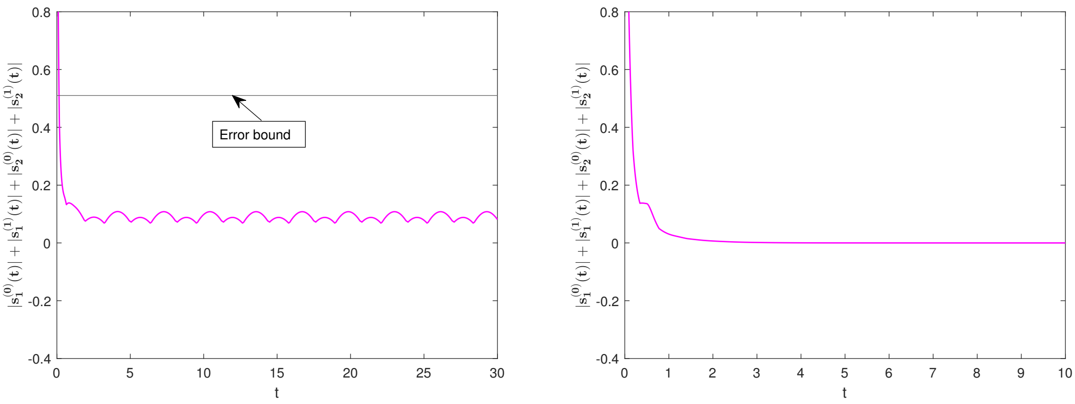

FIGURE 5.2: Plots of stability of CVNNs in (a) Lagrange and (b) Lyapunov sense.

$$
\begin{aligned}
A =& \begin{bmatrix} 1+0.7\psi_1+1.9\psi_2-1.3\psi_3 & -2-2.3\psi_1+2.8\psi_2-1.6\psi_3 \\ -0.6+2.2\psi_1+1.2\psi_2+1.8\psi_3 & 1.6-0.4\psi_1+0.9\psi_2-1.5\psi_3 \end{bmatrix} \\
=& \begin{bmatrix} 1 & -2 \\ -0.6 & 1.6 \end{bmatrix} + \begin{bmatrix} 0.7 & -2.3 \\ 2.2 & -0.4 \end{bmatrix}\psi_1 + \begin{bmatrix} 1.9 & 2.8 \\ 1.2 & 0.9 \end{bmatrix}\psi_2 + \begin{bmatrix} -1.3 & -1.6 \\ 1.8 & -1.5 \end{bmatrix}\psi_3 \\
=& A^{(0)} + A^{(1)}\psi_1 + A^{(2)}\psi_2 + A^{(3)}\psi_3, \\
B =& \begin{bmatrix} 1.1+2.2\psi_1+2.3\psi_2-0.6\psi_3 & 0.6+1.3\psi_1-1.9\psi_2-1.7\psi_3 \\ 2.2-1.4\psi_1+2.1\psi_2-0.8\psi_3 & 2.5-0.7\psi_1+2.3\psi_2-0.8\psi_3 \end{bmatrix} \\
=& \begin{bmatrix} 1.1 & 0.6 \\ 2.2 & 2.5 \end{bmatrix} + \begin{bmatrix} 2.2 & 1.3 \\ -1.4 & -0.7 \end{bmatrix}\psi_1 + \begin{bmatrix} 2.3 & -1.9 \\ 2.1 & 2.3 \end{bmatrix}\psi_2 + \begin{bmatrix} -0.6 & -1.7 \\ -0.8 & -0.8 \end{bmatrix}\psi_3 \\
=& B^{(0)} + B^{(1)}\psi_1 + B^{(2)}\psi_2 + B^{(3)}\psi_3,
\end{aligned}
$$

$C = \begin{bmatrix} 25 & 0 \\ 0 & 19 \end{bmatrix}$, $G = (0.2cost+0.3sint\psi_1+0.4cost\psi_2+0.2sint\psi_3, 0.1sint+0.2cost\psi_1+0.3sint\psi_2+0.4cost\psi_3)^T$, $\sigma(t) = \frac{1.2}{1+e^{-t}}$ with $0 \leq \dot{\sigma}(t) \leq \nu = 0.3$, which satisfies the criterion of Assumption 5.2.4. The activation functions are given by $h_j(s_j^{(i)}) =$

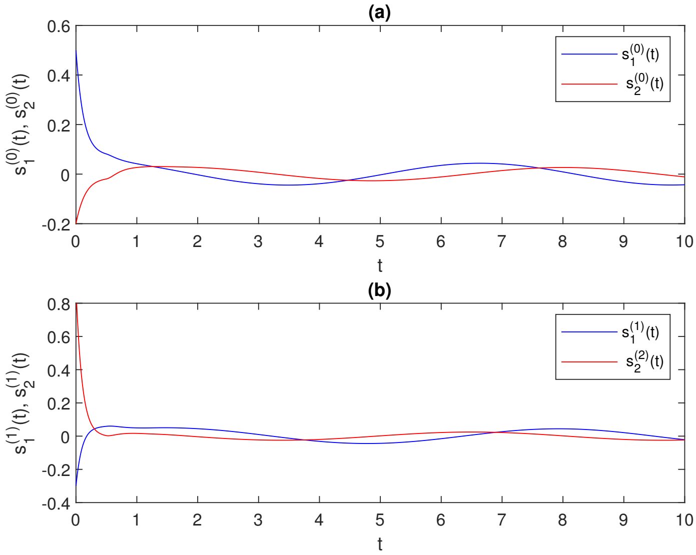

FIGURE 5.3: Plots of trajectories of $s_\jmath^{(i)}(t)$, $\jmath = 1, 2$ and $i = 0, 1, 2, 3$ for the QVNN system (5.1).

$g_\jmath(s_\jmath^{(i)}) = tanh(s_\jmath^{(i)})$ and we obtain $\hat{k}_\jmath = 1$, $\hat{f}_\jmath = 1$ for $\jmath = 1, 2$ and $i = 0, 1, 2, 3$ with the initial conditions $s_1(0) = 0.2 - 0.3\psi_1 + 0.1\psi_2 + 0.2\psi_3$, $s_2(0) = 0.4 + 0.3\psi_1 + 0.5\psi_2 + 0.4\psi_3$. In this case, Figure 5.3 depicts the trajectories of the system (5.1). Now, choosing $\omega_\jmath^{(i)} = 1$, $\beta_\jmath^{(i)} = 1$, we get $\kappa_\jmath^{(0)} = 1.16 > 0$, $\kappa_\jmath^{(1)} = 3.62 > 0$, $\tilde{\kappa}_2 = 2.1$

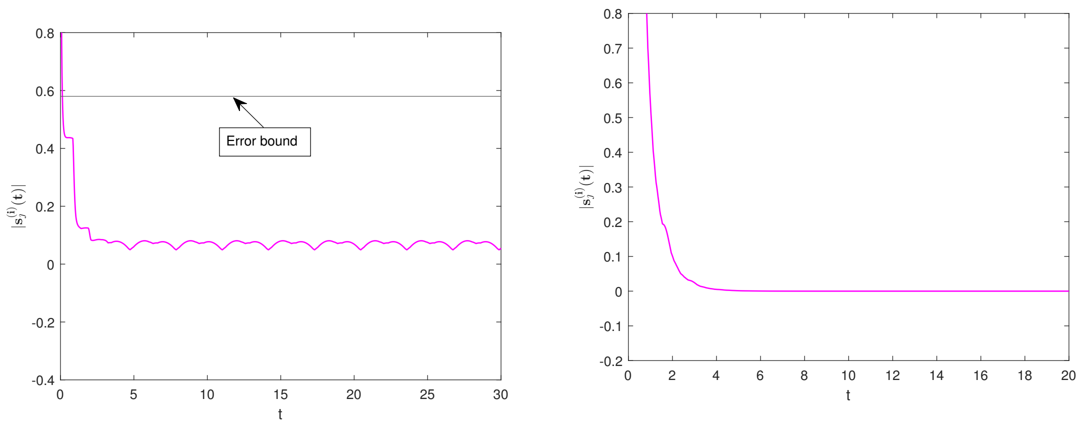

FIGURE 5.4: Plots of stability of QVNNs in (a) Lagrange sense and (b) Lyapunov sense.

and $\mathcal{R}_1 = \frac{\tilde{\kappa}_2}{\beta\kappa_1} = 0.58$ which satisfy all the conditions of Theorem 5.3.1 for $\jmath = 1, 2$

and $i = 0, 1, 2, 3$. Also, the graph of the error system with Lagrange and Lyapunov stabilities is shown through Figure 5.4.

Example 5.4.3. Suppose $n = 7$, then the equation (5.2) becomes OVNN with time-varying delay and multiplication is defined in Table 1.6. For $N = 2$, the parameters of the equation (5.2) are given by

$$
\begin{aligned}
A =& \begin{bmatrix} 1.2 & 2.1 \\ 1.1 & 0.2 \end{bmatrix} + \begin{bmatrix} 1.4 & -1.6 \\ 1.2 & 0.3 \end{bmatrix} \psi_1 + \begin{bmatrix} 1.3 & 2.4 \\ -2.1 & 0.4 \end{bmatrix} \psi_2 + \begin{bmatrix} 1.4 & 2.9 \\ -2.9 & 0.5 \end{bmatrix} \psi_3 \\
&+ \begin{bmatrix} -2 & 1.1 \\ 0.6 & 0.6 \end{bmatrix} \psi_4 + \begin{bmatrix} 0.6 & -3 \\ 1.8 & 0.2 \end{bmatrix} \psi_5 + \begin{bmatrix} -2.1 & -2 \\ 2 & 0.8 \end{bmatrix} \psi_6 + \begin{bmatrix} 1.2 & 1.5 \\ 3.2 & 1.2 \end{bmatrix} \psi_7 \\
=& A^{(0)} + A^{(1)}\psi_1 + A^{(2)}\psi_2 + A^{(3)}\psi_3 + A^{(4)}\psi_4 + A^{(5)}\psi_5 + A^{(6)}\psi_6 + A^{(7)}\psi_7 \\
B =& \begin{bmatrix} 1.7 & 2 \\ 0.8 & 0.2 \end{bmatrix} + \begin{bmatrix} -1.6 & 2.2 \\ -0.9 & -1.3 \end{bmatrix} \psi_1 + \begin{bmatrix} 1.2 & -1.9 \\ 0.2 & 2 \end{bmatrix} \psi_2 + \begin{bmatrix} 2.8 & 1.3 \\ 0.3 & 0.8 \end{bmatrix} \psi_3 \\
&+ \begin{bmatrix} 1.5 & 1.7 \\ 3 & 0.4 \end{bmatrix} \psi_4 + \begin{bmatrix} -0.3 & 2 \\ -1.1 & -0.6 \end{bmatrix} \psi_5 + \begin{bmatrix} 2 & -1.2 \\ 0.4 & 1.8 \end{bmatrix} \psi_6 + \begin{bmatrix} 1.3 & 1.1 \\ 0.2 & 1.3 \end{bmatrix} \psi_7 \\
=& B^{(0)} + B^{(1)}\psi_1 + B^{(2)}\psi_2 + B^{(3)}\psi_3 + B^{(4)}\psi_4 + B^{(5)}\psi_5 + B^{(6)}\psi_6 + B^{(7)}\psi_7
\end{aligned}
$$

$C = \begin{bmatrix} 18 & 0 \\ 0 & 15.8 \end{bmatrix}$, $G = (0.1cost - 0.2sint\psi_1 + 0.2cost\psi_2 - 0.3sint\psi_3 + 0.3cost\psi_4 - 0.4sint\psi_5 + 0.4cost\psi_6 + 0.2sint\psi_7, 0.1sint - 0.2cost\psi_1 + 0.2sint\psi_2 - 0.3cost\psi_3 + 0.3sint\psi_4 - 0.4cost\psi_5 + 0.4sint\psi_6 + 0.2cost\psi_7)^T$, $\sigma(t) = \frac{0.4}{1+e^{-t}}$ with $0 \leq \dot{\sigma}(t) \leq \nu = 0.1$, which have met the Assumption 5.2.4.

The activation functions are given by $h_j(s_j^{(i)}) = g_j(s_j^{(i)}) = tanh(s_j^{(i)})$ and we obtain $l_j = 1$ for $j = 1, 2$ and $i = 0, 1, 2, 3, 4, 5, 6, 7$ with the initial conditions $s_1(0) = -0.4 + 1.3\psi_1 + 1.2\psi_2 + 0.4\psi_3 - 2.2\psi_4 + 1.3\psi_5 + 0.2\psi_6 + 0.8\psi_7$,

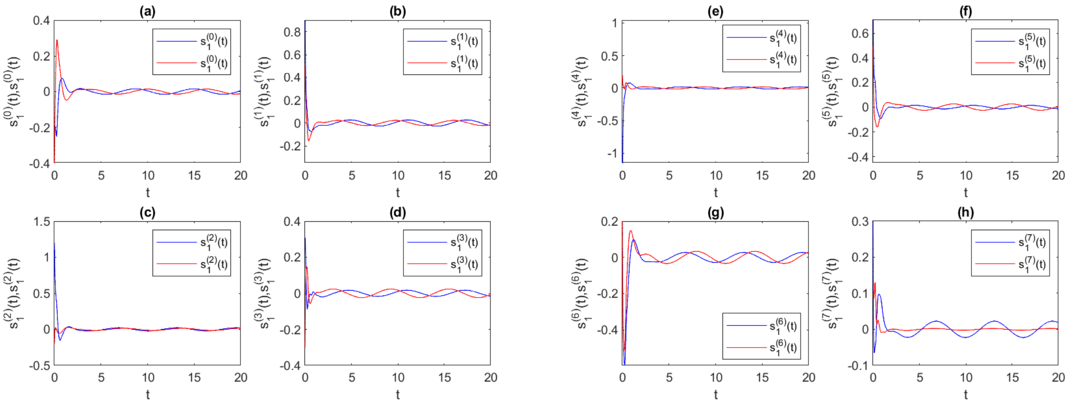

FIGURE 5.5: Plots of trajectories of $s_1^{(i)}(t)$ where $i = 0, 1, 2, 3, 4, 5, 6, 7$, for the system (5.1).

$s_2(0) = -0.3+0.5\psi_1-0.2\psi_2-0.1\psi_3+0.1\psi_4+0.7\psi_5+0.6\psi_6+0.2\psi_7$. In this case, Figure

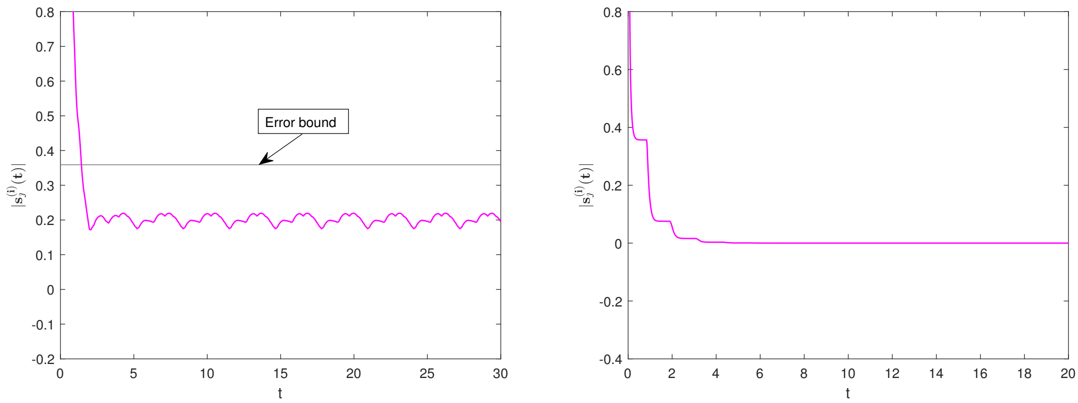

FIGURE 5.6: Plots of stability of OVNNs in (a) Lagrange sense and (b) Lyapunov sense.

5.5 depicts the trajectories of the system (5.1). Now, choosing $\omega_\jmath^{(i)} = 1$, $\beta_\jmath^{(i)} = 1$, we get $\kappa_\jmath^{(0)} = 2.2 > 0$, $\kappa_\jmath^{(1)} = 0.556 > 0$, $\tilde{\kappa}_2 = 0.2$ and $\mathcal{R} = \frac{\tilde{\kappa}_2}{\beta\kappa_1} = 0.359$ which satisfy all the conditions of Theorem 5.3.1 for $\jmath = 1, 2$ and $i = 0, 1, 2, 3, 4, 5, 6, 7$. Also, the graph of the error system with Lagrange and Lyapunov stabilities is shown through in Figure 5.6.

Remark 5.4.1. The LS of RVNNs and CVNNs with constant or time-varying delays had been taken into consideration in [110], [111], [112], [113] and [114], and certain necessary criteria for stability in the Lagrange sense were developed. In [113], the global LS of CVNNs of the neutral type with time-varying delays was investigated by separating real and imaginary components of the activation function. Also, [114] explored the LS of CVNNs with time-varying delays by the LMI approach, and several criteria for determining global stability in the Lagrange sense were derived. The LS of memristive QVNNs with neutral items was covered in [115] and [116] studied the LES of QVNNs with leakage delay and mixed time-varying delays. The global exponential stability of neutral-type OVNNs with time-varying delays was investigated in [109]. However, no findings exist on the LES of OVNN and higher dimensions. As a result, the numerical examples looked at the LES of RVNNs, QVNNs, and OVNNs, which are particular cases of HCNNs, and are generic NNs.

5.5 Conclusion

This chapter explores the Lagrange exponential stability of HCNNs with time-varying delays, marking the first study to address the Lagrange Exponential Stability (LES) criteria for such networks. The research uses the Lyapunov stability theory to investigate the Lagrange global exponential stability for HCNNs with time-varying delays. To tackle hypercomplex numbers' non-commutative and non-associativity, HCNNs are decomposed into $n+1$ equivalent RVNNs, a method inspired by RVNNs research. The framework of the exponential convergence ball is analyzed, and an error bound is derived. Moreover, the Lyapunov sense of exponential stability of HCNNs, a specific case of LES, is explored. The accuracy of the theoretical results is validated through numerical simulations involving CVNNs, QVNNs, and OVNNs.

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