

# A Study on Generalizations of Direct Projective Modules and Direct Projective Covers of Modules



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*MATHEMATICAL SCIENCES*

*by*

**Sonal Gupta**

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DEPARTMENT OF MATHEMATICAL SCIENCES

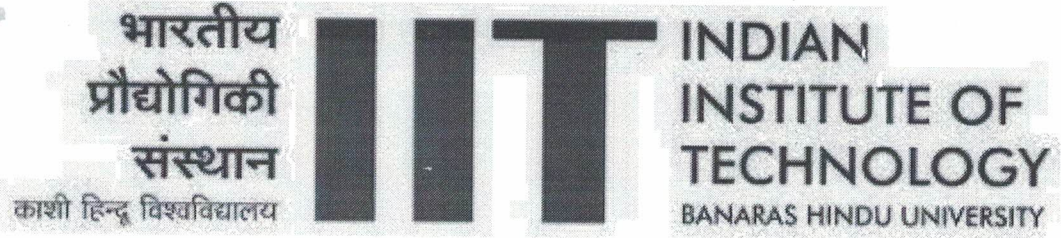
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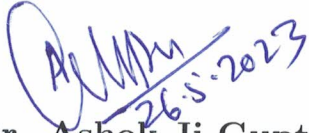
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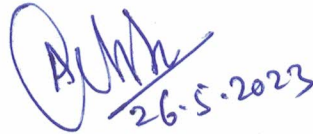
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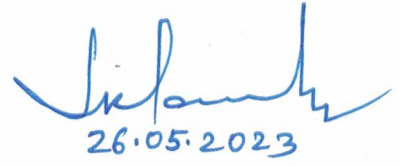
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*TO*  
*MY BELOVED FAMILY*  
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*SUPERVISOR*



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---

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## ABBREVIATION

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$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$
$\mathbb{Z}_{p^\infty}$	The prufer $p$ -group
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\subseteq$	A subset
$\leq$	A submodule
$\leq^\oplus$	A direct summand
$M^{(I)}$	The direct sum of $I$ copies of $M$ indexed by $I$
$M^I$	The direct product of $I$ copies of $M$ indexed by $I$
$\leq^p$	A pure submodule
$\trianglelefteq$	A fully invariant submodule or An ideal
$\leq^e$	An essential submodule
$Mat_n(X)$	$n$ by $n$ matrix over the set $X$

# Contents

<b>Preface</b>	<b>xi</b>
<b>Abbreviation</b>	<b>xiii</b>
<b>Introduction</b>	<b>xv</b>
<b>1 Preliminaries</b>	<b>1</b>
<b>2 Direct Projective Modules and Semi-Simple Direct Projective Modules</b>	<b>15</b>
2.1 Direct Projective Modules . . . . .	16
2.2 Semi-Simple Direct Projective Modules . . . . .	24
<b>3 Finite Direct Projective Modules</b>	<b>31</b>
3.1 Finite Direct Projective Modules . . . . .	32
3.2 Characterization of rings using Finite Direct Projective Modules . . .	36
<b>4 Finite Direct Projective Covers and Envelopes</b>	<b>45</b>
4.1 Finite Direct Projective Covers . . . . .	46
4.2 Finite Direct Projective Envelopes . . . . .	49
<b>5 Pure Direct Projective Modules</b>	<b>55</b>
5.1 Pure Direct Projective Modules . . . . .	55
5.2 Characterizations of rings using Pure Direct Projective Modules . . .	63
<b>Conclusion and Future Scope</b>	<b>67</b>
<b>Bibliography</b>	<b>71</b>
<b>List of Publications</b>	<b>77</b>