Chapter 5

Variational Approach for Solving Regular N-dimensional Fractional Sturm-Liouville Problems of Order Between $(0, 1]$

In this chapter, we study higher dimensional FSLPs. We presents our problem and give the introduction about FLPs, in Section 5.1. In Section 5.2, we develop the main results and demonstrate the existence of orthogonal solutions to the N-dimensional FSLP. An example is provided in Subsection 5.2.1 to explain the results. Section 5.3 concludes the chapter.

5.1 Introduction

In this chapter, we study the variational approach for solving the regular N-dimensional FSLPs (N-DFSLPs) of order $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in (0, 1]$. Consider the N-DFSLP defined in terms of the fractional version of the gradient operator involving the left and right CFDs with mixed boundary conditions (BCs)

$$-\left({}^C\nabla^{\alpha}_{d^-} \cdot \left(\mathcal{P}(t)\, {}^C\nabla^{\alpha}_{c^+} y\right)\right)(t) + \mathcal{Q}(t)y(t) = \lambda w(t)y(t), \tag{5.1}$$

$$\begin{cases} b^{[i]}_1 y(t)\big|_{t_i=c_i} + b^{[i]}_2 I^{1-\alpha_i}_{d_i^-}\left(\mathcal{P}\ {}^C\partial^{\alpha_i}_{c_i^+} y\right)(t)\big|_{t_i=c_i} & = 0, \quad i = 1, 2, ..., N \\ b^{[i]}_3 y(t)\big|_{t_i=d_i} + b^{[i]}_4 I^{1-\alpha_i}_{d_i^-}\left(\mathcal{P}\ {}^C\partial^{\alpha_i}_{c_i^+} y\right)(t)\big|_{t_i=d_i} & = 0, \quad i = 1, 2, ..., N \end{cases} \tag{5.2}$$

where $t = (t_1, t_2, ..., t_N) \in \Pi^n_{i=1}(c_i, d_i) = \Theta \subset \mathbb{R}^N$. $\mathcal{P}$, $\mathcal{Q}$ and w are continuous functions on Θ and w is weight function. Also, $\mathcal{P}(t) > 0$ and $w(t) > 0\ \forall\ t \in \Theta$, and $\lambda \in \mathbb{R}$ is the eigenvalue for which there exists non-trivial solution of the problem (5.1), and "$\cdot$" represents the scalar product between two vectors in $\mathbb{R}^N$. Here, ${}^C\nabla^{\alpha}_{c^+}$ and ${}^C\nabla^{\alpha}_{d^-}$ denote the left and right Caputo fractional gradient operator of order $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ and defined as,

$${}^C\nabla^{\alpha}_{c^+} = \sum_{j=1}^{N} e_j\, {}^C\partial^{\alpha_j}_{c_j^+} \quad \text{and} \quad {}^C\nabla^{\alpha}_{d^-} = \sum_{j=1}^{N} e_j\, {}^C\partial^{\alpha_j}_{d_j^-}, \quad j = 1, 2, ..., N, \tag{5.3}$$

where ${}^C\partial^{\alpha_j}_{c_j^+}$ and ${}^C\partial^{\alpha_j}_{d_j^-}$ are left and right partial CFDs with respect to variable t_j of order $\alpha_j \in (0, 1]$, e_j, $j = 1, 2..., N$ and $I^{1-\alpha_i}_{d_i^-}$ denote the standard unit vector in the direction of t_j and right Riemann-Liouville (R-L) fractional integral with respect to t_i respectively.

The above problem (5.1-5.2) is a generalization in the fractional order of the integer-order eigenvalue problem in higher dimension. Recently, FSLPs have evolved into the most interesting topic for researchers. The emerging interest of several mathematicians in FSLPs is because of the orthogonal eigenfunctions of FSLP, which are used to solve fractional partial differential equations [83]. Furthermore, using the variational technique, the main properties of the EFs and the EVs of the associated one-dimensional FSLPs have been studied in [50, 39, 51, 99, 98]. Later, in 2021, Ferreira et al. [101], presented the fractional variational approach for solving the higher dimensions FSLP and proved the main result of the existence of EVs and corresponding EFs for fractional order $\mu \in (1/2, 1]$. Moreover, they also showed that the eigenfunctions are orthogonal and the eigenvalues are real and simple.

This chapter discusses a regular higher dimensional FSLPs (5.1) of order $\alpha \in (0, 1]$ under the BCs (5.2). Ferreira et al. [101] discussed the variational method for solving the regular higher dimensional FSLPs represented in terms of fractional gradient operator concerning the left and right RLFDs and established the properties of EVs and EFs for a specific range of fractional order $\alpha \in (1/2, 1]$. In this chapter, we extend the fractional variational technique for N-dimensional FSLPs of fractional order $\alpha \in (0, 1]$ and prove EVs existence and the corresponding EFs form an orthonormal set.

5.2 Existence of Eigenvalues for Regular N-Dimensional Fractional Sturm-Liouville Problem

In this section, we will demonstrate the vital result presented here using fractional variational calculus. We show the existence of EVs of N-dimensional FSLP (5.1-5.2) and corresponding EFs. First, we prove some lemmas which are required to establish the main results.

Lemma 5.1. *Let* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ *with each* $\alpha_j \in (0,1]$ *and function* $r(t) = (r_1(t), r_2(t), ..., r_N(t)) \in C(\Theta)$ *such that*

$$^{C}\partial_{c_j^+}^{1+\alpha_j} \sum_{\substack{i=1 \\ i\neq j}}^{N} I_{c_J^+}^{1+\alpha_j} r_i(t) = 0, \;\; j = 1, 2, ..., N. \tag{5.4}$$

If

$$\int_{\Theta} r(t) \cdot {}^{C}\nabla_{c^+}^{1+\alpha} f(t)\, dt = 0, \tag{5.5}$$

for $f \in C^1(\Theta)$ *such that* ${}^{C}\nabla_{c^+}^{1+\alpha} f(t) \in C(\Theta)$ *and* f *fulfills the BCs*

$$\begin{aligned} f(t)\big|_{t_i=c_i} &= 0, & I_{c_j^+}^{1-\alpha_j} f(t)\big|_{t_j=d_j} &= 0 \\ {}^{C}\partial_{c_j^+}^{\alpha_j} f(t)\big|_{t_j=c_j} &= 0, & {}^{C}\partial_{c_j^+}^{\alpha_j} f(t)\big|_{t_j=d_j} &= 0, \quad i,j = 1, ..., N, \quad i \neq j, \end{aligned} \tag{5.6}$$

then for every $j = 1, 2, ..., N$ *there exist real constants* a_0^j *and* a_1^j*, we have* $r_j(t) = a_0^j + a_1^j t_j$.

Proof. Let us take a function $f(t)$ as pursues

$$f(t) = I_{c^+}^{1+\alpha} \cdot \left(\sum_{i=1}^{N} e_i \left(r_i(t) - a_0^i - a_1^i t_i \right) \right) = \sum_{i=1}^{N} I_{c_i^+}^{1+\alpha_i} \left(r_i(t) - a_0^i - a_1^i t_i \right),$$

with fixed by the $2N$ conditions

$$I_{c_j^+}^{2} \left(r_j(t) - a_0^j - a_1^j t_j \right) \big|_{t_j=d_j} \qquad = 0, \tag{5.7}$$

$$I_{c_j^+}^{1} \left(r_j(t) - a_0^j - a_1^j t_j \right) \big|_{t_j=d_j} \qquad = 0, \tag{5.8}$$

$$I_{c_j^+}^{1-\alpha_j} I_{c_i^+}^{1+\alpha_i} \left(r_i(t) - a_0^i - a_1^i t_i \right) \big|_{t_j=d_j} \qquad = 0, \quad i = 1, 2, ..., N \;\; \text{and} \;\; i \neq j, \tag{5.9}$$

$$ {}^{C}\partial^{\alpha_j}_{c_j^+} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right)\big|_{t_j=d_j} \quad = 0, \quad i = 1,2,...,N \text{ and } i \neq j. \tag{5.10} $$

We observe that function $f(t)$ is continuous and satisfies the BCs (5.6). Now, using the definitions of FDs and integrals, we have for $i,j = 1,...,N$,

$$ \begin{aligned} f(t)\big|_{t_i=c_i} &= \sum_{i=1}^{N} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right)\big|_{t_i=c_i} = 0, \\ {}^{C}\partial^{\alpha_j}_{c_j^+} f(t)\big|_{t_j=c_j} &= \sum_{i=1}^{N} {}^{C}\partial^{\alpha_j}_{c_j^+} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right)\big|_{t_i=c_i} = 0. \end{aligned} $$

Now, from Eqs. (5.7-5.10), we have

$$ \begin{aligned} I^{1-\alpha_j}_{c_j^+} f(t)\big|_{t_j=d_j} &= \sum_{\substack{i=1 \\ i\neq j}}^{N} I^{1-\alpha_j}_{c_j^+} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right)\big|_{t_j=d_j} \\ &\quad + I^{2}_{c_j^+} \left(r_j(t) - a_0^j - a_1^j t_j\right)\big|_{t_j=d_j} = 0, \\ {}^{C}\partial^{\alpha_j}_{c_j^+} f(t)\big|_{t_j=d_j} &= \sum_{\substack{i=1 \\ i\neq j}}^{N} {}^{C}\partial^{\alpha_j}_{c_j^+} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right)\big|_{t_j=d_j} \\ &\quad + I^{1}_{c_j^+} \left(r_j(t) - a_0^j - a_1^j t_j\right)\big|_{t_j=d_j} = 0. \end{aligned} $$

In addition, for $j = 1,2,...,N$

$$ \begin{aligned} \partial_{t_j} f(t) &= \sum_{\substack{i=1 \\ i\neq j}}^{N} \partial_{t_j} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right) + I^{\alpha_j}_{c_j^+} \left(r_j(t) - a_0^j - a_1^j t_j\right) \\ &= \sum_{\substack{i=1 \\ i\neq j}}^{N} I^{1+\alpha_i}_{c_i^+} \partial_{t_j} \left(r_i(t)\right) + I^{\alpha_j}_{c_j^+} \left(r_j(t) - a_0^j - a_1^j t_j\right) \in C(\Theta). \end{aligned} $$

Moreover, using the Eqs. (1.5) and (5.4) we have

$$ {}^{C}\nabla^{1+\alpha}_{c^+} f(t) = \sum_{j=1}^{N} e_j {}^{C}\partial^{1+\alpha_j}_{c_j^+} f(t) \left(\sum_{i=1}^{N} I^{1+\alpha_i}_{c_i^+} \left(r_i(t) - a_0^i - a_1^i t_i\right) \right) $$

$$
\begin{aligned}
&= \sum_{j=1}^{N} e_j \sum_{\substack{i=1\\ i\neq j}}^{N} {}^{C}\partial_{c_j^+}^{1+\alpha_j} I_{c_i^+}^{1+\alpha_i} \left(r_i(t) - a_0^i - a_1^i t_i\right) + \sum_{j=1}^{N} e_j \left(r_j(t) - a_0^j - a_1^j t_j\right) \\
&= \sum_{j=1}^{N} e_j {}^{C}\partial_{c_j^+}^{1+\alpha_j} \sum_{\substack{i=1\\ i\neq j}}^{N} I_{c_i^+}^{1+\alpha_i} r_i(t) + \sum_{j=1}^{N} e_j \left(r_j(t) - a_0^j - a_1^j t_j\right) \\
&= \sum_{j=1}^{N} e_j \left(r_j(t) - a_0^j - a_1^j t_j\right) \in C(\Theta).
\end{aligned}
\tag{5.11}
$$

Now from Eqs. (5.5) and (5.6), we obtain for $\widetilde{\Theta} = \Pi_{\substack{i=1\\ i\neq j}}^{N}(c_i, d_i)$

$$
\begin{aligned}
&\int_{\Theta} \left(\sum_{j=1}^{N} e_j \left(r_j(t) - a_0^j - a_1^j t_j\right) \right) \cdot {}^{C}\nabla_{c^+}^{1+\alpha} f(t)\, dt \\
&= \int_{\Theta} r(t) \cdot {}^{C}\nabla_{c^+}^{1+\alpha} f(t)\, dt - \int_{\Theta} \left(\sum_{j=1}^{N} e_j \left(a_0^j + a_1^j t_j\right) \right) \cdot {}^{C}\nabla_{c^+}^{1+\alpha} f(t)\, dt \\
&= -\sum_{j=1}^{N} \int_{\widetilde{\Theta}} \int_{c_j}^{d_j} \left(a_0^j + a_1^j t_j\right) \partial_{t_j} {}^{C}\partial_{c_j^+}^{\alpha_j} f(t)\, dt_j\, d\widetilde{t} \\
&= -\sum_{j=1}^{N} \int_{\widetilde{\Theta}} \left[\left(a_0^j + a_1^j t_j\right) {}^{C}\partial_{c_j^+}^{\alpha_j} f(t) \Big|_{t_j=c_j}^{t_j=d_j} - a_1^j \int_{c_j}^{d_j} \partial_{c_j^+}^{\alpha_j} f(t) dt_j \right] d\widetilde{t} \\
&= 0.
\end{aligned}
\tag{5.12}
$$

Here $\widetilde{t} = t_1, ..., t_{i-1}, t_{i+1}, ..., t_N$. Thus, from the Eqs. (5.11) and (5.12) we get

$$
\begin{aligned}
0 &= \int_{\Theta} \left(\sum_{j=1}^{N} e_j \left(r_j(t) - a_0^j - a_1^j t_i\right) \right) \cdot {}^{C}\nabla_{c^+}^{1+\alpha} f(t)\, dt \\
&= \int_{\Theta} \left(\sum_{j=1}^{N} \left(r_j(t) - a_0^j - a_1^j t_i\right) \right)^2 dt.
\end{aligned}
\tag{5.13}
$$

Eq. (5.13) implies that for every $j = 1, 2, ..., N$, we have

$$
r_j(t) = a_0^j + a_1^j t_j.
$$

□

Lemma 5.2. *Let* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ *with each* $\alpha_j \in (0, 1]$, *and* $r_1(t)$ *is a continuous scalar function in* Θ. *Let* $r_2(t) = (r_1^{(2)}, r_2^{(2)}, ..., r_N^{(2)})$, *for every* $j = 1, ..., N$ *and* $p > 1$, $r_j^{(2)} = I^2_{d_j^-}\psi_j \in L^p(\Theta)$ *with* $\psi_j \in L^p(\Theta)$. *If*

$$\int_{\Theta} \left(r_2(t) \cdot {}^{C}\nabla^{\alpha}_{c^+} f(t) + r_1(t) f(t) \right) \, dt = 0,$$

for every $f \in C^1(\Theta)$ *such that* $\partial^2_{t_j} f \in L^2(\Theta), j = 1, ..., N$, *and* ${}^{C}\nabla^{1+\alpha}_{c^+} f(t) \in C(\Theta)$ *satisfying the BCs (5.6) we have*

$${}^{C}\nabla^{\alpha}_{d^-} f(t) \cdot \psi(t) + r_1(t) = 0.$$

Proof. Since, $r_j^{(2)} \in I^2_{d_j^-}(L^p)$ for $j = 1, ..., N$ and $p > 1$, we have

$$\begin{aligned} \int_{\Theta} \left(r_2(t) \cdot {}^{C}\nabla^{\alpha}_{c^+} f(t) \right) \, dt &= \int_{\Theta} \sum_{j=1}^{N} r_j^{(2)} \, {}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \, dt \\ &= \sum_{j=1}^{N} \int_{\widetilde{\Theta}} \int_{c_j}^{d_j} I^2_{d_j^-}\psi_j \, {}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \, dt_j \, d\widetilde{t}. \end{aligned}$$

Now using the Eq. (1.11), we obtain

$$\begin{aligned} \int_{\Theta} \left(r_2(t) \cdot {}^{C}\nabla^{\alpha}_{c^+} f(t) \right) \, dt &= \sum_{j=1}^{N} \int_{\widetilde{\Theta}} \int_{c_j}^{d_j} I^1_{d_j^-}\psi_j \, I^1_{c_j^+} {}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \, dt_j \, d\widetilde{t} \\ &= \sum_{j=1}^{N} \int_{\widetilde{\Theta}} \int_{c_j}^{d_j} I^1_{d_j^-}\psi_j \, {}^{C}\partial^{1+\alpha_j}_{c_j^+} f(t) \, dt_j \, d\widetilde{t} \\ &= \int_{\Theta} I^1_{d^-}\psi(t) \cdot {}^{C}\partial^{1+\alpha}_{c^+} f(t) \, dt, \end{aligned} \tag{5.14}$$

where, $I^1_{d^-}\psi(t) = \sum\limits_{j=1}^{N} e_j I^1_{d_j^-}\psi_j$. Meanwhile, we have

$$\int_\Theta r_1(t) f(t)\,dt = \frac{1}{N}\int_\Theta r_1(t)\sum_{j=1}^{N} I^{\alpha_j}_{c_j^+}{}^C\partial^{\alpha_j}_{c_j^+} f(t)\,dt$$

Now, from the Eqs. (1.11), (5.6) and using the formula integration by parts, we obtain

$$\begin{aligned}\int_\Theta r_1(t) f(t)\,dt &= \frac{1}{N}\sum_{j=1}^{N}\int_{\widetilde{\Theta}}\int_{c_j}^{d_j} I^{\alpha_j}_{d_j^-} r_1(t)\,{}^C\partial^{\alpha_j}_{c_j^+} f(t)\,dt_j\,d\widetilde{t}\\ &= \frac{-1}{N}\sum_{j=1}^{N}\int_{\widetilde{\Theta}}\int_{c_j}^{d_j} I^{1+\alpha_j}_{d_j^-} r_1(t)\,{}^C\partial^{1+\alpha_j}_{c_j^+} f(t)\,dt_j\,d\widetilde{t}\\ &= \frac{-1}{N}\int_\Theta I^{1+\alpha}_{d^-} r_1(t)\cdot\ {}^C\nabla^{\alpha}_{c^+} f(t)\,dt. \end{aligned} \tag{5.15}$$

Since $r_1(t)$ and $\psi(t)$ are continuous in Θ and

$$I^1_{d^-}\psi(t) - \frac{1}{N} I^{1+\alpha}_{d^-} r_1(t) \in C(\Theta) \subset L^2(\Theta).$$

Now from Lemma (5.1) there exist real constants a_0^j and a_1^j for $j = 1, ..., N$ such that

$$I^1_{d^-}\psi(t) - \frac{1}{N} I^{1+\alpha}_{d^-} r_1(t) = \sum_{j=1}^{N} e_j (a_0^j + a_1^j t_j). \tag{5.16}$$

Applying both side operator ${}^C\nabla^{1+\alpha}_{d^-}$ in Eq. (5.16) and using the Eq. (1.5), we have

$$\begin{aligned} &{}^C\nabla^{1+\alpha}_{d^-}\cdot\left(I^1_{d^-}\psi(t) - \frac{1}{N} I^{1+\alpha}_{d^-} r_1(t)\right) = {}^C\nabla^{1+\alpha}_{d^-}\cdot\left(\sum_{j=1}^{N} e_j (a_0^j + a_1^j t_j)\right)\\ \Longleftrightarrow\ &\sum_{j=1}^{N} {}^C\partial^{1+\alpha_j}_{d_j^-} I^1_{d_j^-}\psi_j(t) + r_1(t) = \sum_{j=1}^{N} {}^C\partial^{1+\alpha_j}_{d_j^-}(a_0^j + a_1^j t_j) \end{aligned}$$

$$\Longleftrightarrow \; {}^{C}\nabla^{\alpha}_{d-} f(t) \cdot \psi(t) + r_1(t) = 0.$$

□

Theorem 5.3. *If* $\alpha = (\alpha_1, ..., \alpha_N) \in (0,1]$ *and* $\mathcal{P}(t)$ *is continuously differentiable in domain* Θ *and* $\sqrt{w(t)}$ *satisfies the Holder inequality, then the FSLP (5.1-5.2), has an countably infinite monotonically increasing sequence of EVs* $\lambda^{(1)}, \lambda^{(2)}, ...,$ *and to every eigenvalue (EV)* $\lambda^{(N)}$ *there correspond unique (up to a constant factor) eigenfunction (EF)* $y^{(N)}$*. In addition, the set of EFs* $y^{(N)}$ *forms an orthogonal set.*

Proof. The steps of the proof is on the pattern of [39, 101]. The proof consists of the following steps.

Step 1. The variational formulation of Eq. (5.1) can be observed as the minimization problem of the quadratic functional,

$$J(y) = \int_{\Theta} \left[\mathcal{P}(t)\left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) \cdot \left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) + \mathcal{Q}(t) y^2(t)\right] dt, \tag{5.17}$$

with respect to the BCs (5.2) and the integral constraint

$$F(y) = \int_{\Theta} w(t) y^2(t)\, dt = 1. \tag{5.18}$$

Since, $w(t) > 0$, $\mathcal{P}(t) > 0$ and $\mathcal{P}(t)\left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) \cdot \left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) > 0$, we will demonstrate that functional $J(y)$ is bounded below.

$$\begin{aligned} J(y) &= \int_{\Theta} \left[\mathcal{P}(t)\left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) \cdot \left({}^{C}\nabla^{\alpha}_{c+} y(t)\right) + \mathcal{Q}(t) y^2(t)\right] dt \\ &\geq \int_{\Theta} \mathcal{Q}(t) y^2(t)\, dt \\ &\geq \min_{t \in \Theta} \frac{\mathcal{Q}(t)}{w(t)} \int_{\Theta} w(t) y^2(t)\, dt = \min_{t \in \Theta} \frac{\mathcal{Q}(t)}{w(t)} = M > -\infty. \end{aligned} \tag{5.19}$$

Now, we approximate the solution of Eqs. (5.2), (5.17) and (5.18) by using Rayleigh-Ritz method. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in $L^2(\Theta)$ and it satisfies the BCs (5.2). Using Rayleigh-Ritz method, we approximate the solution in form of orthonormal basis function

$$y_k(t) = \frac{1}{\sqrt{w(t)}} \sum_{i=1}^{k} \rho_i \xi_i(t), \tag{5.20}$$

where $\rho_i \in \mathbb{R}$, $\forall i = 1, ..., k$, and $y_k(t)$ satisfies the BCs (5.2). Now, putting Eq. (5.20) into Eqs. (5.17) and (5.18), and we obtain the problem of minimizing the functional

$$\begin{aligned}\widehat{J}(\rho_1, \rho_2, ..., \rho_k) = \widehat{J}([\rho]) = \int_\Theta \Bigg[\mathcal{P}(t) \sum_{j=1}^{N} \left(\sum_{i=1}^{k} \rho_k {}^C\partial^{\alpha_j}_{c_j^+} \left(\frac{\xi_i(t)}{\sqrt{w(t)}} \right) \right)^2 \\ + \frac{\mathcal{Q}(t)}{w(t)} \left(\sum_{i=1}^{k} \rho_i \xi_i(t) \right)^2 \Bigg] \, dt, \end{aligned} \tag{5.21}$$

subject to the integral constraint

$$\widehat{I}(\rho_1, \rho_2, ..., \rho_k) = \widehat{I}([\rho]) = \int_\Theta \left(\sum_{i=1}^{k} \rho_i \xi_i(t) \right)^2 dt = \sum_{i=1}^{k} {\rho_i}^2 \int_\Theta \xi_i^2(t) \, dt = \sum_{i=1}^{k} {\rho_i}^2 = 1. \tag{5.22}$$

Now the problem is to minimize the functional $\widehat{J}([\rho])$ on the surface of $k-$dimensional unit sphere with Eq. (5.22). Due to $\widehat{J}[\rho]$ being continuous and Eq. (5.22) is compact, so $\widehat{J}([\rho])$ reaches its minimum, say $\lambda_k^{(1)}$ at some point $\rho^{(1)} = (\rho_1^{(1)}, ..., \rho_k^{(1)})$. If we continue this procedure for $k = 1, 2, 3, ...$, we get a sequence of real numbers $\lambda_1^{(1)}, \lambda_2^{(1)}, ...$ Since we have the relation $\lambda_{k+1}^{(1)} \leq \lambda_k^{(1)}$ and the functional $J(y)$ is bounded from below, we can obtain the limit

$$\lim_{k\to\infty} \lambda_k^{(1)} = \lambda^{(1)}.$$

Step 2. In this step we will show that $\{y_k^{(1)}\}_{k\in\mathbb{N}}$ has a subsequence $\{y_{k_l}\}_{l\in\mathbb{N}}$ which is uniformly convergent. Let

$$y_k^{(1)}(t) = \frac{1}{\sqrt{w(t)}} \sum_{i=1}^{k} \rho_i^{(1)} \xi_i(t), \tag{5.23}$$

be the linear combination (5.20) and attaining the minimum $\lambda_k^{(1)}$. For simplicity, we write y_k rather than $y_k^{(1)}$. Since, the sequence $\lambda_k^{(1)}$ given by

$$\lambda_k^{(1)} = \int_\Theta \left[\mathcal{P}(t)\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\cdot\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right) + \mathcal{Q}(t) y_k^2(t)\right] dt,$$

is convergent, then it is bounded and $\exists$ constant C_0 such that $\forall\, k \in \mathbb{N}$,

$$\int_\Theta \left[\mathcal{P}(t)\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\cdot\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right) + \mathcal{Q}(t) y_k^2(t)\right] dt \le C_0.$$

So, $\forall\, k \in \mathbb{N}$, and from Eq. (5.18), it holds

$$\begin{aligned} 0 &\le \int_\Theta \left[\mathcal{P}(t)\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\cdot\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\right] dt \\ &\le C_0 + \left|\int_\Theta \mathcal{Q}(t) y_k^2(t)\, dt\right| \\ &\le C_0 + \max_{t\in\Theta}\left|\frac{\mathcal{Q}(t)}{w(t)}\right| \int_\Theta w(t)(y_k(t))^2\, dt \le C_1. \end{aligned}$$

In addition, since $\mathcal{P}(t) > 0$, we have

$$\min_{t\in\Theta} \mathcal{P}(t) \int_\Theta \left[\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\cdot\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\right] dt \le \int_\Theta \left[\mathcal{P}(t)\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\cdot\left({}^C\nabla_{c^+}^{\alpha} y_k(t)\right)\right] dt$$

$$\leq C_1,$$

which give us

$$\int_{\Theta}\left[\left({}^{C}\nabla^{\alpha}_{c^+}y_k(t)\right)\cdot\left({}^{C}\nabla^{\alpha}_{c^+}y_k(t)\right)\right]\,dt \leq \frac{C_1}{\min_{t\in\Theta}\mathcal{P}(t)} \leq C_2. \tag{5.24}$$

Now using the property that $I^{\alpha}_{c^+}\cdot{}^{C}\nabla^{\alpha}_{c^+}y_k(t) = Ny_k(t)$, we get the following

$$\begin{aligned}\|Ny_k(t)\|_{L^2(\Theta)} &= \left\|I^{\alpha}_{c^+}\cdot{}^{C}\nabla^{\alpha}_{c^+}y_k(t)\right\|_{L^2(\Theta)}\\ &= \sum_{j=1}^{N}\frac{1}{(\Gamma\alpha_j)}\left\|\int_{c_j}^{t_j}(t_j-x)^{\alpha_j-1}\left({}^{C}\partial^{\alpha_j}_{c_j^+}y_k\right)(\tilde{t})\,dx\right\|_{L^2}\\ &= \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha_j)}\left\|t_j^{\alpha_j-1}*\left({}^{C}\partial^{\alpha_j}_{c_j^+}y_k(t)\right)\right\|_{L^2},\end{aligned}$$

where $\tilde{t}=(t_1,...,t_{j-1},x,t_{j+1},...,t_N)$ and "$*$" denotes the convolution of the integral. Now, using Eq. (5.24) and Young's inequality [[110], Theorem 3.1], we get

$$\begin{aligned}\|Ny_k(t)\|_{L^2(\Theta)} &\leq \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha_j)}\left\|t_j^{\alpha_j-1}\right\|_{L^1}\left\|\left({}^{C}\partial^{\alpha_j}_{c_j^+}y_k(t)\right)\right\|_{L^2}\\ &\leq \sum_{j=1}^{N}\frac{\sqrt{C_2}}{\Gamma(\alpha_j+1)}(d_j^{\alpha_j}-c_j^{\alpha_j}).\end{aligned} \tag{5.25}$$

Hence from Eq. (5.25), we can say that sequence $(y_k(t))_{k\in\mathbb{N}}$ is uniformly bounded. We have, for $j=1,...,N$, $c_j<t_1^{(j)}<t_2^{(j)}\leq d_j$

$$\begin{aligned}|Ny_k(x_2)-Ny_k(x_1)| &= \left|I^{\alpha}_{c^+}\cdot{}^{C}\nabla^{\alpha}_{c^+}y_k(x_2)-I^{\alpha}_{c^+}\cdot{}^{C}\nabla^{\alpha}_{c^+}y_k(x_1)\right|\\ &\leq \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha_j)}\left|\int_{c_j}^{t_2^{(j)}}\left(t_2^{(j)}-x\right)^{\alpha_j-1}\left({}^{C}\partial^{\alpha_j}_{c_j^+}y_k\right)(\tilde{t_2})\,dx\right.\\ &\quad\left.-\int_{c_j}^{t_1^{(j)}}\left(t_1^{(j)}-x\right)^{\alpha_j-1}\left({}^{C}\partial^{\alpha_j}_{c_j^+}y_k\right)(\tilde{t_1})\,dx\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{N} \frac{1}{\Gamma(\alpha_j)} \left[\sup_{\tilde{t}_2 \in \Theta} \left| {}^{C}\partial_{c_j^+}^{\alpha_j} y_k(\tilde{t}_2) \right| \int_{c_j}^{t_2^{(j)}} \left| \left(t_2^{(j)} - x \right)^{\alpha_j - 1} \right| dx \right. \\
&\left. + \sup_{\tilde{t}_1 \in \Theta} \left| {}^{C}\partial_{c_j^+}^{\alpha_j} y_k(\tilde{t}_1) \right| \int_{c_j}^{t_1^{(j)}} \left| \left(t_1^{(j)} - x \right)^{\alpha_j - 1} \right| dx \right] \\
&= \sum_{j=1}^{N} \frac{1}{\Gamma(\alpha_j + 1)} \left[C_j^{'} (t_2^{(j)} - c_j)^{\alpha_j} + C_j^{''} (t_1^{j} - c_j)^{\alpha_j} \right] \\
&\leq \sum_{j=1}^{N} \frac{(C_j^{'} + C_j^{''})}{\Gamma(\alpha_j + 1)} (d_j - c_j)^{\alpha_j}, \qquad (5.26)
\end{aligned}
$$

where $\sup_{\tilde{t_2} \in \Theta} \left| {}^{C}\partial_{c_j^+}^{\alpha_j} y_k(\tilde{t_2}) \right| = C_j^{'}$ and $\sup_{\tilde{t_1} \in \Theta} \left| {}^{C}\partial_{c_j^+}^{\alpha_j} y_k(\tilde{t_1}) \right| = C_j^{''}$. Hence from Eq. (5.26), $y_k(t)$ is equicontinuous. So, by the Arzala-Ascoli theorem there exist a uniformly convergent subsequence $\{y_{k_l}\}_{l \in \mathbb{N}}$ such that

$$\lim_{l \to \infty} y_{k_l} = y^{(1)}.$$

Step 3. In this step, we will demonstrate that $y^{(1)}$ is the solution of the problem. From the Lagrange multipliers rule at $[\rho] = [\tilde{\rho}]$, we have

$$\frac{\partial}{\partial \rho_i} \left[\widehat{J}_k[\tilde{\rho}] - \lambda_k^{(1)} \widehat{I}_k[\tilde{\rho}] \right]_{\tilde{\rho} = \tilde{\rho}^{(1)}} = 0, \quad i = 1, ..., k. \qquad (5.27)$$

Multiplying Eq. (5.27) by an arbitrary constants A_i and summing over i from 1 (1) k, we get

$$\sum_{i=1}^{k} A_i \frac{\partial}{\partial \rho_i} \left[\widehat{J}_k[\tilde{\rho}] - \lambda_k^{(1)} \widehat{I}_k[\tilde{\rho}] \right]_{\tilde{\rho} = \tilde{\rho}^{(1)}} = 0. \qquad (5.28)$$

By considering

$$f_k(t) = \frac{1}{\sqrt{w(t)}} \sum_{i=1}^{k} A_i \xi_i(t),$$

Eq. (5.28) can be rewritten as

$$\int_{\Theta} \left[\mathcal{P}(t)\left({}^{C}\nabla^{\alpha}_{c^+} y_k(t)\right)\cdot\left({}^{C}\nabla^{\alpha}_{c^+} f_k(t)\right) + y_k(t) f_k(t)\left(\mathcal{Q}(t) - \lambda_k^{(1)} w(t)\right)\right] dt = 0. \tag{5.29}$$

Now, using the properties of fractional differentiation and integrating by parts the first term of Eq. (5.29), we obtain

$$\begin{aligned}
\int_{\Theta} \mathcal{P}(t)\left({}^{C}\nabla^{\alpha}_{c^+} y_k(t)\right)\cdot\left({}^{C}\nabla^{\alpha}_{c^+} f_k(t)\right) dt &= \sum_{j=1}^{N} \int_{\tilde{\Theta}} \int_{c_j}^{d_j} \mathcal{P}(t)\,\partial_{t_j} I^{1-\alpha_j}_{c_j^+} y_k(t)\, {}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t)\, dt_j\, d\tilde{t} \\
&= \sum_{j=1}^{N} \Bigg\{ \int_{\tilde{\Theta}} \left[\mathcal{P}(t)\, {}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) I^{1-\alpha_j}_{c_j^+} y_k(t)\right]_{t=c_i}^{t=d_i} d\tilde{t} \\
&\quad - \int_{\tilde{\Theta}} \int_{c_j}^{d_j} \Big[\partial_{t_j}(\mathcal{P}(t))\, {}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) \\
&\quad + \mathcal{P}(t)\partial_{t_j}\left({}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t)\right)\Big]\, I^{1-\alpha_j}_{c_j^+} y_k(t)\, dt_j\, d\tilde{t} \Bigg\},
\end{aligned} \tag{5.30}$$

where, $d\tilde{t} = dt_1 ... dt_{j-1} dt_{j+1} ... dt_N$ and $\widetilde{\Theta} = \Pi_{\substack{i=1 \\ i \neq j}}^{N}(c_j, d_j)$. Using Eq. (5.30), we can rewrite (5.29) in following manner

$$\begin{aligned}
0 &= -\sum_{j=1}^{N} \int_{\tilde{\Theta}} \int_{c_j}^{d_j} \left[\partial_{t_j}(\mathcal{P}(t))\, {}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) + \mathcal{P}(t)\partial_{t_j}\left({}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t)\right)\right] I^{1-\alpha_j}_{c_j^+} y_k(t)\, dt_j\, d\tilde{t} \\
&\quad + \sum_{j=1}^{N} \int_{\tilde{\Theta}} \left[\mathcal{P}(t)\, {}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) I^{1-\alpha_j}_{c_j^+} y_k(t)\right]_{t=c_i}^{t=d_i} d\tilde{t} + \int_{\Theta} y_k(t) f_k(t)\left(\mathcal{Q}(t) - \lambda_k^{(1)} w(t)\right) dt \\
&= I_k.
\end{aligned} \tag{5.31}$$

For a convergent subsequence $\{y_{k_l}\}_{l\in\mathbb{N}}$, we can obtain in the limit the relation

$$0 = -\sum_{j=1}^{N} \int_{\tilde{\Theta}} \int_{c_j}^{d_j} \left[\partial_{t_j}(\mathcal{P}(t))\, {}^{C}\partial^{\alpha_j}_{c_j^+} f(t) + \mathcal{P}(t)\partial_{t_j}\left({}^{C}\partial^{\alpha_j}_{c_j^+} f(t)\right)\right] I^{1-\alpha_j}_{c_j^+} y^{(1)}(t)\, dt_j\, d\tilde{t}$$

$$
\begin{aligned}
&+\sum_{j=1}^{N}\int_{\widetilde{\Theta}}\left[\mathcal{P}(t)\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right]_{t=c_i}^{t=d_i}d\tilde{t}+\int_{\Theta}y^{(1)}(t)f(t)\left(\mathcal{Q}(t)-\lambda^{(1)}w(t)\right)\,dt\\
&=I. \qquad (5.32)
\end{aligned}
$$

Now, we investigate the convergence of integrals (5.31) and (5.32) directly

$$
\begin{aligned}
|I_k-I|\le{}&\sum_{j=1}^{N}\int_{\Theta}\left|-\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)I^{1-\alpha_j}_{c_j^+}y_k(t)+\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right|\,dt\\
&+\sum_{j=1}^{N}\int_{\Theta}\left|-\mathcal{P}(t)\,\partial_{t_j}\left({}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)\right)I^{1-\alpha_j}_{c_j^+}y_k(t)+\mathcal{P}(t)\partial_{t_j}\left({}^{C}\partial^{\alpha_j}_{c_j^+}f(t)\right)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right|\,dt\\
&+\sum_{j=1}^{N}\int_{\widetilde{\Theta}}\left|\left[\mathcal{P}(t)\,{}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)I^{1-\alpha_j}_{c_j^+}y_k(t)\right]_{t=d_i}-\left[\mathcal{P}(t)\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right]_{t=d_i}\right|\,d\tilde{t}\\
&+\sum_{j=1}^{N}\int_{\widetilde{\Theta}}\left|\left[\mathcal{P}(t)\,{}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)I^{1-\alpha_j}_{c_j^+}y_k(t)\right]_{t=c_i}-\left[\mathcal{P}(t)\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right]_{t=c_i}\right|\,d\tilde{t}\\
&+\int_{\Theta}\left|y_k(t)f_k(t)\left(\mathcal{Q}(t)-\lambda_k^{(1)}w(t)\right)-y^{(1)}(t)f(t)\left(\mathcal{Q}(t)-\lambda^{(1)}w(t)\right)\right|\,dt.
\end{aligned}
\tag{5.33}
$$

For the first term in Eq. (5.33), we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_{\Theta}\left|-\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)I^{1-\alpha_j}_{c_j^+}y_k(t)+\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\right|\,dt\\
&=\sum_{j=1}^{N}\int_{\Theta}\Big|-\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f_k(t)I^{1-\alpha_j}_{c_j^+}y_k(t)+\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y_k(t)\\
&\quad-\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y_k(t)+\partial_{t_j}(\mathcal{P}(t))\,{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)I^{1-\alpha_j}_{c_j^+}y^{(1)}(t)\Big|\,dt\\
&\le\sum_{j=1}^{N}\left\|\partial_{t_j}(\mathcal{P}(t))\right\|\Bigg(\left\|I^{1-\alpha_j}_{c_j^+}y_k(t)\right\|_{L^2}\left\|{}^{C}\partial^{\alpha_j}_{c_j^+}\left(f(t)-f_k(t)\right)\right\|_{L^2}\\
&\quad+\left\|{}^{C}\partial^{\alpha_j}_{c_j^+}f(t)\right\|_{L^2}\left\|I^{1-\alpha_j}_{c_j^+}\left(y_k(t)-y^{(1)}(t)\right)\right\|_{L^2}\Bigg)\\
&\le\sum_{j=1}^{N}\left\|\partial_{t_j}(\mathcal{P}(t))\right\|\Bigg(K_{1-\alpha_j}M_1\left\|{}^{C}\partial^{\alpha_j}_{c_j^+}\left(f(t)-f_k(t)\right)\right\|_{L^2}
\end{aligned}
$$

$$+\left\| {}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \right\|_{L^2} \left\| I^{1-\alpha_j}_{c_j^+}\left(y_k(t) - y^{(1)}(t)\right) \right\|_{L^2} \Bigg),$$

where, $M_1 = \sup_{\substack{k\in\mathbb{N}\\ t\in\Theta}} \|y_k(t)\|$ and $K_{1-\alpha_j} = \frac{(d_j - c_j)^{1-\alpha_j}}{\Gamma(-\alpha_j)}$.

Continuing in the same way, we get the second term

$$\begin{aligned}
&\sum_{j=1}^{N} \int_{\Theta} \left| -\mathcal{P}(t)\, \partial_{t_j} \left({}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) \right) I^{1-\alpha_j}_{c_j^+} y_k(t) + \mathcal{P}(t) \partial_{t_j} \left({}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \right) I^{1-\alpha_j}_{c_j^+} y^{(1)}(t) \right| dt \\
&\le \sum_{j=1}^{N} \|(\mathcal{P}(t))\| \Bigg(K_{1-\alpha_j} M_1 \left\| \partial_{t_j} \left({}^{C}\partial^{\alpha_j}_{c_j^+} \left(f(t) - f_k(t)\right) \right) \right\|_{L^2} \\
&\quad + \left\| \partial_{t_j} \left({}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \right) \right\|_{L^2} \left\| I^{1-\alpha_j}_{c_j^+} \left(y_k(t) - y^{(1)}(t) \right) \right\|_{L^2} \Bigg).
\end{aligned}$$

For the third and fourth terms, we have, for $j = 1, ..., N$ $I^{1-\alpha_j}_{c_j^+} y_k(a) \to I^{1-\alpha_j}_{c_j^+} y^{(1)}(a)$, $I^{1-\alpha_j}_{c_j^+} y_k(b) \to I^{1-\alpha_j}_{c_j^+} y^{(1)}(b)$, and $I^{1-\alpha_j}_{c_j^+} f_k(t) \to I^{1-\alpha_j}_{c_j^+} f(t)$, because the sequence $\left\|y_k - y^{(1)}\right\| \to 0$ and $\|f_k - f\| \to 0$ converges as $k \to \infty$. Further, we have

$$\begin{aligned}
I^{1-\alpha_j}_{c_j^+} f_k(t) \to I^{1-\alpha_j}_{c_j^+} f(t) &\implies \lim_{k\to\infty} \left\| {}^{C}\partial^{\alpha_j}_{c_j^+} \left(f(t) - f_k(t)\right) \right\|_{L^2} = 0 \\
&\implies \lim_{k\to\infty} \left\| \partial_{t_j} \left({}^{C}\partial^{\alpha_j}_{c_j^+} \left(f(t) - f_k(t)\right) \right) \right\|_{L^2} = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_{t_j} \left[{}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) \right]_{t_j = c_j} &\to \partial_{t_j} \left[{}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \right]_{t_j = c_j}, \\
\partial_{t_j} \left[{}^{C}\partial^{\alpha_j}_{c_j^+} f_k(t) \right]_{t_j = d_j} &\to \partial_{t_j} \left[{}^{C}\partial^{\alpha_j}_{c_j^+} f(t) \right]_{t_j = d_j}.
\end{aligned}$$

Hence, due to the above pointwise convergence, the third and fourth terms converge to zero when $k \to \infty$. For the last term in (5.33), we get

$$\begin{aligned}
&\int_{\Theta} \left| y_k(t) f_k(t) \left(\mathcal{Q}(t) - \lambda_k^{(1)} w(t) \right) - y^{(1)}(t) f(t) \left(\mathcal{Q}(t) - \lambda^{(1)} w(t) \right) \right| dt \\
&\leq \int_{\Theta} \left| \mathcal{Q}(t) \left(y_k(t) f_k(t) - y^{(1)}(t) f(t) \right) \right| dt + \int_{\Theta} \left| w(t \left(\lambda_k^{(1)} y_k(t) f_k(t) - \lambda^{(1)} y^{(1)}(t) f(t) \right) \right| dt \\
&\leq \|q\| \left[M_1 \|f_k(t) - f(t)\|_{L^2} + \|f(t)\|_{L^2} \left\| y_k(t) - y^{(1)}(t) \right\|_{L^2} \right] + \|w\| \left[R \left(M_1 \|f_k(t) - f(t)\| \right.\right. \\
&\quad \left.\left. + \|f(t)\| \left\| y_k(t) - y^{(1)}(t) \right\| \right) + \left\| y^{(1)}(t) f(t) \right\| |\lambda_k^{(1)} - \lambda^{(1)}| \right],
\end{aligned}$$

where $M_1 = \sup_{\substack{k \in \mathbb{N} \\ t \in \Theta}} \|y_k(t)\|$ and $R = \sup_{k \in \mathbb{N}} |\lambda_k^{(1)}|$.
We conclude that

$$0 = \lim_{k \to \infty} I_k = I,$$

and Eq. (5.32) is satisfied for $y^{(1)}(t)$ being the limit of subsequence y_{k_l} of sequence $\{y_k\}_{k \in \mathbb{N}}$.
The remaining part of the proof can be done similarly to [101]. We demonstrate that $y^{(1)}(t)$ is also solution of the N-dimensional FSLP (5.1)-(5.2) in domain Θ and $\{y_k(t)\}_{k \in \mathbb{N}}$ converges to $y^{(1)}(t)$. Further, we find EF $y^{(2)}(t)$ and corresponding EV $\lambda^{(2)}$ such that

$$\lambda^{(1)} < \lambda^{(2)}.$$

Eventually, if we continue the above process, we can find EVs $\lambda^{(3)}, \lambda^{(4)}, \ldots$ and corresponding EFs $y^{(3)}(t), y^{(4)}(t), \ldots$. We can see that if $N = 1$, the presented result reduces to the case studied in [39]. $\square$

5.2.1 Example

Let us consider the following fractional Laplace eigenvalue problem

$$-\left({}^{C}\nabla^{\alpha}_{1-}\cdot\left({}^{C}\nabla^{\alpha}_{0+}y\right)\right)(t)=\lambda y(t)\ \text{in}\ \Theta, \tag{5.34}$$

with boundary conditions

$$y=0,\ \text{ on }\partial\Theta, \tag{5.35}$$

where $\alpha=(\alpha_1,\alpha_2)\in(0,1]$ and $t=(t_1,t_2)\in\Theta=(0,1)\times(0,1)$ and $\partial\Theta$ is the boundary of the domain Θ. The problem (5.34) becomes a special case of N-DFSLP (5.1-5.2) with $N=2$, $p=1$, $q=0$ and $w=1$. Hence, Eq. (5.34) can be re-written as

$$J_\alpha(y)=\int_\Theta\left[\left({}^{C}\nabla^{\alpha}_{0+}y(t)\right)\cdot\left({}^{C}\nabla^{\alpha}_{0+}y(t)\right)\right]dt=\sum_{j=1}^{2}\left\|{}^{C}\partial^{\alpha_j}_{0+}y(t)\right\|^2_{L^2(\Theta)}, \tag{5.36}$$

with respect to integral constraint

$$\int_\Theta y^2(t)\,dt=1. \tag{5.37}$$

Let us assume that $\alpha^{(1)}=(\alpha^{(1)}_1,\alpha^{(1)}_2)$ and $\alpha^{(2)}=(\alpha^{(2)}_1,\alpha^{(2)}_2)$ satisfy the conditions with $0<\alpha^{(1)}<\alpha^{(2)}<1$, then for functionals $J_{\alpha^{(1)}}$, $J_{\alpha^{(2)}}$, we obtain

$$\begin{aligned}J_{\alpha^{(1)}}(y)&=\sum_{j=1}^{2}\left\|{}^{C}\partial^{\alpha^{(1)}_j}_{0+}y(t)\right\|^2_{L^2}=\sum_{j=1}^{2}\left\|I^{1-\alpha^{(1)}_j}_{0+}\partial_{t_j}y(t)\right\|^2_{L^2}\\&=\sum_{j=1}^{2}\left\|I^{\alpha^{(2)}_j-\alpha^{(1)}_j}_{0+}I^{1-\alpha^{(2)}_j}_{0+}\partial_{t_j}y(t)\right\|^2_{L^2}\\&\le K_{\alpha^{(2)}-\alpha^{(1)}}J_{\alpha^{(2)}}(y),\end{aligned} \tag{5.38}$$

where $K_{\alpha^{(2)}-\alpha^{(1)}} = \left(\frac{1}{\left(\Gamma(\alpha_1^{(2)}-\alpha_1^{(1)}+1)\right)^2} + \frac{1}{\left(\Gamma(\alpha_2^{(2)}-\alpha_2^{(1)}+1)\right)^2}\right)$. Since, we have $J_{\alpha^{(1)}}(y) \leq K_{\alpha^{(2)}-\alpha^{(1)}} J_{\alpha^{(2)}}(y)$, therefore in terms of eigenvalues $\lambda_k^{(j)}$ for every $j, k \in \mathbb{N}$, we obtain

$$\lambda_k^{(j)}(\alpha^{(1)}) \leq K_{\alpha^{(2)}-\alpha^{(1)}} \lambda_k^{(j)}(\alpha^{(2)}).$$

In particular, If we compare the EVs for classical and fractional Laplace eigenvalue problems, we see that the EVs of the integer-order case are greater than the corresponding fractional problem, for each $j, k \in \mathbb{N}$

$$\lambda_k^{(j)}(\alpha^{(1)}) \leq K_{\alpha^{(2)}-\alpha^{(1)}} \lambda_k^{(j)}(1) = K_{\alpha^{(2)}-\alpha^{(1)}} \pi^2(j^2+k^2). \tag{5.39}$$

5.2.2 Numerical Validation

Here, we also present the numerical simulation of the given problem (5.34) to validate our theoretical results. We choose $\xi_k(t) = t_1^k t_2^k(1-t_1)(1-t_2), k = 1, 2, ..., n$ as basis function and approximate $y_n(t) = \sum_{k=1}^{n} A_k \Psi_k(t)$ with the BCs $y_n = 0$ on $\partial\Theta$, where $\Psi_k(t), k = 1, ..., n$ are orthonormal basis of $\xi_k(t)$. We evaluate EVs numerically for $n = 4$. From Table (5.1), we observe that eigenvalues increases wih $\alpha \in (0, 1)$ monotonically. For $\alpha = 1$, the first exact eigenvalue is 19.7392.

α	0.1	0.3	0.5	0.7	0.9	0.99
$\lambda^{(1)}$	2.1924	2.7971	4.1504	7.3213	14.1426	19.1922

TABLE 5.1: Lowest (First) eigenvalues $\lambda^{(1)}$ for fixed $n = 4$ and various α.

5.3 Conclusions

We proposed a variational approach to solve the regular N-DFSLP of order $\alpha = (\alpha_1, ..., \alpha_N) \in (0, 1]$ defined in terms of the fractional gradient operator of the left and right Caputo fractional derivatives. We demonstrated that N-DFSLP has countably many EVs, and every EV corresponds to an EF. Ferreira et al. [101] showed similar results for the N-DFSLP defined with RLFDs of fractional order $\alpha = (\alpha_1, ..., \alpha_N) \in (1/2, 1]$. Here, we extended the result for N-DFSLP defined with CFDs of order $\alpha \in (0, 1]$. Similarly, the similar result can also be proved for the N-DFSLPs defined with RLFDs or the combination of RLFD and CFD. Furthermore, Using an example, we implemented our theoretical results and computed eigenvalue for different values of α. Numerically, we demonstrate that approximated results validate our analytical predictions.

****