Study on Numerical Schemes for Solving Coupled Fractional Order Diffusion Equations

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DOCTOR OF PHILOSOPHY

by

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Chapter 6

Conclusion and future work

6.1 Conclusion

The diffusion phenomenon is the most occurring phenomenon in nature and can be observed in many areas. It is governed by a PDE known as a diffusion equation. Diffusion equation has numerous applications in the fields of engineering and science. Experimental evidence has demonstrated that we are unable to use the classical diffusion equation to model every diffusion phenomenon. Here the fractional order diffusion equation is considered with different types of fractional derivatives in constant and variable forms. When fractional calculus came into light, the researchers found a more effective way to model the physical and biological diffusion phenomena accurately, but the complexity of the model will be increased simultaneously.

Things become more complicated when dealing with nonlinear coupled PDEs. The classical methods became almost useless for dealing with nonlinear coupled fractional order models. This became a challenging task for the researchers to develop new methods to deal with the complex models accurately. Due to complexity, it is not always possible to find the exact solution of those complex models, so the spectral method along with the operational matrix and the non-standard finite difference method have been used by using different types of polynomials.

In chapter 2, a very efficient nonstandard method has been developed with the help of Fibonacci polynomials to deal with nonlinear coupled spatial fractional order

Burgers' equations. The accuracy of the developed scheme has been shown numerically. Through comparison, it has been shown that the proposed numerical method is performing more accurately as compared to the previously used methods. Also, the behavior of the diffusion model for different spatial fractional-order for different particular cases has been observed.

In chapter 3, the numerical technique with the operational matrix has been discussed with the help of Bernstein polynomials to deal with the nonlinear variable order coupled system of a reaction-diffusion equation with the prescribed initial and boundary conditions. The validation of the method has been shown by solving different types of nonlinear examples. The error bound of any function with the Bernstein expansion has been derived. Overshoots of solute concentration for different variable orders have also been justified.

In chapter 4, a generalized Gray-Scott model which has nonlinearity in reaction terms is studied with initial and boundary conditions. This model contains time derivatives in the form of variable order. It has been solved with the help of an operational matrix and the collocation method. The accuracy and efficiency of the method are validated by comparing the exact solutions and numerical solutions of test problems through error analysis.

In chapter 5, a variable fractional order advection-diffusion-reaction predator-prey equation has been taken. An operational matrix with collocation scheme is developed with the help of the shifted airfoil polynomials. It presents an analysis of the existence and uniqueness of a solution for the model under consideration. Furthermore, it examines the Ulam-Hyers stability of the solution. The effectiveness of the proposed method is demonstrated through illustrative examples, revealing that the suggested method is both efficient and sufficiently accurate.

6.2 Future work

There are some interesting and challenging research problems that can be studied in the near future. This section provides future work to consolidate the study presented in this thesis. The present study is restricted to one-dimensional nonlinear coupled fractional order diffusion equations. However, this analysis can be extended to two or higher-dimensional nonlinear fractional order diffusion equations. The key areas that can be focused on for future research are identified here. Some of the proposed extensions of the work made in the thesis are highlighted as follows:

- Future work may focus on numerical solutions for the more complex models like two-dimensional nonlinear variable order reaction-advection-diffusion and the two-dimensional variable order coupled system of equations.
- The proposed techniques can be applied to various nonlinear systems, such as the disease model, epidemic model and ecological model, among many others.
- The more accurate numerical technique can be discovered for coupled advection-reaction-diffusion equations with nonlinearity in diffusion terms by using the different fractional variable order operators and generalized operators.
- One can develop a scheme with different polynomials or wavelets to study the multi-dimensional reaction-advection-diffusion equations with other non-singular fractional variable order derivatives.

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