Chapter 4

Existence and Approximate Controllability Results for Fractional Integro-Differential Equations

4.1 Introduction

In the previous Chapter 3, we explored the existence and uniqueness results of the system (3.1) as well as controllability results for the control system (3.2). In this chapter, we extend the work done in Chapter 3 and discuss the existence and uniqueness of a mild solution to the problem (4.10) and the approximate controllability for the considered control system. In particular, this chapter focuses on the existence of mild solutions and the approximate controllability of a class of time-fractional integro-differential equations with delay by using the techniques of fractional calculus, semigroup theory, and fixed-point theory approaches.

In Chapters 2 and 3, we discussed the concept of controllability and its applications. However, many continuous infinite-dimensional systems encountered in practical scenarios, such as heat and wave equations, are not controllable. Consequently, the authors [108, 109] have focused on studying the approximate controllability of such models. Approximate controllability refers to the ability to steer the system's state from a given initial condition to a state arbitrarily close to a desired target state using an appropriate control. The concept of approximate controllability is a natural extension of exact controllability, addressing situations where achieving an exact target state is either impossible or impractical. While exact controllability requires the system to reach a precise final state from any initial condition, approximate controllability relaxes this requirement by ensuring the system can get arbitrarily close to the desired state within a finite time. This extension is particularly relevant in infinite-dimensional systems governed by partial differential equations, where constraints, nonlinearity, or system dynamics make exact control unattainable.

In [110], the authors established that the relaxation equation is approximately controllable if the corresponding linear system is approximately controllable. Chang et al. [111] considered the fractional differential systems of the Sobolev type in Banach spaces. By using the properties of resolvent operators and fixed-point technique, they proved approximate controllability results for Sobolev type fractional differential systems in the Caputo and Riemann-Liouville fractional derivatives. In [112], the authors proved sufficient conditions for the existence of mild solutions and approximate controllability for the control system of parabolic type along with nonlocal conditions. Apart from this, there has been significant development in the concept of control systems of fractional order in abstract spaces [113, 114, 115, 116].

We consider the following nonautonomous fractional functional integro-differential control system:

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big), \ t \in J \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{4.1}$$

where ${}_0^C D_t^\eta$ denotes the Caputo fractional operator of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The delays $\mu_i : J \to J$ are continuous functions and satisfy the conditions $0 \le \mu_i(t) \le t$, $(i = 1, 2, \ldots, n, n+1)$, $n \in \mathbb{N}$. The function $\mathsf{f} : J \times X^n \times U \to X$ is continuous. Further, $\{\mathcal{A}(t)\}_{t\ge 0}$ form a family of closed linear operators defined on the domain $D(\mathcal{A})$, where $D(\mathcal{A})$ is dense in X. Additionally, the integral operator $\mathcal{H}$ is defined by

$$\mathcal{H}\mathsf{u}(t) = \int_0^t \mathsf{h}\big(t, p, \mathsf{u}(p)\big) dp,$$

where $\mathsf{h} : \Delta \times U \to U$ is nonlinear and continuous such that $\Delta = \{(t, \mathsf{x}) : 0 \le \mathsf{x} \le t \le T\}$. Further, $\mathcal{G} \in BL\,(U, X)$. We note that U is a Banach space and $\mathsf{u}(\cdot)$ is a control function in a Banach space of admissible control functions $L^2\,(J, U)$.

The goal of this chapter is to expand on the work done in Chapter 3. This chapter investigates the existence, uniqueness, and approximate controllability of the control system (4.10) in Banach space along with the Caputo fractional derivative. Employing FC techniques, the Krasnoselskii fixed-point theorem, the Banach fixed-point theorem, and Gronwall's inequality, we explore our main results under the suitable assumption on $\mathsf{f}, \mathcal{G}$ and g.

The structure of this chapter is as follows: To begin, in Section 4.2, we discuss some fundamental definitions, theorem, and lemma that are essential to establishing our main results. Further, we establish a few existence and uniqueness results for the mild solutions in Section 4.3 for the system (4.10) under various assumptions on f. We establish the approximate controllability results for the control system (4.10) in Section 4.4.

4.2 Preliminary Results

Before delving into the main results, let us first discuss a few key definitions and foundational results that will be essential for the subsequent discussion.

Definition 4.1. (Mild Solution) [6, 73] A function $\mathsf{x} \in C(J, X)$ is said to be a mild solution of the control system (4.10) for every $\mathsf{u} \in L^2(J, U)$ if it satisfies the following integral equation:

$$\mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1} \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big]dp. \tag{4.2}$$

Lemma 4.2. ***(Krasnoselskii fixed-point theorem)*** *Let $\mathcal{S}$ be a closed, bounded, convex, and non-empty subset of a Banach space X. Let ψ_1 and ψ_2 be two operator such that*

(a) $\psi_1\mathsf{x} + \psi_2\mathsf{y} \in \mathcal{S}$ whenever $\mathsf{x}, \mathsf{y} \in \mathcal{S}$.

(b) ψ_1 is contraction mapping.

(c) ψ_2 is compact and continuous.

Then, there exists $\mathsf{z} \in \mathcal{S}$ such that $\psi_1\mathsf{z} + \psi_2\mathsf{z} = \mathsf{z}$.

Definition 4.3. [117] The set

$$K_T(\mathsf{f}) = \{\mathsf{x}(T) \in X : \mathsf{x}(\cdot) \text{ is a mild solution of } (4.10)\}$$

is said to be the reachable set of the problem (4.10).

The set $K_T(0)$ represents the reachable set corresponding to the linear system of the problem.

Definition 4.4. [117] If $\overline{K_T(\mathsf{f})} = X$, the problem (5.1) is approximately controllable on J, where $\overline{K_T(\mathsf{f})}$ represents the closure of $K_T(\mathsf{f})$. The corresponding linear system is approximately controllable on J, provided $\overline{K_T(0)} = X$.

Lemma 4.5. *[118] Suppose X and U are two normed linear spaces over a field F and ψ is an onto linear operator from X to U. Then, there exists a constant $\mathcal{L} > 0$ such that $\|\psi\mathsf{x}\|_U \geq \mathcal{L}\|\mathsf{x}\|_X, \ \forall\, \mathsf{x} \in X$ if and only if ψ^{-1} exists and is a bounded linear operator.*

4.3 Existence and Uniqueness Results

In this section, we discuss the existence and uniqueness of mild solutions for the fractional control system (4.10) under a few assumptions on f, $\mathcal{G}$, and g.

Theorem 4.6. *Suppose that the following conditions hold:*

(H_1) $\mathsf{f} : J \times X^n \times U \to X$ *is continuous, and there exist non-negative Lebesgue integrable functions* $\mathsf{k}_i(t) \in L^1(J, \mathbb{R}^+), (i = 1, 2, \ldots, n, n+1)$ *such that*

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big)\big\| \leq \sum_{i=1}^{n+1} \mathsf{k}_i(t)\|\mathsf{x}_i\|.$$

(H_2) *There exist a non-negative Lebesgue integrable function* $\mathsf{k}^*(t) \in L^1(J, \mathbb{R}^+)$ *such that* $\|\mathcal{H}\mathsf{u}(\mathsf{t})\| \leq \mathsf{k}^*(t), \ \forall\ \mathsf{u} \in L^2(J, U)$.

(H_3) *The resolvent operator* $\mathcal{R}_\eta(t,p)$ *is compact for* $t, p > 0$ *such that*

$$\mathcal{P}^* = \max_{0 \leq p \leq t \leq T} \|\mathcal{R}_\eta(t,p)\|$$

(H_4) *There exists* $\mathsf{w}(t) \in L^1(J, \mathbb{R}^+)$ *such that* $\|\mathcal{G}\mathsf{u}(t)\| \leq \mathsf{w}(t)$.

(H_5) *The function* g *is continuous, and there exist positive constants* $\mathcal{N}$ *and* c *such that* $\|\mathsf{g}(\mathsf{x})\| \leq \mathsf{c}$ *and* $\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\| \leq \mathcal{N}\|\mathsf{x} - \mathsf{y}\|$ *along with* $\mathcal{P}^*\mathcal{N} < 1$.

Then, the system (3.1) *has a mild solution.*

Proof. Let us consider an operator $\Phi : \mathcal{S}_m \to \mathcal{S}_m$, where $\mathcal{S}_m = \{\mathsf{x} \in X : \|\mathsf{x}\| \leq m,\ \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{g}(\mathsf{x})\}$, defined as:

$$\Phi\mathsf{x}(t) = \Phi_1\mathsf{x}(t) + \Phi_2\mathsf{x}(t), \tag{4.3}$$

where

$$\begin{aligned} \Phi_1\mathsf{x}(t) &= \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big), \\ \Phi_2\mathsf{x}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t R_\eta(t,p)(t-p)^{\eta-1} \\ &\quad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big]\,dp. \end{aligned}$$

To demonstrate that the system (4.10) has a mild solution, it is sufficient to show that the equation (4.3) possesses a fixed-point. The proof involves the following steps:

Step 1: First of all, we show that $\Phi_1\mathsf{x}(t) + \Phi_2\mathsf{y}(t) \in \mathcal{S}_m$ for every $\mathsf{x}, \mathsf{y} \in \mathcal{S}_m$. For this, we have

$$\begin{aligned}
&\big\|\Phi_1\mathsf{x}(t) + \Phi_2\mathsf{y}(t)\big\| \\
&\quad \leq \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big)\big\| + \Bigg\|\frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1} \\
&\qquad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\big]dp.\Bigg\| \\
&\quad \leq \mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{c}\big) + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1} \\
&\qquad \times \left[\mathsf{w}(p) + \sum_{i=1}^n \mathsf{k}_i(p)\|\mathsf{y}_i\| + \mathsf{k}_{n+1}(p)\big\|\mathcal{H}\mathsf{u}(\mathsf{p})\big\|\right]dp \\
&\quad \leq \mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{c}\big) + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1} \\
&\qquad \times \left[\mathsf{w}(p) + \sum_{i=1}^n \mathsf{k}_i(p)\|\mathsf{y}_i\| + \mathsf{k}_{n+1}(p)\mathsf{k}^*(p)\right]dp \\
&\quad \leq \mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{c}\big) + \frac{\mathcal{P}^* M}{\Gamma(\eta)} \\
&\quad \leq m,
\end{aligned}$$

where

$$M = \frac{1}{\Gamma(\eta)}\sup_{t\in J}\int_0^t (t-p)^{\eta-1}\left[\mathsf{w}(p) + \sum_{i=1}^n \mathsf{k}_i(p)\|\mathsf{y}_i\| + \mathsf{m}^*(p)\right]d\mathsf{p},$$

with $\mathsf{m}^*(p) = \mathsf{k}_{n+1}(p)\mathsf{k}^*(p)$. Therefore, Φ maps $\mathcal{S}_m$ to itself.

Step 2: We show that Φ_1 is a contraction mapping. Now for every $\mathsf{x}, \mathsf{y} \in \mathcal{S}_m$, we have

$$\begin{aligned}
\big\|\Phi_1(\mathsf{x}) - \Phi_1(\mathsf{y})\big\| &= \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) - \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{y})\big)\big\| \\
&\leq \big\|\mathcal{R}_\eta(t,0)\big\|\big\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\big\| \\
&\leq \mathcal{P}^*\mathcal{N}\big\|\mathsf{x} - \mathsf{y}\big\|,
\end{aligned}$$

where $\mathcal{P}^*\mathcal{N} < 1$, and hence Φ_1 is a contraction mapping.

Step 3: We demonstrate that Φ_2 is continuous and compact. To prove continuity, we consider a sequence $\{\mathsf{x}_k\}_{k\in\mathbb{N}} \subset \mathcal{S}_m$ with $\mathsf{x}_k \to \mathsf{x}$, where $\mathsf{x} \in \mathcal{S}_m$. By the continuity of the function f, for any $t \in J$, and noting that $0 \le \mu_i(t) \le t$ $(i = 1, 2\ldots, n, n+1)$, there exists a constant $N_0 \in \mathbb{N}$ such that for any $k > N_0$, we have

$$\begin{aligned}\Big\|\mathsf{f}\big(t,&\mathsf{x}_k(\mu_1(t)), \mathsf{x}_k(\mu_2(t)), \ldots, \mathsf{x}_k(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big)\\ &- \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big)\Big\| < \frac{\Gamma(\eta)}{\mathcal{P}^*T^\eta}\epsilon. \end{aligned} \tag{4.4}$$

Hence, for $k > N_0$, $t \in J$, we have

$$\begin{aligned}\big\|\Phi_2(\mathsf{x}_k) - \Phi_2(\mathsf{x})\big\| \le{}& \frac{1}{\Gamma(\eta)}\int_0^t \big\|R_\eta(t,p)\big\|\big\|(t-p)^{\eta-1}\big\|\\ &\times \Bigg\|\Big[\mathsf{f}\big(p, \mathsf{x}_k(\mu_1(p)), \mathsf{x}_k(\mu_2(p)), \ldots, \mathsf{x}_k(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\\ &- \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\Big]dp\Bigg\|\\ \le{}& \frac{\mathcal{P}^*T^{\eta-1}}{\Gamma(\eta)}\int_0^t \Bigg\|\Big[\mathsf{f}\big(p, \mathsf{x}_k(\mu_1(p)), \mathsf{x}_k(\mu_2(p)), \ldots, \mathsf{x}_k(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\\ &- \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\Big]dp\Bigg\|\\ <{}& \epsilon.\end{aligned}$$

Next, we demonstrate that Φ_2 is compact. To achieve this, we first establish that Φ_2 is uniformly bounded. For $\mathsf{x} \in \mathcal{S}_m$, $t \in J$, we have

$$\begin{aligned}\big\|\Phi_2\mathsf{x}\big\| \le{}& \Bigg\|\frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1}\\ &\times\Big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\Big]dp\Bigg\|\\ \le{}& \frac{1}{\Gamma(\eta)}\int_0^t \big\|R_\eta(t,p)\big\|(t-p)^{\eta-1}\end{aligned}$$

$$\times \left[\left\| \mathcal{G}\mathsf{u}(p) \right\| + \left\| \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \right\| \right] dp$$
$$\leq \frac{\mathcal{P}^*}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \left[\mathsf{w}(p) + \sum_{i=1}^{n} \mathsf{k}_i(p) \|\mathsf{x}_i\| + \mathsf{m}^*(p) \right] dp$$
$$\leq \frac{\mathcal{P}^* M}{\Gamma(\eta)}.$$

Thus, it follows that Φ_2 is uniformly bounded. Further, we show that Φ_2 is equicontinuous. For this, let $t_1, t_2 \in J, t_1 < t_2$, and $\mathsf{x} \in \mathcal{S}_m$, we have

$$\begin{aligned}
&\big\| \Phi_2 \mathsf{x}(t_2) - \Phi_2 \mathsf{x}(t_1) \big\| \\
&\quad \leq \Bigg\| \frac{1}{\Gamma(\eta)} \int_0^{t_2} (t_2-p)^{\eta-1} \mathcal{R}_\eta(t_2,p) \\
&\qquad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \big] dp \\
&\qquad - \frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1-p)^{\eta-1} \mathcal{R}_\eta(t_1,p) \\
&\qquad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \big] dp \Bigg\| \\
&\quad \leq \Bigg\| \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big[(t_2-p)^{\eta-1} \mathcal{R}_\eta(t_2,p) - (t_1-p)^{\eta-1} \mathcal{R}_\eta(t_1,p) \Big] \\
&\qquad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \big] dp \Bigg\| \\
&\qquad + \Bigg\| \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2-p)^{\eta-1} \mathcal{R}_\eta(t_2,p) \\
&\qquad \times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \big] dp \Bigg\| \\
&\quad = I_1 + I_2.
\end{aligned}$$

Now, by using the hypothesis (H_1), we have

$$\begin{aligned}
I_1 = &\Bigg\| \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big[(t_2-p)^{\eta-1} \mathcal{R}_\eta(t_2,p) - (t_1-p)^{\eta-1} \mathcal{R}_\eta(t_1,p) \Big] \\
&\times \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \big] dp \Bigg\|
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big[(t_2 - p)^{\eta-1} \mathcal{R}_\eta(t_2, p) - (t_1 - p)^{\eta-1} \mathcal{R}_\eta(t_1, p) \Big]$$
$$\times \Big[\mathsf{w}(p) + \sum_{i=1}^{n+1} \mathsf{k}_i(p) \|\mathsf{x}_i\| + \mathsf{m}^*(p) \Big] dp$$
$$\leq \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big[(t_2 - p)^{\eta-1} \mathcal{R}_\eta(t_2, p) - (t_1 - p)^{\eta-1} \mathcal{R}_\eta(t_1, p) \Big] M dp$$
$$\to 0 \quad \text{as} \quad t_1 \to t_2.$$

Clearly, $I_2 \to 0$ as $t_1 \to t_2$, consequently, $\big\|\Phi_2 \mathsf{x}(t_2) - \Phi_2 \mathsf{x}(t_1)\big\| \to 0$ as $t_1 \to t_2$. Hence, Φ_2 is equicontinuous. Therefore, $\Phi_2 : \mathcal{S}_m \to \mathcal{S}_m$ is compact as an application of the Arzelá-Ascoli theorem. Thus, by using Lemma 4.2, the fractional control system (4.10) has a solution. □

In the previous Theorem 4.6, we established the existence of a mild solution for the system (4.10) using Krasnoselskii's fixed point theorem. Subsequently, we demonstrate the uniqueness of the mild solution to the control system (4.10) by applying the Banach contraction mapping theorem.

Theorem 4.7. *If the condition (H_3) of the Theorem 4.6 holds true and suppose that the following conditions staisfy:*

(R_1) *The mapping $\mathsf{f} : J \times X^n \times U \to X$ is continuous, and there exist non-negative Lebesgue integrable functions $\mathsf{k}_i(t) \in L^1(J, \mathbb{R}^+), (i = 1, 2, \ldots, n, n+1)$ such that*

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(t, \mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_n, \mathsf{y}_{n+1}\big)\big\| \leq \sum_{i=1}^{n+1} \mathsf{k}_i(t) \|\mathsf{x}_i - \mathsf{y}_i\|.$$

(R_2) *The function g is continuous and there exists a constant $\mathcal{N} > 0$ such that $\big\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\big\| \leq \mathcal{N}\big\|\mathsf{x} - \mathsf{y}\big\|$.*

(R_3) *Let* $\mathcal{K} = \left[\mathcal{P}^*\mathcal{N} + \frac{\mathcal{P}^*}{\Gamma(\eta)}\sup_{t\in J}\int_0^t (t-p)^{\eta-1}\mathsf{k}(p)dp\right]$ *be such that* $\mathcal{K} < 1$*, where* $\mathsf{k}(t) = \sum_{i=1}^{n}\mathsf{k}_i(t)$.

Then, the system (4.10) *has a unique mild solution.*

Proof. Let us define the operator $\Phi : C(J,X) \to C(J,X)$ given by

$$\Phi\mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1} \times \left[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\right]dp.$$

Here, we will show that Φ is a contraction mapping on $C(J,X)$. For all $\mathsf{x},\mathsf{y} \in C(J,X)$, we have

$$\begin{aligned}
&\big\|\Phi\mathsf{x}(t) - \Phi\mathsf{y}(t)\big\| \\
&\quad\le \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) - \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{y})\big)\big\| \\
&\qquad + \bigg\|\frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1} \\
&\qquad \times \Big\{\big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\big] \\
&\qquad - \big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\big]\Big\}dp\bigg\| \\
&\quad\le \mathcal{P}^*\mathcal{N}\|\mathsf{x}-\mathsf{y}\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1} \\
&\qquad \times \bigg\|\Big\{\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big) \\
&\qquad - \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big)\Big\}\bigg\|dp \\
&\quad\le \mathcal{P}^*\mathcal{N}\|\mathsf{x}-\mathsf{y}\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\left[\sum_{i=1}^{n}\mathsf{k}_i(p)\|\mathsf{x}_i - \mathsf{y}_i\|\right]dp \\
&\quad\le \left[\mathcal{P}^*\mathcal{N} + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\sum_{i=1}^{n}\mathsf{k}_i(p)dp\right]\|\mathsf{x}-\mathsf{y}\|
\end{aligned}$$

$$\leq \mathcal{K}\|\mathsf{x}-\mathsf{y}\|.$$

Thus, we have

$$\left\|\Phi\mathsf{x}(t)-\Phi\mathsf{y}(t)\right\| \leq \mathcal{K}\|\mathsf{x}-\mathsf{y}\|.$$

Hence, Φ is a contraction mapping on $C\left(J,X\right)$, and therefore, it has a unique mild solution. □

4.4 Approximate Controllability Results

In the proceeding Section 4.3, we investigated the existence and uniqueness results of a mild solution to the equation (4.10). In this section, we discuss another crucial concept known as approximate controllability. Our discussion mainly concentrates on the proof of approximate controllability of the system (4.10) under suitable assumptions of f. First, we suppose that the control operator $\mathcal{G}$ is the identity operator. In this case, the control system (4.10) can be written as

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathsf{u}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big), \;\; t \in J \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X. \end{cases} \tag{4.5}$$

Consider the following linear control system

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{y}(t) = \mathcal{A}(t)\mathsf{y}(t) + \mathsf{v}(t) \\ \mathsf{y}(0) = \mathsf{x}_0 - \mathsf{g}(\mathsf{x}). \end{cases} \tag{4.6}$$

Consider the hypothesis:

(H_*) Let $\big\|\mathsf{h}(t,p,\mathsf{u})-\mathsf{h}(t,p,\mathsf{v})\big\| \le \mathcal{M}\|\mathsf{u}-\mathsf{v}\|$, for some positive constant $\mathcal{M}>0$.

Theorem 4.8. *Let the hypotheses of Theorem 4.7 and the condition (H_*) be satisfied. Then, the control system (4.5) is approximately controllable provided the system (4.6) is approximately controllable along with $\mathcal{M}T\big\|\mathsf{k}_{n+1}(t)\big\|<1$.*

Proof. Consider the following nonlinear system:

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathsf{f}\big(t,\mathsf{x}(\mu_1(t)),\mathsf{x}(\mu_2(t)),\dots,\mathsf{x}(\mu_n(t)),\mathcal{H}\mathsf{u}(t)\big) \\ \qquad\qquad + \mathsf{v}(t) - \mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}(t)\big), \\ \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{g}(\mathsf{x}), \quad \mathsf{x}_0 \in X. \end{cases} \tag{4.7}$$

By comparing equations (4.5) and (4.7), we get the following

$$\mathsf{u}(t) = \mathsf{v}(t) - \mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}(t)\big).$$

Now, we show that there exists a $\mathsf{u}(\cdot)\in L^2([0,T],U)$ such that

$$\mathsf{u}(t) = \mathsf{v}(t) - \mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}(t)\big)$$

Let $\mathsf{u}_0\in U$ and $\mathsf{u}_{m+1} = \mathsf{v}(t) - \mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}_m\big)$. Then, we have

$$\begin{aligned} \big\|\mathsf{u}_{m+1}-\mathsf{u}_m\big\| &\le \big\|\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}_m\big) \\ &\qquad - \mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}_{m-1}\big)\big\| \\ &\le \big\|\mathsf{k}_{n+1}(t)\big\|\big\|\mathcal{H}\mathsf{u}_m-\mathcal{H}\mathsf{u}_{m-1}\big\| \\ &\le \mathcal{M}T\big\|\mathsf{k}_{n+1}(t)\big\|\big\|\mathsf{u}_m-\mathsf{u}_{m-1}\big\| \end{aligned}$$

$$\begin{aligned}&\leq \mathcal{M}^2T^2\big\|\mathsf{k}_{n+1}(t)\big\|^2\big\|\mathsf{u}_{m-1}-\mathsf{u}_{m-2}\big\|\\&\quad\vdots\\&\leq \mathcal{M}^mT^m\big\|\mathsf{k}_{n+1}(t)\big\|^m\big\|\mathsf{u}_1-\mathsf{u}_0\big\|.\end{aligned}$$

Since $\mathcal{M}T\big\|\mathsf{k}_{n+1}(t)\big\| < 1$, it follows that $\big\|\mathsf{u}_{m+1}-\mathsf{u}_m\big\| \to 0$ as $m\to\infty$. Consequently, it $\{\mathsf{u}_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence and hence converges to an element $\mathsf{u}\in U$. Now,

$$\begin{aligned}&\Big\|\big(\mathsf{v}-\mathsf{u}_{m+1}\big)-\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}\big)\Big\|\\&\quad=\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}_m\big)\\&\qquad-\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}\big)\\&\quad\leq \mathcal{M}T\big\|\mathsf{k}_{m+1}(t)\big\|\|\mathsf{u}_m-\mathsf{u}\|\to 0\ \text{ as } m\to\infty.\end{aligned}$$

Since $\mathsf{y}_m\to\mathsf{y}$ as $m\to\infty$. Thus, we obtain

$$\begin{aligned}\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}\big)&=\lim_{m\to\infty}(\mathsf{v}-\mathsf{u}_{m+1})\\&=\mathsf{v}-\mathsf{u}.\end{aligned}$$

Hence, $\mathsf{u}(t)=\mathsf{v}(t)-\mathsf{f}\big(t,y(\mu_1(t)),\mathsf{y}(\mu_2(t)),\dots,\mathsf{y}(\mu_n(t)),\mathcal{H}\mathsf{u}(t)\big)$. It is easy to show that this control $\mathsf{u}(t)$ exists uniquely. For a control $\mathsf{v}(t)$, the mild solution $\mathsf{y}(t)$ of (4.6) is

$$\mathsf{y}(t)=\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0-\mathsf{g}(\mathsf{x})\big)+\frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1}\mathsf{v}(p)dp,$$

where $\mathsf{y}\in C([0,T],U)$. Further, the mild solution of (4.7) is given by

$$\begin{aligned}\mathsf{x}(t)=\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0-\mathsf{g}(\mathsf{x})\big)+\frac{1}{\Gamma(\eta)}\int_0^t R_\eta(t,p)(t-p)^{\eta-1}\\\times\Big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\dots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{u}(p)\big)\end{aligned}$$

$$+\mathsf{v}(p)-\mathsf{f}\big(p,\mathsf{y}(\mu_1(p)),\mathsf{y}(\mu_2(p)),\ldots,\mathsf{y}(\mu_n(p)),\mathcal{H}\mathsf{u}(p)\big)\big]dp.$$

Thus, we have

$$\begin{aligned}\big\|\mathsf{x}(t)-\mathsf{y}(t)\big\| &= \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0-\mathsf{g}(\mathsf{x})\big)-\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0-\mathsf{g}(\mathsf{x})\big)\big\| \\ &\quad+\frac{1}{\Gamma(\eta)}\int_0^t \big\|R_\eta(t,p)\big\|(t-p)^{\eta-1} \\ &\quad\times\bigg\|\big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{u}(p)\big) \\ &\quad-\mathsf{f}\big(p,\mathsf{y}(\mu_1(p)),\mathsf{y}(\mu_2(p)),\ldots,\mathsf{y}(\mu_n(p)),\mathcal{H}\mathsf{u}(p)\big)\big]\bigg\|dp. \\ &\le \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\sum_{i=1}^n \mathsf{k}_i(t)\|\mathsf{x}(\mu_i(p))-\mathsf{y}(\mu_i(p))\|dp \\ &\le \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{k}(p)\|\mathsf{x}-\mathsf{y}\|dp.\end{aligned}$$

Hence,

$$\|\mathsf{x}-\mathsf{y}\| \le \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{k}(p)\|\mathsf{x}-\mathsf{y}\|dp.$$

Therefore, by Gronwall's inequality $\mathsf{x}(t)=\mathsf{y}(t), \forall\ t\in[0,T]$. Since every solution of the linear system (4.6) with control v is also a solution of the nonlinear system with control u. Thus, $\mathcal{R}_T(0)\subset\mathcal{R}_T(\mathsf{f})$. Since $\mathcal{R}_T(0)$ is dense in U as the linear system (4.6) is approximately controllable, it follows that $\mathcal{R}_T(\mathsf{f})$ is dense in U. Consequently, the problem (4.5) is approximately controllable. This concludes the proof. □

In the previous Theorem 4.8, we addressed the approximate controllability of the fractional control system (4.10) under the condition that the controller is the identity operator I. Now, in the following theorem, we extend our discussion to any control operator. For this, we consider the following linear system:

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{y}(t) = \mathcal{A}(t)\mathsf{y}(t) + \mathcal{G}\mathsf{u}(t), \ \ t \in J \\ \\ \mathsf{y}(0) = \mathsf{x}_0 - \mathsf{g}(\mathsf{x}), \end{cases} \tag{4.8}$$

Theorem 4.9. *Let $\mathcal{M} > 0$ be a constant such that $\left\|\mathcal{G}\mathsf{u}\right\| \geq \mathcal{L}\mathsf{u}$, $\forall\ \mathsf{u} \in U$ and $\mathcal{R}(\mathsf{f}) \subseteq \mathcal{R}(\mathcal{G})$. Suppose the hypotheses of the Theorem (4.7) and the condition (H_*) hold. Then, the control system (4.10) is approximately controllable, provided the corresponding linear control System 4.8 is approximately controllable and $\dfrac{\mathcal{M}T\|\mathsf{k}_{n+1}(t)\|}{\mathcal{L}} < 1$.*

Proof. Consider the following nonlinear system:

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \dots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big) \\ \qquad\qquad + \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \dots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big), \\ \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{g}(\mathsf{x}), \quad \mathsf{x}_0 \in X. \end{cases} \tag{4.9}$$

On comparing the system (4.10) and (4.9), we have

$$\mathcal{G}\mathsf{u}(t) = \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \dots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big)$$

In the beginning, we demonstrate that there is a $\mathsf{u}(\cdot) \in L^2\left([0,T], U\right)$ such that

$$\mathcal{G}\mathsf{u}(t) = \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \dots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big)$$

Let $\mathsf{u}_0 \in U$ and $\mathcal{G}\mathsf{u}_{m+1} = \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_m\big),\ m \in \mathbb{N}$. Then, by using Lemma 4.5, we get

$$\begin{aligned}
\big\|\mathsf{u}_{m+1} - \mathsf{u}_m\big\| &\leq \bigg\|\mathcal{G}^{-1}\Big\{\mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_m\big) \\
&\quad - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_{m-1}\big)\Big\}\bigg\| \\
&\leq \frac{1}{\mathcal{L}}\big\|\mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_m\big) \\
&\quad - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_{m-1}\big)\big\| \\
&\leq \frac{\mathcal{M}T\|\mathsf{k}_{n+1}(t)\|}{\mathcal{L}}(t)\big\|\mathsf{u}_m - \mathsf{u}_{m-1}\big\| \\
&\leq \frac{\big(\mathcal{M}T\big)^2\|\mathsf{k}_{n+1}(t)\|^2}{\mathcal{L}^2}\big\|\mathsf{u}_{m-1} - \mathsf{u}_{m-2}\big\| \\
&\ \ \vdots \\
&\leq \left(\frac{\mathcal{M}T\|\mathsf{k}_{n+1}(t)\|}{\mathcal{L}}\right)^m\big\|\mathsf{u}_1 - \mathsf{u}_0\big\|.
\end{aligned}$$

Since $\dfrac{\mathcal{M}T\|\mathsf{k}_{n+1}(t)\|}{\mathcal{L}} < 1$, it shows that $\left(\dfrac{\mathcal{M}T\|\mathsf{k}_{n+1}(t)\|}{\mathcal{L}}\right)^m \to 0$ as $m \to \infty$. Therefore, the right-hand side of the above inequality converges to zero. Thus, the sequence $\{\mathsf{u}_m\}$ is a Cauchy sequence in U. Therefore, $\{\mathsf{u}_m\}$ converges to $\mathsf{u} \in U$, because U is a Banach space. Now,

$$\begin{aligned}
&\big\|\mathbf{G}\big(\mathsf{v} - \mathsf{u}_{m+1}\big) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}\big)\big\| \\
&\qquad = \big\|\mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}_m\big) \\
&\qquad\quad - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}\big)\big\| \\
&\qquad \leq \mathcal{M}T\big\|\mathsf{k}_{n+1}(t)\big\|\|\mathsf{u}_m - \mathsf{u}\|.
\end{aligned}$$

Since $\mathsf{u}_m \to \mathsf{u}$ as $m \to \infty$, the right-hand side of the above inequality tends to zero when $m \to \infty$, we have

$$\begin{aligned} \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathcal{H}\mathsf{u}\big) &= \lim_{m\to\infty} \mathcal{G}(\mathsf{v} - \mathsf{u}_{m+1}) \\ &= \mathcal{G}\mathsf{v} - \mathcal{G}\mathsf{u}. \end{aligned}$$

Hence, $\mathcal{G}\mathsf{u}(t) = \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(t, y(\mu_1(t)), \mathsf{y}(\mu_2(t)), \ldots, \mathsf{y}(\mu_n(t)), \mathsf{u}(t)\big)$, It is easy to show that this control $\mathsf{u}(t)$ exists uniquely.

Corresponding to the control $\mathsf{v}(t)$, let $\mathsf{y}(t)$ be a solution of (4.8) , which is given as:

$$\mathsf{y}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)} \int_0^t R_\eta(t,p)(t-p)^{\eta-1} Gv(p)dp,$$

and the mild solution of the system (4.9) is

$$\begin{aligned} \mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) &+ \frac{1}{\Gamma(\eta)} \int_0^t R_\eta(t,p)(t-p)^{\eta-1} \\ &\times \big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p) \\ &+ \mathcal{G}\mathsf{v}(t) - \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big] dp. \end{aligned}$$

Thus, for $t \in [0,T]$, we obtain

$$\begin{aligned} \mathsf{x}(t) - \mathsf{y}(t) = \frac{1}{\Gamma(\eta)} &\int_0^t R_\eta(t,p)(t-p)^{\eta-1} \\ &\times \big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{u}(p) \\ &- \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{u}(p)\big] dp. \end{aligned}$$

Now, by applying the technique from the Theorem 4.8, we get $\mathsf{x}(t) = \mathsf{y}(t),\ \forall\ t \in J$. It follows that every solution to the problem (4.8) with control v is also a solution to the Problem (4.10) with control u, *i.e.*, the set of solutions to the nonlinear system with control u and the set of solutions to the System (4.8) are the same.

Thus, $\mathcal{R}_T(\mathsf{f}) \supset \mathcal{R}_T(0)$, which is dense in U since $\mathcal{R}_T(0)$ is dense in U. Consequently, the System (4.10) is approximately controllable. This concludes the proof. □

4.5 Applications

Example 4.1. *Let's consider the following fractional differential equation:*

$$\begin{cases} {}_0^C D_t^{\eta}\mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) + e^t \sin \mathsf{x}\big(\mu_1(t)\big) \\ \qquad\qquad + \dfrac{1 - e^{-\mathsf{x}(\mu_2(t))}}{1+t^2} + (1+t^2)\displaystyle\int_0^T \frac{t\ p^3\ \mathsf{u}(p)}{3} dp, \quad t \in [0, T], \\ \mathsf{x}(0) + \tanh(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{4.10}$$

where $\mathsf{u}(t) \geq 0$ *a.e.* $t \in [0, T]$*, and* $X = \mathbb{R}$*.*

Here,

$$\mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \mathcal{H}\mathsf{u}(t)\big) = e^t \sin \mathsf{x}\big(\mu_1(t)\big) + \frac{1 - e^{-\mathsf{x}(\mu_2(t))}}{1+t^2} + (1+t^2)\int_0^T \frac{t\ p^3\ \mathsf{u}(p)}{3} dp$$

Clearly,

$$\big\|\mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \mathcal{H}\mathsf{u}(t)\big)\big\| \leq e^t \big\|\mathsf{x}\big(\mu_1(t)\big)\big\| + \frac{\big\|\mathsf{x}\big(\mu_2(t)\big)\big\|}{1+t^2} + (1+t^2)\big\|\mathcal{H}\mathsf{u}(t)\big\|$$

Further, we have

$$\mathcal{H}\mathsf{u}(t) = \int_0^T \frac{t\ p^3\ \mathsf{u}(p)}{3} dp.$$

Since $\mathsf{u}(t) \in L^2\big(J, \mathbb{R}\big)$, and $\mathsf{u}(t) \geq 0$ $a.e.$ $t \in [0, T]$ so we have $\mathsf{u}(t) \in L^1\big(J, \mathbb{R}^+\big)$ for $t \in [0, T]$ and consequently there exists a non-negative Lebesgue integrable function $\mathsf{k}^*(t) \in L^1(J, \mathbb{R}^+)$ such that $\big\|\mathcal{H}\mathsf{u}(t)\big\| \leq \mathsf{k}^*(t)$.

Again, $\mathsf{g}(\mathsf{x}) = \tanh \mathsf{x}$, which is a continuous function satisfying $\|\mathsf{g}(\mathsf{x})\| \leq 1$. Moreover,

$$\big\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\big\| \leq \|\mathsf{x} - \mathsf{y}\|$$

which implies that $\mathcal{N} = 1$.

Let the Resolvent operator $\mathcal{R}_\eta(t, p)$ be compact for all $t, p > 0$, and satisfies

$$\max_{0 \leq p \leq t \leq T} \big\|\mathcal{R}_\eta(t, p)\big\| = \mathcal{P}^* < 1,$$

and there exists a function $\mathsf{w}(t) \in L^1\left(J, \mathbb{R}^+\right)$ such that

$$\big\|\mathcal{G}\mathsf{u}(t)\big\| \leq \mathsf{w}(t).$$

Then, all the assumptions of Theorem 4.6 are satisfied, and consequently, the system (4.10) has a mild solution.

4.6 Conclusion

In this chapter, we explore the existence and uniqueness of mild solutions for the fractional control system (4.10) under various assumptions on the force function f, control term $\mathcal{G}$, and nonlocal term g. Additionally, we investigate the system's approximate controllability, initially assuming the controller to be the identity operator and subsequently extending the results to any control term under different hypotheses on f, $\mathcal{G}$, and g. To establish these results, we employ the theory of FC, semigroup theory, nonlinear analysis, the Krasnoselskii fixed-point theorem, and Gronwall's inequality.

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