# Chapter 2

# Pure C1 Modules

In 1965, Utumi[53] observed the $C1$ condition on a ring, which is satisfied if the ring is self-injective. Later, similarly $C1$ condition on a module $N$ is defined as every submodule of an $R$-module $N$ is essential in a direct summand of $N$. A module satisfying the $C1$ condition is known as a $C1$ module or extending module. As we know that every submodule of a module $N$ need not be a pure submodule (In fact no submodule is a pure submodule of $\mathbb{Z}$ as $\mathbb{Z}$-module). Motivated by the above facts, in this chapter, we study the class of modules which have the property that every pure submodule is essential in a direct summand. These modules are termed as pure $C1$ modules which is a proper generalization of $C1$ modules. Examples and counterexamples are given. We study some properties of pure $C1$ modules and characterize von Neumann regular rings, semisimple rings, local rings, and PDS rings in terms of pure $C1$ modules and introduce the pure continuous modules as the generalization of the continuous modules.

## 2.1 Pure C1 Modules

In this section, we introduce pure $C1$ modules and study their properties.

**Definition 2.1.1.** *An $R$-Module $N$ is called a pure $C1$ (or pure extending) module if every pure submodule of $N$ is essential in a direct summand of $N$.*

**Example 2.1.2.** *(i) Since, only pure submodules of $\mathbb{Z}$-module $\mathbb{Z}$ are $\{0\}$ and $\mathbb{Z}$ itself. Therefore $\mathbb{Z}$ as a $\mathbb{Z}$-module is a pure $C1$.*

*(ii) Any pure injective module $N$ is a pure $C1$ module (since every pure injective module is quasi pure injective and by Proposition 2.1.16).*

*(iii) Semisimple modules and injective modules are pure $C1$.*

*(iv) Any finitely generated module over a Noetherian ring is a pure $C1$ module. Since its pure submodules are just direct summands [Corollary 4.91,[32]]. In particular, every finitely generated $\mathbb{Z}$-module is pure $C1$.*

The example below shows that every $R$-module (in particular $\mathbb{Z}$-module) need not be pure $C1$.

**Example 2.1.3.** *Consider a $\mathbb{Z}$-module $N$ such that $N = \Pi_{p\in\mathbb{P}}\mathbb{Z}/\langle p\rangle$ where $p$ varies through all primes. Let $P = \oplus_{p\in\mathbb{P}}\mathbb{Z}/\langle p\rangle$ be a pure submodule of $N$ which is not essential in a direct summand of $N$.*

**Proposition 2.1.4.** *The direct summand of a pure $C1$ module is a pure $C1$ module.*

*Proof.* Let $K$ be a direct summand of pure $C1$ module $N$ such that $N = K \oplus K'$ where $K'$ is a some submodule of $N$ and $P$ is a pure submodule of $K$. Since $P \leq K \leq^{\oplus} N$, $P$ is a pure submodule of $N$. As $N$ is a pure $C1$ module, therefore $P$ is essential in a direct summand of $N$. Since $P \leq K$ and $P \cap K' = 0$ then $P$ is essential in $K$. Hence, $K$ is pure $C1$. $\square$

**Corollary 2.1.5.** *Over a PDS ring, every pure submodule of a pure $C1$ module is a pure $C1$ module.*

*Proof.* Over a PDS ring, every pure submodule of pure $C1$ $R$-module is a pure $C1$ $R$-module. □

**Remark 2.1.6.** *Submodule of a pure $C1$ module need not be pure $C1$.*
*Let $N$ be a module that is not a pure $C1$ and $PE(N)$ be a pure injective hull of $N$. Then $PE(N)$ be a pure injective module which implies $PE(N)$ is pure $C1$ while $N \leq PE(N)$ is not pure $C1$. While over the von Neumann regular ring, every submodule of a pure $C1$ module is pure $C1$.*

The following examples show that the class of pure $C1$ modules is a proper generalization of the class of $C1$ modules.

**Example 2.1.7.** *Consider $N = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as $\mathbb{Z}$-module, where $p$ is any prime. In particular for $p = 2$, $P = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$-module. Since $N$ is a finitely generated module over a Noetherian ring, each pure submodule is a direct summand [Corollary 4.91,[32]], so $P$ is pure $C1$ whereas $P$ is not $C1$. In fact, its submodule $\mathbb{Z}(1+2\mathbb{Z}, 2+8\mathbb{Z})$ is not essential in any direct summand of $P$.*

**Example 2.1.8.** *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $R_R$ is finitely generated and Noetherian module, it is a pure $C1$ module whereas it is not a $C1$ module [Example 6.2,[10]].*

In the following Proposition, we give a sufficient condition for the pure $C1$ modules to be $C1$ modules.

**Proposition 2.1.9.** *A ring $R$ is a von-Neumann regular if and only if every pure $C1$ $R$-module is $C1$ module.*

*Proof.* Let $N$ be a pure $C1$ module. Then every pure submodule of $N$ is essential in a direct summand of $N$. As $R$ is a von-Neumann regular ring, every submodule of an $R$-module $N$ is pure. Hence $N$ is $C1$ module.
Conversely, every pure $C1$ $R$-module is $C1$ module, which implies every submodule of an $R$-module $N$ is pure. Hence, $R$ is a von Neumann regular ring. □

**Proposition 2.1.10.** *Let $N$ be a module that is fully invariant in its pure injective hull. Then $N$ is a pure $C1$ module.*

*Proof.* Proof follows from [Lemma 3.1,[21]].

□

**Proposition 2.1.11.** *Let Rand $S$ be rings for which there is a Morita equivalence $f$ : Mod-$R \to$ Mod-$S$ and let $A \in$ Mod-$R$. Then $A$ is pure $C1$ if and only if $f(A)$ is pure $C1$.*

*Proof.* It follows from the Morita invariant property of pure submodule. □

**Definition 2.1.12.** *[21]. A submodule $K$ of a module $N$ is said to be pure essential in $N$ if $K$ is pure in $N$ and for any non zero submodule $S$ of $N$ either $K \cap S \neq 0$ or $(K \oplus S)/S$ is not pure in $N/S$*

**Proposition 2.1.13.** *If every submodule of a module $N$ is pure essential in $N$, then $N$ is pure $C1$.*

*Proof.* Let $K$ be a pure submodule of $N$. Since every submodule is pure essential in $N$, it follows that every pure submodule is essential. Then $K$ is essential in $N$. Hence, $N$ is a pure $C1$ module. □

The converse of the above statement need not be true in general. Now, we provide the sufficient condition when it holds true.

**Proposition 2.1.14.** *If $N$ is a pure simple pure $C1$ module, then every submodule of $N$ is a pure essential submodule.*

*Proof.* If $N$ is a pure simple module, then it has no nonproper nontrivial pure submodule. Hence, every submodule of $N$ is pure essential. □

**Corollary 2.1.15.** *If $N$ is an indecomposable pure $C1$ module, then every submodule of $N$ is a pure essential submodule.*

**Proposition 2.1.16.** *In any quasi pure injective module $N$, every pure submodule of $N$ is essential in a direct summand of $N$.*

*Proof.* Let $P$ be a pure submodule of $N$ and write $PE(N) = PE(P) \oplus L$ where $L$ is some submodule of $N$. The quasi pure injectivity of $N$, it follows that $N \cap PE(N) = (N \cap PE(P)) \oplus (N \cap L)$ and $P \leq^e (N \cap PE(P))$. Hence, $P$ is essential in a direct summand of $N$. □

**Proposition 2.1.17.** *Let $N$ be a $\mathbb{Z}$-module. If $N$ satisfies any one of the following conditions, then $N$ is a pure $C1$ module.*

*(i) $N$ is finitely generated.*

*(ii) $N$ is divisible.*

*Proof.* $(i)$ If $N$ is a finitely generated module then by Corollary 4.91, [32], every pure submodule of $N$ is a direct summand, as $\mathbb{Z}$ is a Noetherian ring. Thus, every pure submodule is essential in itself. Hence, $N$ is pure $C1$.

$(ii)$ Let $N$ be a divisible module. Then it is a $C1$ $\mathbb{Z}$-module. Hence $N$ is a pure $C1$ module.

□

**Proposition 2.1.18.** *Every pure split module is pure $C1$.*

*Proof.* Let $N$ be a pure split $R$-module. Every pure submodule of a pure split module $N$ is a direct summand. Hence, $N$ is a pure $C1$ module. □

**Proposition 2.1.19.** *Let $N = N_1 \oplus N_2$ be a direct sum decomposition of a module $N$. If $P = P_1 \oplus P_2$ is a pure submodule of $N$ then each $P_i$ is pure submodule of $N_i$, where $P_i = P \cap N_i$ for $i = \{1, 2\}$.*

*Proof.* Since $P_i \leq^{\oplus} P$, therefore $P_i$ is a pure submodule of $P$. As $P$ is pure in $N$, then each $P_i = P \cap N_i$, is pure in $N_i$ for $i = \{1, 2\}$. □

**Theorem 2.1.20.** *Let $N = \oplus_{i\in\Lambda} N_i$, where $\Lambda$ be a finite index set. Then, $N$ is pure $C1$ if and only if $N_i$ is pure $C1$ module for every $i \in \Lambda$.*

*Proof.* Since $N$ is a pure $C1$ module. So, by proposition 2.1.4 $N_i$ is pure $C1$ module for each $i \in \Lambda$.

Conversely, Let $P$ be a pure submodule of $N$. From proposition (2.1.19), $P = \oplus_{i\in\Lambda}(P_i)$ such that $P_i$ is a pure submodule of $N_i$ for each $i \in \Lambda$ . Since $N_i$ is pure $C1$, so every pure submodule of $N_i$ is essential in a direct summand of $N_i$. As $P_i$ is a pure submodule of $N_i$, so $P_i$ is also essential in a direct summand of $N_i$ i.e., $P_i \leq^e K_i \leq^{\oplus} N_i$. Therefore $P = \oplus_{i\in\Lambda} P_i \leq^e \oplus_{i\in\Lambda} K_i \leq^{\oplus} N$. Hence, $N$ is a pure $C1$ module. □

**Definition 2.1.21** ([58, 34.8(c), pp. 290])**.** *A submodule $P$ of a module $N$ is RD (relatively divisible) pure if for every $r \in R$, $rP = rN \cap P$.*

It is obvious that every pure submodule is $RD$-pure submodule. We say an $R$-module $N$ is $RD$-pure $C1$ if every $RD$- pure submodule is essential in a direct summand of $N$.

**Lemma 2.1.22.** *Every pure $C1$ module is RD-pure $C1$.*

*Proof.* Let $N$ be a pure $C1$ module and $K$ be a pure submodule of $N$. Since $N$ is pure $C1$, so $K$ is essential in a direct summand. But we already know that every pure submodule is $RD$-pure, therefore $K$ is $RD$-pure submodule and essential in a direct summand. Hence, the module $N$ is $RD$-pure $C1$. □

**Example 2.1.23.** *In [page no.-159,[32]], Lam has given an example that shows that every RD-pure submodule need not be a pure submodule. Therefore, it will tend to happen that there exists a RD-pure $C1$ module which will not be pure $C1$, thou any specific example couldn't be found.*

**Proposition 2.1.24.**

1. *Direct summand of RD-pure $C1$ module is RD-pure $C1$.*
2. *RD-pure submodule of RD-pure $C1$ module is RD-pure $C1$.*

*Proof.* The proof is similar to proposition 2.1.5 and 2.1.4. □

**Proposition 2.1.25.** *A free (projective) R-module $N$ is RD-pure $C1$ if and only if $N$ is pure $C1$.*

*Proof.* The proof follows from the [page no.159, [32]] □

**Corollary 2.1.26.** *Let $R$ be a right perfect ring. Then, a flat R-module $N$ is RD-pure $C1$ if and only if $N$ is pure $C1$.*

*Proof.* We know that if $R$ is flat, both projective and flatness will coincide [Theorem 24.25, [31]]. Therefore the result will hold by Proposition 2.1.25. □

## 2.2 Characterization of rings using pure C1 modules

In the next proposition, we characterize von Neumann regular rings in terms of pure $C1$ modules.

**Proposition 2.2.1.** *For a ring $R$, the following conditions are equivalent:*

1. *$R$ is a von Neumann regular ring.*
2. *Every pure $C1$ $R$-module is flat.*

*Proof.* (1) $\Rightarrow$ (2) It is clear from [Theorem 4.21,[32]]
(2) $\Rightarrow$ (1) Let $N$ be an right $R$-module and $PE(N)$ be the pure injective hull of $N$. Then $0 \to N \to PE(N) \to PE(N)/N \to 0$ is a pure exact sequence. By the given hypothesis, $PE(N)$ is a flat module so by [Corollary 4.86 (1),[32]], $PE(N)/N$ is a flat module. Therefore, by [Corollary 4.86(2),[32]], $N$ is a flat module which implies that $R$ is von Neumann regular ring [Theorem 4.21,[32]]. □

Recall that an $R$-module $N$ is called as a pure $C2$ (pure-direct-injective) module if $A$ is a pure submodule of $N$ and $B$ is some submodule of $N$ such that $A \cong B \leq^{\oplus} N$ then $A \leq^{\oplus} N$.
Again, recall that an $R$-module $N$ is called pure $C3$ if $A \oplus B$ is a pure submodule of $N$ for submodules $A$ and $B$ of $N$ with $A \leq^{\oplus} N$, $B \leq^{\oplus} N$ and $A \cap B = 0$, then $A \oplus B \leq^{\oplus} N$. In the next theorem, we characterize semisimple rings.

**Theorem 2.2.2.** *For a ring $R$, the following conditions are equivalent:*

1. *$R$ is a semisimple ring.*

2. *Every pure $C3$ $R$-module is projective.*

3. *Every pure $C2$ $R$-module is projective*

4. *Every quasi pure injective $R$-module is projective.*

5. *Every pure injective $R$-module is projective.*

6. *Every pure $C1$ $R$-module is projective.*

*Proof.* (1) $\Rightarrow$ (2) Let $R$ be a right semisimple ring then every $R$-module $N$ is projective [Proposition 20.7,[57]]. So (2) holds.

(2) $\Rightarrow$ (3) Every pure $C2$ module is pure $C3$, so (3) holds.

(3) $\Rightarrow$ (4) Every quasi pure Injective right $R$-module is pure $C2$, so (4) holds.

(4) $\Rightarrow$ (5) Every pure injective right $R$-module is quasi pure injective. so (5) holds.

(5) $\Rightarrow$ (1) As we know, the pure injective module is a generalization of the injective module. So by the given hypothesis, every injective $R$-module is projective. Therefore (1) holds by [Proposition 20.7,[57]]

(1) $\Leftrightarrow$ (4) It follows from [Proposition 6,[38]].

(1) $\Rightarrow$ (6) By the given hypothesis, every $R$-module over semisimple is projective. Hence, every pure $C1$ module is projective. By [Proposition 20.7,[57]], (6) holds.

(6) $\Rightarrow$ (1) By the given hypothesis, every pure $C1$ module is projective. As every injective module is a pure $C1$ module, it follows that every injective is projective. Hence, $R$ is a right semisimple ring [Proposition 20.7,[57]]. □

**Proposition 2.2.3.** *Let $R$ be a local ring. Then $R_R$ is a pure $C1$ module.*

*Proof.* Since every local ring is pure simple, hence indecomposable [Theorem 4,[18]], it follows that $R_R$ and 0 are its only pure submodules. Hence, $R_R$ is a pure $C1$ module. □

A ring $R$is called a left PDS ring if a pure submodule of a left $R$-module is a direct summand and ring $R$ is said to be a PDS if it is both left and right PDS [17].

**Proposition 2.2.4.** *For a PDS ring $R$, every $R$-module is pure $C1$.*

*Proof.* Let $N$ be an $R$-module. By the given hypothesis, $R$ is a PDS ring, it follows that every pure submodule is a direct summand. So, every pure submodule is essential in a direct summand. Hence, $N$ be a pure $C1$ module. □

**Proposition 2.2.5.** *Let $N$ be a flat cotorsion $R$-module. Then $N$ is pure $C1$.*

*Proof.* Let $N$ be a flat and cotorsion module. Then by [Proposition 3.2,[36]], $N$ is quasi-pure injective. Therefore $N$ is a pure $C1$ module. □

**Corollary 2.2.6.** *If $R$ is a right cotorsion ring then $R_R$ is pure $C1$ $R$-module.*

## 2.3 Pure Continuous Modules

In this section, we define pure continuous modules which is the generalization of the continuous modules.

**Definition 2.3.1.** *A pure continuous module is a module $N$ in which every pure submodule is essential in a direct summand of $N$ and every pure submodule $K$ of $N$ isomorphic to a direct summand of $N$ is itself a direct summand of $N$.*

*In other words, a module that is pure $C_1$ and pure $C_2$ is known as a pure continuous module.*

**Example 2.3.2.** *Every continuous module is a pure continuous module.*

It is easily seen from the definition of pure continuous modules that continuous modules are pure continuous modules whereas the converse is not true. Now, we show this in the following example.

**Example 2.3.3.** *Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module, it is pure $C_1$ and pure $C_2$ since its only pure submodules are 0 and $\mathbb{Z}$. Hence it is a pure continuous module moreover it is not $C_2$ since $2\mathbb{Z}$ is isomorphic to a direct summand of $\mathbb{Z}$ but it is itself, not a direct summand of $\mathbb{Z}$, which implies it is not a continuous module.*

**Proposition 2.3.4.** *Every direct summand of a pure $C2$ module is a pure $C2$ module [38].*

**Proposition 2.3.5.** *Every direct summand of a pure continuous module is a pure continuous module.*

*Proof.* From Proposition 2.3.4 and Proposition 2.1.4, the proof is obvious. □